Pre-AP® Algebra 1

TEACHER RESOURCES

Units 3 and 4

ABOUT COLLEGE BOARD

College Board is a mission-driven not-for-profit organization that connects students to college success and opportunity. Founded in 1900, College Board was created to expand access to higher education. Today, the membership association is made up of over 6,000 of the world's leading educational institutions and is dedicated to promoting excellence and equity in education. Each year, College Board helps more than seven million students prepare for a successful transition to college through programs and services in college readiness and college success—including the SAT® and the Advanced Placement Program®. The organization also serves the education community through research and advocacy on behalf of students, educators, and schools.

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PRE-AP EQUITY AND ACCESS POLICY

College Board believes that all students deserve engaging, relevant, and challenging grade-level coursework. Access to this type of coursework increases opportunities for all students, including groups that have been traditionally underrepresented in AP and college classrooms. Therefore, the Pre-AP program is dedicated to collaborating with educators across the country to ensure all students have the supports to succeed in appropriately challenging classroom experiences that allow students to learn and grow. It is only through a sustained commitment to equitable preparation, access, and support that true excellence can be achieved for all students, and the Pre-AP course designation requires this commitment.

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The sentence-writing strategies used in Pre-AP lessons are based upon The Writing Revolution, Inc., a national nonprofit organization that trains educators to implement The Hochman Method, an evidence-based approach to teaching writing. The strategies included in Pre-AP materials are meant to support students' writing, critical thinking, and content understanding, but they do not represent The Writing Revolution's full, comprehensive approach to teaching writing. More information can be found at **www.thewritingrevolution.org**.

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Introduction to Pre-AP Algebra 1

About Pre-AP

Introduction to Pre-AP

Every student deserves classroom opportunities to learn, grow, and succeed. College Board developed Pre-AP® to deliver on this simple premise. Pre-AP courses are designed to support all students across varying levels of readiness. They are not honors or advanced courses.

Participation in Pre-AP courses allows students to slow down and focus on the most essential and relevant concepts and skills. Students have frequent opportunities to engage deeply with texts, sources, and data as well as compelling higher-order questions and problems. Across Pre-AP courses, students experience shared instructional practices and routines that help them develop and strengthen the important critical thinking skills they will need to employ in high school, college, and life. Students and teachers can see progress and opportunities for growth through varied classroom assessments that provide clear and meaningful feedback at key checkpoints throughout each course.

DEVELOPING THE PRE-AP COURSES

Pre-AP courses are carefully developed in partnership with experienced educators, including middle school, high school, and college faculty. Pre-AP educator committees work closely with College Board to ensure that the course resources define, illustrate, and measure grade-level-appropriate learning in a clear, accessible, and engaging way. College Board also gathers feedback from a variety of stakeholders, including Pre-AP partner schools from across the nation who have participated in multiyear pilots of select courses. Data and feedback from partner schools, educator committees, and advisory panels are carefully considered to ensure that Pre-AP courses provide all students with grade-level-appropriate learning experiences that place them on a path to college and career readiness.

PRE-AP EDUCATOR NETWORK

Similar to the way in which teachers of Advanced Placement® (AP®) courses can become more deeply involved in the program by becoming AP Readers or workshop consultants, Pre-AP teachers also have opportunities to become active in their educator network. Each year, College Board expands and strengthens the Pre-AP National Faculty—the team of educators who facilitate Pre-AP Readiness Workshops and Pre-AP Summer Institutes. Pre-AP teachers can also become curriculum and assessment contributors by working with College Board to design, review, or pilot the course resources.

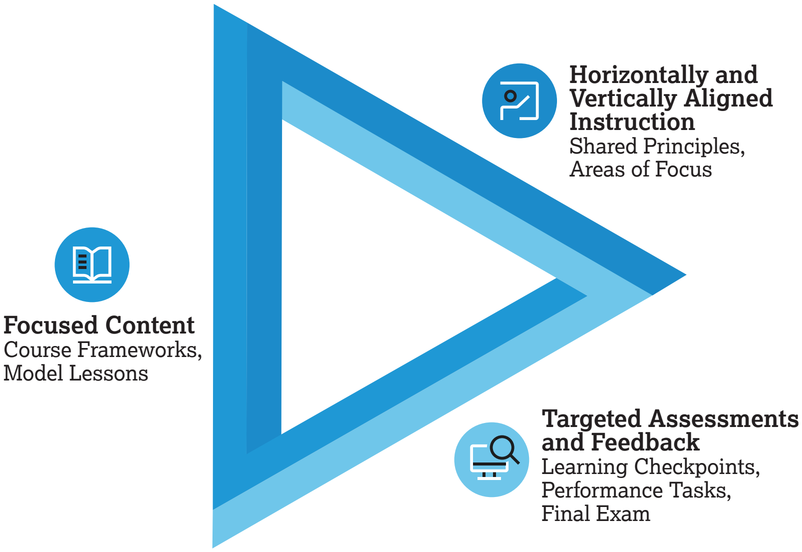
HOW TO GET INVOLVED

Schools and districts interested in learning more about participating in Pre-AP should visit **preap.org/join** or contact us at **preap@collegeboard.org**.

Teachers interested in becoming members of Pre-AP National Faculty or participating in content development should visit **preap.org/national-faculty** or contact us at **preap@collegeboard.org**.

Pre-AP Approach to Teaching and Learning

Pre-AP courses invite all students to learn, grow, and succeed through focused content, horizontally and vertically aligned instruction, and targeted assessments for learning. The Pre-AP approach to teaching and learning, as described below, is not overly complex, yet the combined strength results in powerful and lasting benefits for both teachers and students. This is our theory of action.



FOCUSED CONTENT

Pre-AP courses focus deeply on a limited number of concepts and skills with the broadest relevance for high school coursework and college and career success. The course framework serves as the foundation of the course and defines these prioritized concepts and skills. Pre-AP model lessons and assessments are based directly on this focused framework. The course design provides students and teachers with intentional permission to slow down and focus.

HORIZONTALLY AND VERTICALLY ALIGNED INSTRUCTION

Shared principles cut across all Pre-AP courses and disciplines. Each course is also aligned to discipline-specific areas of focus that prioritize the critical reasoning skills and practices central to that discipline.

SHARED PRINCIPLES

All Pre-AP courses share the following set of research-supported instructional principles. Classrooms that regularly focus on these cross-disciplinary principles allow students to effectively extend their content knowledge while strengthening their critical thinking skills. When students are enrolled in multiple Pre-AP courses, the horizontal alignment of the shared principles provides students and teachers across disciplines with a shared language for their learning and investigation, and multiple opportunities to practice and grow. The critical reasoning and problem-solving tools students develop through these shared principles are highly valued in college coursework and in the workplace.

Close Observation and Analysis

Students are provided time to carefully observe one data set, text, image, performance piece, or problem before being asked to explain, analyze, or evaluate. This creates a safe entry point to simply express what they notice and what they wonder. It also encourages students to slow down and capture relevant details with intentionality to support more meaningful analysis, rather than rush to completion at the expense of understanding.

Higher-Order Questioning

Students engage with questions designed to encourage thinking that is elevated beyond simple memorization and recall. Higher-order questions require students to make predictions, synthesize, evaluate, and compare. As students grapple with these questions, they learn that being inquisitive promotes extended thinking and leads to deeper understanding.

Evidence-Based Writing

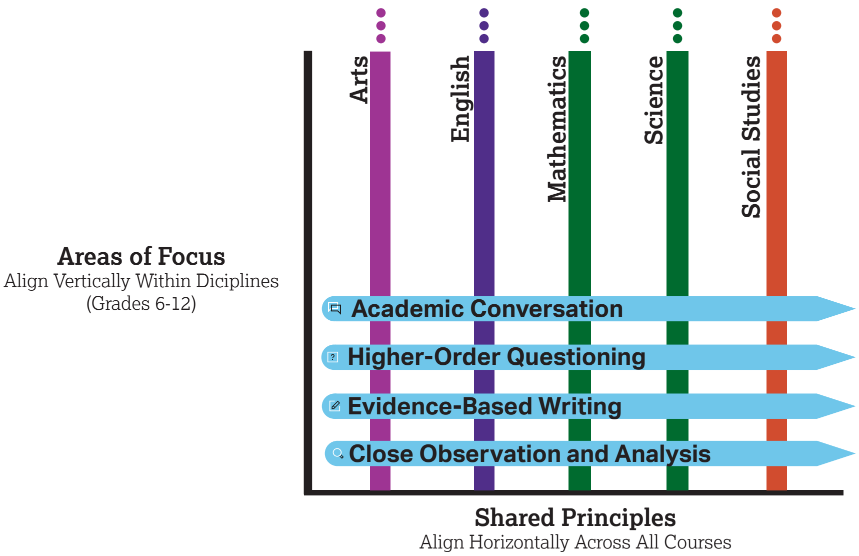
With strategic support, students frequently engage in writing coherent arguments from relevant and valid sources of evidence. Pre-AP courses embrace a purposeful and scaffolded approach to writing that begins with a focus on precise and effective sentences before progressing to longer forms of writing.

Academic Conversation

Through peer-to-peer dialogue, students' ideas are explored, challenged, and refined. As students engage in academic conversation, they come to see the value in being open to new ideas and modifying their own ideas based on new information. Students grow as they frequently practice this type of respectful dialogue and critique and learn to recognize that all voices, including their own, deserve to be heard.

AREAS OF FOCUS

The areas of focus are discipline-specific reasoning skills that students develop and leverage as they engage with content. Whereas the shared principles promote horizontal alignment across disciplines, the areas of focus provide vertical alignment within a discipline, giving students the opportunity to strengthen and deepen their work with these skills in subsequent courses in the same discipline.



For information about the Pre-AP mathematics areas of focus, see page 15.

TARGETED ASSESSMENTS FOR LEARNING

Pre-AP courses include strategically designed classroom assessments that serve as tools for understanding progress and identifying areas that need more support. The assessments provide frequent and meaningful feedback for both teachers and students across each unit of the course and for the course as a whole. For more information about assessments in Pre-AP Algebra 1, see page 55.

Pre-AP Professional Learning

Pre-AP teachers are required to engage in two professional learning opportunities. The first requirement is designed to help prepare them to teach their specific course. There are two options to meet the first requirement: the Pre-AP Summer Institute (Pre-APSI) and the Online Foundational Module Series. Both options provide continuing education units to educators who complete them.

- The Pre-AP Summer Institute is a four-day collaborative experience that empowers participants to prepare and plan for their Pre-AP course. While attending, teachers engage with Pre-AP course frameworks, shared principles, areas of focus, and sample model lessons. Participants are given supportive planning time where they work with peers to begin to build their Pre-AP course plan.
- The Online Foundational Module Series is available to all teachers of Pre-AP courses. This 12- to 20-hour course supports teachers in preparing for their Pre-AP course. Teachers explore course materials and experience model lessons from the student's point of view. They also begin to plan and build their own course so they are ready on day one of instruction.

The second professional learning requirement is to complete at least one of the Online Performance Task Scoring Modules, which offer guidance and practice applying Pre-AP scoring guidelines to student work.

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About the Course

Introduction to Pre-AP Algebra 1

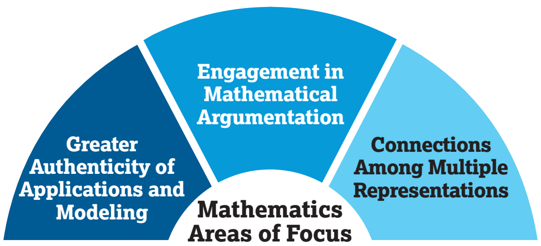
The Pre-AP Algebra 1 course is designed to deepen students' understanding of linear relationships by emphasizing patterns of change, multiple representations of functions and equations, modeling real world scenarios with functions, and methods for finding and representing solutions of equations and inequalities. Taken together, these ideas provide a powerful set of conceptual tools that students can use to make sense of their world through mathematics.

Rather than seeking to cover all topics traditionally included in a standard algebra textbook, this course focuses on the foundational algebraic knowledge and skills that matter most for college and career readiness. The Pre-AP Algebra 1 Course Framework highlights how to guide students to connect core ideas within and across the units of the course, promoting a coherent understanding of linear relationships.

The components of this course have been crafted to prepare not only the next generation of mathematicians, scientists, programmers, statisticians, and engineers, but also a broader base of mathematically informed citizens who are well equipped to respond to the array of mathematics-related issues that impact our lives at the personal, local, and global level.

PRE-AP MATHEMATICS AREAS OF FOCUS

The Pre-AP mathematics areas of focus, shown below, are mathematical practices that students develop and leverage as they engage with content. They were identified through educator feedback and research about where students and teachers need the most curriculum support. These areas of focus are vertically aligned to the mathematical practices embedded in other mathematics courses in high school, including AP, and in college, giving students multiple opportunities to strengthen and deepen their work with these skills throughout their educational career. They also support and align to the AP Calculus Mathematical Practices, the AP Statistics Course Skills, and the mathematical practices listed in various state standards.



Greater Authenticity of Applications and Modeling

Students create and use mathematical models to understand and explain authentic scenarios.

Mathematical modeling is a process that helps people explore, represent, analyze, and explain the world. In Pre-AP Algebra 1, students explore real-world contexts where mathematics can be used to make sense of a situation. They engage in the modeling process by making choices about what aspects of the situation to model, assessing how well the model represents the available data, drawing conclusions from their model, justifying decisions they make through the process, and identifying what the model helps clarify and what it does not.

In addition to mathematical modeling, Pre-AP Algebra 1 students engage in mathematics through authentic applications. Applications are similar to modeling problems in that they are drawn from real-world phenomena, but they differ because the applications dictate the appropriate mathematics to use to solve the problem. Pre-AP Algebra 1 balances these two types of real-world tasks.

Engagement in Mathematical Argumentation

Students use evidence to craft mathematical conjectures and prove or disprove them.

Conjecture and proof lie at the heart of the discipline of mathematics. Mathematics is both a way of thinking and a set of tools for solving problems. Pre-AP Algebra 1 students gain experience, comfort, and proficiency with mathematical thinking by observing and generalizing patterns in number sequences, graphs, equations, operations, and functions. They harness their curiosity to create problems to solve and conjectures to prove or disprove. Through mathematical argumentation, students learn how to be critical of their own reasoning and the reasoning of others.

Connections Among Multiple Representations

Students represent mathematical concepts in a variety of forms and move fluently among the forms.

Mathematical concepts can be represented in a variety of forms. Pre-AP Algebra 1 students learn how the multiple representations of a concept are connected to each other and how to fluently translate between graphical, numerical, algebraic, and verbal representations. Every mathematical representation illuminates certain characteristics of a concept while also obscuring other aspects. With experience that begins to develop in Pre-AP Algebra 1, students develop a nuanced understanding of which representations best serve a particular purpose.

PRE-AP ALGEBRA 1 AND CAREER READINESS

The Pre-AP Algebra 1 course resources are designed to expose students to a wide range of career opportunities that depend upon Algebra 1 knowledge and skills. Examples include not only field-specific careers such as mathematician or statistician but also other endeavors where algebraic knowledge is relevant and applicable, such as actuaries, engineers, programmers, carpenters, and HVAC technicians.

Career clusters that involve mathematics, along with examples of careers in mathematics and other careers that require the use of algebra, are provided below. Teachers should consider discussing these with students throughout the year to promote motivation and engagement.

Career Clusters Involving Mathematics
arts, A/V technology, and communications architecture and construction business management and administration finance government and public administration health science information technology manufacturing marketing STEM (science, technology, engineering, and math)

Examples of Mathematics Careers	**Examples of Algebra 1 Related Careers**
actuary financial analyst mathematician mathematics teacher professor programmer statistician	carpenter computer programmer economist electrician engineer HVAC technician operations research analyst programmer

Source for Career Clusters: "Advanced Placement and Career and Technical Education: Working Together." Advance CTE and the College Board. October 2018. **https://careertech.org/resource/ap-cte-working-together**.

For more information about careers that involve mathematics, teachers and students can visit and explore the College Board's Big Future resources: **https://bigfuture.collegeboard.org/majors/math-statistics-mathematics**.

SUMMARY OF RESOURCES AND SUPPORTS

Teachers are strongly encouraged to take advantage of the full set of resources and supports for Pre-AP Algebra 1, which is summarized below. Some of these resources must be used for a course to receive the Pre-AP Course Designation. To learn more about the requirements for course designation, see details below and on page 64.

COURSE FRAMEWORK

The framework defines what students should know and be able to do by the end of the course. It serves as an anchor for model lessons and assessments, and it is the primary document teachers can use to align instruction to course content. **Use of the course framework is required**. *For more details see page 22.*

MODEL LESSONS

Teacher resources, available in print and online, include a robust set of model lessons that demonstrate how to translate the course framework, shared principles, and areas of focus into daily instruction. **Use of the model lessons is encouraged but not required**. *For more details see page 53.*

LEARNING CHECKPOINTS

Accessed through Pre-AP Classroom (the Pre-AP digital platform), these short formative assessments provide insight into student progress. They are automatically scored and include multiple-choice and technology-enhanced items with rationales that explain correct and incorrect answers. **Use of one learning checkpoint per unit is required**. *For more details see page 55.*

PERFORMANCE TASKS

Available in the printed teacher resources as well as on Pre-AP Classroom, performance tasks allow students to demonstrate their learning through extended problem-solving, writing, analysis, and/or reasoning tasks. Scoring guidelines are provided to inform teacher scoring, with additional practice and feedback suggestions available in online modules on Pre-AP Classroom. **Use of each unit's performance task is required**. *For more details see page 56.*

PRACTICE PERFORMANCE TASKS

Available in the student resources, with supporting materials in the teacher resources, these tasks provide an opportunity for students to practice applying skills and knowledge as they would in a performance task, but in a more scaffolded environment. **Use of the practice performance tasks is encouraged but not required**. *For more details see page 57.*

FINAL EXAM

Accessed through Pre-AP Classroom, the final exam serves as a classroom-based, summative assessment designed to measure students' success in learning and applying the knowledge and skills articulated in the course framework. **Administration of the final exam is encouraged but not required**. *For more details see page 58.*

PROFESSIONAL LEARNING

Both the four-day Pre-AP Summer Institute (Pre-APSI) and the Online Foundational Module Series support teachers in preparing and planning to teach their Pre-AP course. **All Pre-AP teachers are required to either attend the Pre-AP Summer Institute or complete the module series. In addition, teachers are required to complete at least one Online Performance Task Scoring module**. *For more details see page 11.*

Course Map

PLAN

The course map shows how components are positioned throughout the course. As the map indicates, the course is designed to be taught over 140 class periods (based on 45-minute class periods), for a total of 28 weeks.

Model lessons are included for approximately 50% of the total instructional time, with the percentage varying by unit. Each unit is divided into key concepts.

TEACH

The model lessons demonstrate how the Pre-AP shared principles and mathematics areas of focus come to life in the classroom.

Shared Principles
Close observation and analysis
Higher-order questioning
Evidence-based writing
Academic conversation

Areas of Focus
Greater authenticity of applications and modeling
Engagement in mathematical argumentation
Connections among multiple representations

ASSESS AND REFLECT

Each unit includes two learning checkpoints and a performance task. These formative assessments are designed to provide meaningful feedback for both teachers and students.

Note: The final exam, offered during a six-week window in the spring, is not represented in the map.

UNIT 1 Linear Functions and Linear Equations

~45 Class Periods

Pre-AP model lessons provided for approximately 55% of instructional time in this unit

KEY CONCEPT 1.1
Constant Rate of Change and Slope

KEY CONCEPT 1.2
Linear Functions

Learning Checkpoint 1

KEY CONCEPT 1.3
Linear Equations

KEY CONCEPT 1.4
Linear Models of Nonlinear Scenarios

Learning Checkpoint 2

KEY CONCEPT 1.5
Two-Variable Linear Inequalities

Performance Task for Unit 1

UNIT 2 Systems of Linear Equations and Inequalities

~25 Class Periods

Pre-AP model lessons provided for approximately 30% of instructional time in this unit

KEY CONCEPT 2.1
The Solution to a System of Equations

KEY CONCEPT 2.2
Solving a System of Linear Equations Algebraically

KEY CONCEPT 2.3
Modeling with Systems of Linear Equations

Learning Checkpoint 1

KEY CONCEPT 2.4
Systems of Linear Inequalities

Learning Checkpoint 2

Performance Task for Unit 2

UNIT 3 Quadratic Functions

~45 Class Periods

Pre-AP model lessons provided for approximately 45% of instructional time in this unit

KEY CONCEPT 3.1
Functions with a Linear Rate of Change

KEY CONCEPT 3.2
The Algebra and Geometry of Quadratic Functions

Learning Checkpoint 1

KEY CONCEPT 3.3
Solving Quadratic Equations

KEY CONCEPT 3.4
Modeling with Quadratic Functions

Learning Checkpoint 2

Performance Task for Unit 3

UNIT 4 Exponent Properties and Exponential Functions

~25 Class Periods

Pre-AP model lessons provided for approximately 40% of instructional time in this unit

KEY CONCEPT 4.1
Exponent Rules and Properties

KEY CONCEPT 4.2
Roots of Real Numbers

Learning Checkpoint 1

KEY CONCEPT 4.3
Sequences with Multiplicative Patterns

KEY CONCEPT 4.4
Exponential Growth and Decay

Learning Checkpoint 2

Performance Task for Unit 4

Pre-AP Algebra 1 Course Framework

INTRODUCTION

Based on the Understanding by Design® (Wiggins and McTighe) model, the Pre-AP Algebra 1 Course Framework is back mapped from AP expectations and aligned to essential grade-level expectations. The course framework serves as a teacher's blueprint for the Pre-AP Algebra 1 instructional resources and assessments.

The course framework was designed to meet the following criteria:

- **Focused:** The framework provides a deep focus on a limited number of concepts and skills that have the broadest relevance for later high school, college, and career success.
- **Measurable:** The framework's learning objectives are observable and measurable statements about the knowledge and skills students should develop in the course.
- **Manageable:** The framework is manageable for a full year of instruction, fosters the ability to explore concepts in depth, and enables room for additional local or state standards to be addressed where appropriate.
- **Accessible:** The framework's learning objectives are designed to provide all students, across varying levels of readiness, with opportunities to learn, grow, and succeed.

COURSE FRAMEWORK COMPONENTS

The Pre-AP Algebra 1 Course Framework includes the following components:

Big Ideas

The big ideas are recurring themes that allow students to create meaningful connections between course concepts. Revisiting the big ideas throughout the course and applying them in a variety of contexts allows students to develop deeper conceptual understandings.

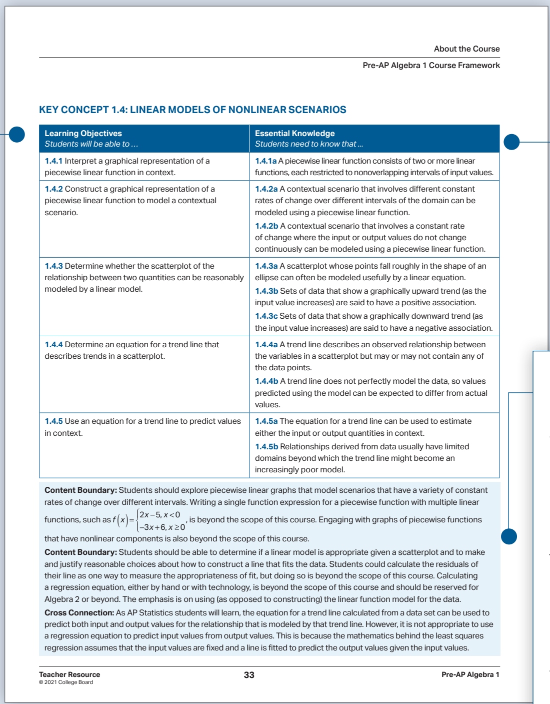
Enduring Understandings

Each unit focuses on a small set of enduring understandings. These are the long-term takeaways related to the big ideas that leave a lasting impression on students. Students build and earn these understandings over time by exploring and applying course content throughout the year.

Key Concepts

To support teacher planning and instruction, each unit is organized by key concepts. Each key concept includes relevant **learning objectives** and **essential knowledge statements** and may also include **content boundary and cross connection statements**. These are illustrated and defined below.

Learning Objectives:
These objectives define what a student needs to be able to do with essential knowledge to progress toward the enduring understandings. The learning objectives serve as actionable targets for instruction and assessment.



About the Course
Pre-AP Algebra 1 Course Framework

KEY CONCEPT 1.4: LINEAR MODELS OF NONLINEAR SCENARIOS

Learning Objectives *Students will be able to ...*	Essential Knowledge *Students need to know that ...*
1.4.1 Interpret a graphical representation of a piecewise linear function in context.	1.4.1a A piecewise linear function consists of two or more linear functions, each restricted to nonoverlapping intervals of input values.
1.4.2 Construct a graphical representation of a piecewise linear function to model a contextual scenario.	1.4.2a A contextual scenario that involves different constant rates of change over different intervals of the domain can be modeled using a piecewise linear function. 1.4.2b A contextual scenario that involves a constant rate of change where the input or output values do not change continuously can be modeled using a piecewise linear function.
1.4.3 Determine whether the scatterplot of the relationship between two quantities can be reasonably modeled by a linear model.	1.4.3a A scatterplot whose points fall roughly in the shape of an ellipse can often be modeled usefully by a linear equation. 1.4.3b Sets of data that show a graphically upward trend (as the input value increases) are said to have a positive association. 1.4.3c Sets of data that show a graphically downward trend (as the input value increases) are said to have a negative association.
1.4.4 Determine an equation for a trend line that describes trends in a scatterplot.	1.4.4a A trend line describes an observed relationship between the variables in a scatterplot but may or may not contain any of the data points. 1.4.4b A trend line does not perfectly model the data, so values predicted using the model can be expected to differ from actual values.
1.4.5 Use an equation for a trend line to predict values in context.	1.4.5a The equation for a trend line can be used to estimate either the input or output quantities in context. 1.4.5b Relationships derived from data usually have limited domains beyond which the trend line might become an increasingly poor model.

Content Boundary: Students should explore piecewise linear graphs that model scenarios that have a variety of constant rates of change over different intervals. Writing a single function expression for a piecewise function with multiple linear functions, such as $f(x)=\begin{cases}2x-5, x<0\\-3x+6, x\geq 0\end{cases}$, is beyond the scope of this course. Engaging with graphs of piecewise functions that have nonlinear components is also beyond the scope of this course.

Content Boundary: Students should be able to determine if a linear model is appropriate given a scatterplot and to make and justify reasonable choices about how to construct a line that fits the data. Students could calculate the residuals of their line as one way to measure the appropriateness of fit, but doing so is beyond the scope of this course. Calculating a regression equation, either by hand or with technology, is beyond the scope of this course and should be reserved for Algebra 2 or beyond. The emphasis is on using (as opposed to constructing) the linear function model for the data.

Cross Connection: As AP Statistics students will learn, the equation for a trend line calculated from a data set can be used to predict both input and output values for the relationship that is modeled by that trend line. However, it is not appropriate to use a regression equation to predict input values from output values. This is because the mathematics behind the least squares regression assumes that the input values are fixed and a line is fitted to predict the output values given the input values.

Teacher Resource
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Pre-AP Algebra 1

Essential Knowledge Statements:
Each essential knowledge statement is linked to a learning objective. One or more essential knowledge statements describe the knowledge required to perform each learning objective.

Content Boundary and Cross Connection Statements:
When needed, content boundary statements provide additional clarity about the content and skills that lie within versus outside of the scope of this course.

Cross connection statements highlight important connections that should be made between key concepts within and across the units.

BIG IDEAS IN PRE-AP ALGEBRA 1

While the Pre-AP Algebra 1 framework is organized into four core units of study, the content is grounded in four big ideas, which are cross-cutting concepts that build conceptual understanding and spiral throughout the course. Since these ideas cut across units, they serve as the underlying foundation for the enduring understandings, key concepts, and learning objectives that make up the focus of each unit. A deep and productive understanding of the concepts presented in Pre-AP Algebra 1 relies on these four big ideas:

- **Patterns of Change:** Families of functions are uniquely defined by their patterns of change. Linear functions have a constant rate of change, quadratic functions have a linear rate of change, and exponential functions have a constant multiplicative rate of change.
- **Representations:** Functions and equations can be represented graphically, numerically (in tables), algebraically (with symbols), or verbally (in words). Algebraic forms of functions and equations can be purposefully manipulated into equivalent forms to reveal certain aspects of the function/equation.
- **Modeling with Functions:** Functions can be used to model real-world phenomena. A function derived from a real-world context can be manipulated free of its context, but the solution must be translated back in order to interpret its meaning in context.
- **Solutions:** A solution to an equation or inequality is a value or set of values that makes the equation or inequality true. Solutions can be found by applying rules of algebra to symbolic expressions, examining a graph of the equation or inequality, or testing numerical values.

OVERVIEW OF PRE-AP ALGEBRA 1 UNITS AND ENDURING UNDERSTANDINGS

Unit 1: Linear Functions and Linear Equations	Unit 2: Systems of Linear Equations and Inequalities
▪ A linear relationship has a constant rate of change, which can be visualized as the slope of the associated graph. ▪ There are many ways to algebraically represent a linear function and each form reveals different aspects of the function. ▪ Linear functions can be used to model contextual scenarios that involve a constant rate of change or data whose general trend is linear. ▪ A solution to a two-variable linear equation or inequality is an ordered pair that makes the equation or inequality true.	▪ A solution to a system of linear equations or inequalities is an ordered pair of numbers that satisfies all the equations or inequalities simultaneously. ▪ Solving a system of linear equations or inequalities is a process of determining the value or values that make the equation or inequality true. ▪ Systems of linear equations or inequalities can be used to model scenarios that include multiple constraints, such as resource limitations, goals, comparisons, and tolerances.
Unit 3: Quadratic Functions	**Unit 4: Exponent Properties and Exponential Functions**
▪ Quadratic functions have a linear rate of change. ▪ Quadratic functions can be expressed as a product of linear factors. ▪ Quadratic functions can be used to model scenarios that involve a linear rate of change and symmetry around a unique minimum or maximum. ▪ Every quadratic equation, $ax^2 + bx + c = 0$, where a is not zero, has at most two real solutions. These solutions can be determined using the quadratic formula.	▪ Properties of exponents are derived from the properties of multiplication and division. ▪ An exponential function has constant multiplicative growth or decay. ▪ Exponential functions can be used to model contextual scenarios that involve constant multiplicative growth or decay. ▪ Graphs and tables can be used to estimate the solution to an equation that involves exponential expressions.

Unit 1: Linear Functions and Linear Equations

Suggested Timing: Approximately 9 weeks

Linear relationships are among the most prevalent and useful relationships in mathematics and the real world. Any equality in two variables that exhibits a constant rate of change for these variables is linear. Real-world contexts that have a constant rate of change and data sets with a nearly constant rate of change can be effectively modeled by a linear function. Students explore all aspects of linear relationships in this unit: contextual problems that involve constant rate of change, lines in the coordinate plane, arithmetic sequences, and algebraic means of expressing a linear relationship between two quantities. Through this unit, students develop deep skills with linear functions and equations and an appreciation for the simplicity and power of linear functions as building blocks of all higher mathematics.

ENDURING UNDERSTANDINGS

Students will understand that ...

- A linear relationship has a constant rate of change, which can be visualized as the slope of the associated graph.
- There are many ways to algebraically represent a linear function and each form reveals different aspects of the function.
- Linear functions can be used to model contextual scenarios that involve a constant rate of change or data whose general trend is linear.
- A solution to a two-variable linear equation or inequality is an ordered pair that makes the equation or inequality true.

KEY CONCEPTS

- 1.1: Constant rate of change and slope
- 1.2: Linear functions
- 1.3: Linear equations
- 1.4: Linear models of nonlinear scenarios
- 1.5: Two-variable linear inequalities

KEY CONCEPT 1.1: CONSTANT RATE OF CHANGE AND SLOPE

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
1.1.1 Determine whether two quantities vary directly given a relationship represented graphically, numerically, algebraically, or verbally.	**1.1.1a** The graph of a direct variation whose domain is all real numbers is a non-vertical and non-horizontal line that contains the origin. **1.1.1b** Direct variation is a special case of a linear relationship where one quantity is proportional to another quantity. Two quantities vary directly if the ratio $\frac{y}{x}$ is constant for all (x, y) pairs. **1.1.1c** A direct variation can be expressed in the algebraic form $y = kx$, where k is a non-zero constant.
1.1.2 Calculate the constant rate of change of a linear relationship.	**1.1.2a** The constant rate of change of a linear relationship is the slope of the line of its associated graph. **1.1.2b** The constant rate of change of a linear relationship, whose associated line is non-vertical, can be graphically interpreted as the ratio of the vertical change of the line to the corresponding horizontal change of the line. **1.1.2c** The constant rate of change of a linear relationship can be calculated by finding the ratio of the change in the output to the change in the input using any two distinct ordered pairs and the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$. **1.1.2d** Rate of change describes how two quantities change together. The unit for rate of change is the unit of the output variable per the unit of the input variable.
1.1.3 Create a graphical or numerical representation of a linear relationship given its constant rate of change.	**1.1.3a** Given any point, the slope of a line can be used to generate all points on the graph of the line that passes through the point. **1.1.3b** Given any initial condition, the constant rate of change of a linear relationship can be used to generate all other pairs of values that satisfy the relationship. **1.1.3c** If the relationship represented in a table of values has a constant rate of change, then the points on the associated graph will lie on a line. **1.1.3d** If the output values of a linear function differ by m when the input values differ by 1, then the output values differ by mk when the input values differ by k, where k is a real number.

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
1.1.4 Determine whether a relationship presented graphically or numerically is linear by examining the rate of change.	**1.1.4a** In a graph of ordered pairs that are linearly related, the ratio of the vertical change between any two points to the corresponding horizontal change between the same two points is constant. **1.1.4b** In a table of linearly related values where successive input values differ by a constant amount (e.g., differ by 1), successive output values will also differ by a constant amount. **1.1.4c** In a table of linearly related values where the input values differ by varying amounts, the associated output values will differ proportionally to these varying amounts.

Content Boundary: Direct variation is an extension of reasoning with proportional relationships, which students explored extensively in middle school. Students will have already solved context-free proportions in prior grades. Here, the focus is on analyzing the proportional nature of the relationship and using it to solve real-world problems.

Cross Connection: Students may be familiar with the slope formula from their previous courses. The focus here is on developing a thorough understanding of the rate of change of a linear relationship.

KEY CONCEPT 1.2: LINEAR FUNCTIONS

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
1.2.1 Determine whether a relationship is linear or nonlinear based on a numerical sequence whose indices increase by a constant amount.	**1.2.1a** An arithmetic sequence is a linear relationship whose domain consists of consecutive integers. **1.2.1b** The differences between successive terms of an arithmetic sequence are equal. **1.2.1c** An arithmetic sequence can be determined using the constant difference and any term in the sequence.
1.2.2 Convert a given representation of an arithmetic sequence to another representation of the arithmetic sequence.	**1.2.2a** The graph of an arithmetic sequence is a set of discrete points that lie on a line. **1.2.2b** Successive terms in an arithmetic sequence are obtained by adding the common difference to the previous term. To find the value of the term that occurs *n* terms after a specified term, add the common difference *n* times to the term. **1.2.2c** An arithmetic sequence can be algebraically expressed with the formula $a_n = a_k + d(n - k)$ where a_n is the *n*th term, a_k is the *k*th term, and *d* is the constant difference between successive terms. **1.2.2d** A verbal representation of an arithmetic sequence describes a discrete domain and a constant rate of change.
1.2.3 Use function notation to describe the relationship between an input–output pair of a function.	**1.2.3a** A function is a type of relationship between two quantities where each input is related to one (and only one) value of the output. **1.2.3b** The domain of a function is the set of all inputs for the function. The range of a function is the set of all outputs for the function resulting from the set of inputs. **1.2.3c** The notation "$f(x)$" is read as "*f* of *x*"; "*f*" is the name of a function, "*x*" stands for any input value in the domain of the function, and "$f(x)$" represents the output value in the range of the function that corresponds to the input value. **1.2.3d** Any solution (x, y) to the equation $y = f(x)$ represents a point that lies on the graph of function *f*.

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
1.2.4 Convert a given representation of a linear function to another representation of the linear function.	**1.2.4a** A graphical representation of a linear function displays ordered pairs satisfying the relationship. The exact coordinates of the ordered pairs may or may not be evident from the graph of the function. **1.2.4b** A numerical representation of a linear function usually consists of only a subset of the ordered pairs that satisfy the relationship. Any two distinct ordered pairs can be used to generate a graph of the relationship and compute the constant rate of change. **1.2.4c** An algebraic representation of a linear function contains the complete information about the function because any output value can be determined from a given input value. **1.2.4d** A verbal representation of a linear function describes the constant rate of change and known values of the function.
1.2.5 Translate between algebraic forms of a linear function, using purposeful algebraic manipulation.	**1.2.5a** Common algebraic forms of linear functions include point–slope form, $y = y_1 + m(x - x_1)$, and slope–intercept form, $y = mx + b$. **1.2.5b** Slope–intercept form is a special case of point–slope form where $(x_1, y_1) = (0, b)$. **1.2.5c** Purposeful algebraic manipulation can reveal information about how the quantities in a linear function relate to each other.
1.2.6 Model a contextual scenario with a linear function.	**1.2.6a** Linear functions can be used to model contextual scenarios that involve a constant rate of change of a dependent variable (the output) with respect to an independent variable (the input). **1.2.6b** A linear function derived from a contextual scenario can be solved free of context, but the solution must be interpreted in context to be correctly understood. **1.2.6c** Two distinct input–output pairs from a contextual scenario that involves a constant rate of change can be used to determine a linear function that models the scenario. **1.2.6d** A constant rate of change and a corresponding input–output pair from a contextual scenario can be used to determine a linear function that models the scenario.

Content Boundary: Students will use arithmetic sequences to help them understand linear functions. By the end of the unit, students should understand that a sequence is a function with whole-number inputs, however knowing formulas associated with arithmetic sequences is beyond the scope of this course.

Cross Connection: Students will come to Algebra 1 with some prior knowledge about linear relationships. However, this knowledge might be procedural (e.g., how to calculate slope) or fragmented (e.g., not connecting the value of b in $y = mx + b$ with the y-intercept of a line). This course guides students to consolidate and make connections among the disparate pieces of information they have relating to linear functions by understanding that a constant rate of change is the defining feature of a linear relationship.

KEY CONCEPT 1.3: LINEAR EQUATIONS

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
1.3.1 Convert a given representation of a linear equation to another representation of that linear equation.	**1.3.1a** A graphical representation of a linear equation is a set of ordered pairs that satisfy the relationship. The exact coordinates of the ordered pairs may or may not be evident from the graph of the equation. **1.3.1b** A numerical representation of a linear equation consists of only a subset of the ordered pairs that satisfy the relationship. It can be used to generate a graph of the relationship. **1.3.1c** An algebraic representation of a linear equation often takes the form $Ax+By=C$, where the parameters *A*, *B*, and *C* are non-zero constants. This form is called the standard form of a linear equation. **1.3.1d** A linear equation in two variables can be used to represent contextual scenarios where there exists a constraint or condition on the variables and neither variable is necessarily considered an input or output.
1.3.2 Interpret the solutions to a two-variable linear equation.	**1.3.2a** A solution to a linear equation, $Ax+By=C$, is an ordered pair $\left(x, y\right)$ that makes the equation true. **1.3.2b** A linear equation derived from a contextual scenario can be solved free of context, but the solution must be interpreted in context to be correctly understood. **1.3.2c** The solution to a linear equation derived from a contextual scenario should use the same units as the variables in the contextual scenario.
1.3.3 Rewrite a two-variable linear equation in terms of one of the variables to preserve the solution set, using purposeful algebraic manipulation.	**1.3.3a** The solution set to a linear equation, $Ax+By=C$, is the set of all ordered pairs $\left(x, y\right)$ that make the equation true. **1.3.3b** Purposeful algebraic manipulation can reveal information about how the quantities in a linear equation relate to each other.
1.3.4 Construct representations of parallel or perpendicular lines.	**1.3.4a** The slopes of parallel lines are equal, and two distinct lines with equal slopes are parallel. **1.3.4b** The slopes of non-vertical and non-horizontal perpendicular lines are multiplicative inverses of each other with opposite signs. **1.3.4c** A vertical line is perpendicular to a horizontal line, and vice versa. **1.3.4d** An equation for a line parallel or perpendicular to a given line can be determined using the slope of the given line and a point not on the given line.

Cross Connection: In this key concept, students regard the two variables in a linear equation as two independent quantities related by a constraint; this is distinct from the input–output thinking that characterized the relationships between the quantities in the previous key concept. Students should make connections with the one-variable equations they solved in middle school, understanding that both one-variable and two-variable equations are statements that can be either true or false.

KEY CONCEPT 1.4: LINEAR MODELS OF NONLINEAR SCENARIOS

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
1.4.1 Interpret a graphical representation of a piecewise linear function in context.	**1.4.1a** A piecewise linear function consists of two or more linear functions, each restricted to nonoverlapping intervals of input values.
1.4.2 Construct a graphical representation of a piecewise linear function to model a contextual scenario.	**1.4.2a** A contextual scenario that involves different constant rates of change over different intervals of the domain can be modeled using a piecewise linear function. **1.4.2b** A contextual scenario that involves a constant rate of change where the input or output values do not change continuously can be modeled using a piecewise linear function.
1.4.3 Determine whether the scatterplot of the relationship between two quantities can be reasonably modeled by a linear model.	**1.4.3a** A scatterplot whose points fall roughly in the shape of an ellipse can often be modeled usefully by a linear equation. **1.4.3b** Sets of data that show a graphically upward trend (as the input value increases) are said to have a positive association. **1.4.3c** Sets of data that show a graphically downward trend (as the input value increases) are said to have a negative association.
1.4.4 Determine an equation for a trend line that describes trends in a scatterplot.	**1.4.4a** A trend line describes an observed relationship between the variables in a scatterplot but may or may not contain any of the data points. **1.4.4b** A trend line does not perfectly model the data, so values predicted using the model can be expected to differ from actual values.
1.4.5 Use an equation for a trend line to predict values in context.	**1.4.5a** The equation for a trend line can be used to estimate either the input or output quantities in context. **1.4.5b** Relationships derived from data usually have limited domains beyond which the trend line might become an increasingly poor model.

Content Boundary: Students should explore piecewise linear graphs that model scenarios that have a variety of constant rates of change over different intervals. Writing a single function expression for a piecewise function with multiple linear functions, such as $f(x)=\begin{cases}2x-5, x<0\\-3x+6, x\geq 0\end{cases}$, is beyond the scope of this course. Engaging with graphs of piecewise functions that have nonlinear components is also beyond the scope of this course.

Content Boundary: Students should be able to determine if a linear model is appropriate given a scatterplot and to make and justify reasonable choices about how to construct a line that fits the data. Students could calculate the residuals of their line as one way to measure the appropriateness of fit, but doing so is beyond the scope of this course. Calculating a regression equation, either by hand or with technology, is beyond the scope of this course and should be reserved for Algebra 2 or beyond. The emphasis is on using (as opposed to constructing) the linear function model for the data.

Cross Connection: As AP Statistics students will learn, the equation for a trend line calculated from a data set can be used to predict both input and output values for the relationship that is modeled by that trend line. However, it is not appropriate to use a regression equation to predict input values from output values. This is because the mathematics behind the least squares regression assumes that the input values are fixed and a line is fitted to predict the output values given the input values.

KEY CONCEPT 1.5: TWO-VARIABLE LINEAR INEQUALITIES

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
1.5.1 Convert a given representation of a linear inequality to another representation of the linear inequality.	**1.5.1a** A graphical representation of a linear inequality is a set of ordered pairs that satisfy the relationship. The exact coordinates of the ordered pairs may or may not be evident from the graph of the inequality. **1.5.1b** A numerical representation of a linear inequality usually consists of only a subset of the ordered pairs that satisfy the relationship. It can be used to generate a graph of the relationship. **1.5.1c** An algebraic representation of a linear inequality usually relates the expressions $Ax + By$ and C, where the parameters A, B, and C are non-zero constants, with an inequality symbol, $<$, $\leq$, $>$, or $\geq$. **1.5.1d** A linear inequality is useful for modeling contextual scenarios that include resource limitations, goals, constraints, comparisons, and tolerances.
1.5.2 Determine solutions to a two-variable inequality.	**1.5.2a** A solution to an inequality in two variables is an ordered pair that makes the inequality true. **1.5.2b** The solution set to a two-variable inequality can be displayed graphically by a half-plane. Any coordinate in the half-plane, or on its boundary if the boundary is included, is a solution to the inequality. **1.5.2c** A solution to a two-variable linear inequality that represents a contextual scenario is a pair of numbers that satisfies the constraints of the contextual scenario.
1.5.3 Rewrite a linear inequality in terms of one of the variables to preserve the solution set, using purposeful algebraic manipulation.	**1.5.3a** The solution set to an inequality in two variables is the set of all ordered pairs that make the inequality true. **1.5.3b** Adding the same real number to or subtracting the same real number from both sides of an inequality does not change the inequality relationship. **1.5.3c** Multiplying both sides of an inequality by the same positive real number or dividing both sides of an inequality by the same positive real number does not change the inequality relationship. **1.5.3d** Multiplying both sides of an inequality by the same negative real number or dividing both sides of an inequality by the same negative real number reverses the direction of the inequality relationship.

Content Boundary: Applications of two-variable linear inequalities are beyond the scope of this unit. They are addressed at the end of Unit 2: Systems of Linear Equations and Inequalities. This key concept focuses on determining if a given ordered pair is a solution to a linear inequality and graphing the solution set on a coordinate plane.

Unit 2: Systems of Linear Equations and Inequalities

Suggested Timing: Approximately 5 weeks

Across Unit 2, students are asked to solve systems of equations in support of two goals: to determine the solution to the system of equations and to become strategic and efficient in choosing a method to solve the system. Students use systems of linear equations and systems of linear inequalities to model physical phenomena, especially those with multiple constraints where an optimal solution to an objective function is desired. Through these contexts students build upon their prior knowledge of solving systems of equations and develop more sophisticated understandings about what the solution(s) to a system means in the context of the problem.

ENDURING UNDERSTANDINGS

Students will understand that ...

- A solution to a system of linear equations or inequalities is an ordered pair of numbers that satisfies all the equations or inequalities simultaneously.
- Solving a system of linear equations or inequalities is a process of determining the value or values that make the equation or inequality true.
- Systems of linear equations or inequalities can be used to model scenarios that include multiple constraints, such as resource limitations, goals, comparisons, and tolerances.

KEY CONCEPTS

- 2.1: The solution to a system of equations
- 2.2: Solving a system of linear equations algebraically
- 2.3: Modeling with systems of linear equations
- 2.4: Systems of linear inequalities

KEY CONCEPT 2.1: THE SOLUTION TO A SYSTEM OF EQUATIONS

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
2.1.1 Use a graph or tables of values to estimate the solution to a system of equations.	**2.1.1a** A solution to a system of linear equations, if one exists, is an intersection point of the lines corresponding to the equations. **2.1.1b** A solution to a system of linear equations, if one exists, corresponds to the solutions that the equations have in common.
2.1.2 Determine the number of real solutions to a system of two linear equations.	**2.1.2a** A system of two linear equations can have no solution, one solution, or infinitely many solutions. **2.1.2b** If the graphs of two linear equations in a system are parallel, then the system has no solutions. If the graphs of two linear equations are not parallel and do not coincide, then the system has one solution. If the graphs of two linear equations coincide, then the system has infinitely many solutions.
2.1.3 Determine the intersection point(s) of the graphs of two functions *f* and *g*.	**2.1.3a** The graphs of functions *f* and *g* intersect at $x=k$ if $f(k)=g(k)$. **2.1.3b** The graphs of the functions *f* and *g* intersect at $x=k$ if $f(k)-g(k)=0$.

Content Boundary: In this unit, students will work with systems of two linear equations in two variables. In Unit 3: Quadratic Functions, students will be exposed to systems of one quadratic and one linear equation and to systems of two quadratic equations. Systems of three linear equations in three variables are beyond the scope of this course.

KEY CONCEPT 2.2: SOLVING A SYSTEM OF LINEAR EQUATIONS ALGEBRAICALLY

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
2.2.1 Solve a system of linear equations using algebraic methods.	**2.2.1a** Algebraic methods of solving a system of equations include the substitution method and the elimination method. **2.2.1b** An efficient method of solving a system of equations should be based on the forms of the equations in the system.
2.2.2 Justify the steps used to algebraically solve a system of linear equations.	**2.2.2a** Algebraically equivalent expressions can be used interchangeably in equations. **2.2.2b** In a system of equations, substituting one equation with a multiple of that equation or substituting one equation with the sum of non-zero multiples of the equations will result in a system with the same solutions as the original system.

KEY CONCEPT 2.3: MODELING WITH SYSTEMS OF LINEAR EQUATIONS

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
2.3.1 Model a contextual scenario with a system of linear equations.	**2.3.1a** A system of linear equations can be used to determine when two linear functions that model a contextual scenario have the same input–output pair. **2.3.1b** A system of linear equations can be used to model a contextual scenario in which two quantities are subject to multiple constraints. **2.3.1c** A system of linear equations derived from a contextual scenario can be solved free of context, but the solution must be interpreted in context to be correctly understood. **2.3.1d** The solution to a system of linear equations used to model a contextual scenario should use the same units as the variables in the contextual scenario.
2.3.2 Use the rates of change to draw conclusions about a contextual scenario modeled by a system of linear equations.	**2.3.2a** If the multiple constraints in a contextual scenario modeled by a system of linear equations each have a different constant rate of change, then the scenario will have one solution. The graph associated with the system will consist of a pair of intersecting lines. **2.3.2b** If the multiple constraints in a contextual scenario modeled by a system of linear equations each have the same constant rate of change and the linear equations in the system share at least one solution, then the scenario will have infinitely many solutions. The graph associated with the system will consist of a pair of coinciding lines. **2.3.2c** If the multiple constraints in a contextual scenario modeled by a system of linear equations each have the same constant rate of change and the linear equations in the system differ in at least one solution, then the scenario will have no solution. The graph associated with the system will consist of a pair of parallel lines.

Content Boundary: The focus of this key concept is to become strategic and efficient about choosing a particular method to solve a system of linear equations. Students should come to understand that the graphical and tabular methods of solving a system of equations are inefficient and imprecise in most cases. Students should appreciate that the elimination method is convenient for linear equations written in standard form. However, elimination is not generally applicable for nonlinear systems. The substitution method has wider utility for nonlinear systems. In this unit, students will only explore linear systems, but they need to develop fluency with the substitution method because it will become more important later in the course and in future courses. Other algebraic techniques, such as those involving matrices, are beyond the scope of this course.

KEY CONCEPT 2.4: SYSTEMS OF LINEAR INEQUALITIES

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
2.4.1 Use algebra to determine if an ordered pair is a solution to a system of linear inequalities.	**2.4.1a** If an ordered pair is a solution to a system of inequalities, it will make all inequalities that constitute the system true.
2.4.2 Graphically represent the solution to a system of two-variable inequalities.	**2.4.2a** The solution to a system of linear inequalities is the intersection of half-planes that correspond to the individual inequalities in the system. Every point located in the solution region, or on the boundary if the boundary is included, is a solution to the system.
2.4.3 Model a contextual scenario with a system of linear inequalities.	**2.4.3a** A system of linear inequalities can be used to model a contextual scenario in which the relationship between two quantities is subject to multiple constraints, such as resource limitations, goals, comparisons, and tolerances. **2.4.3b** A solution, if one exists, to a system of inequalities used to model a contextual scenario is a set of values that satisfies the constraints of the scenario. **2.4.3c** A system of linear inequalities that models a contextual scenario can be solved free of context, but any solution must be interpreted in context to be correctly understood. **2.4.3d** A solution to a system of linear inequalities used to model a contextual scenario should involve the same units as the variables in the contextual scenario.

Cross Connection: Students should make connections with their knowledge from the previous unit and prior courses to understand that any situation that involves an inequality is often best represented by a graphical representation. That is, the solution to a one-variable linear inequality can be displayed as a shaded portion of a number line, the solution to a two-variable linear inequality can be displayed as a half-plane, and the solution to a system of two-variable linear inequalities can be displayed as the intersection of the associated half-planes.

Content Boundary: Systems of inequalities that involve nonlinear functions, such as quadratics or other polynomials, are beyond the scope of this course.

Unit 3: Quadratic Functions

Suggested Timing: Approximately 9 weeks

In this unit, students develop a strong foundation in the important concept of quadratic functions. Students should understand that quadratic functions have a linear rate of change and are often formed by multiplying two linear expressions, and therefore are not linear. Quadratic functions are useful for modeling phenomena that have a linear rate of change and symmetry around a unique minimum or maximum. This foundational understanding of quadratics helps students build their conceptual knowledge of nonlinear functions and prepares them for further study of polynomial and rational functions in Algebra 2.

ENDURING UNDERSTANDINGS

Students will understand that …

- Quadratic functions have a linear rate of change.
- Quadratic functions can be expressed as a product of linear factors.
- Quadratic functions can be used to model scenarios that involve a linear rate of change and symmetry around a unique minimum or maximum.
- Every quadratic equation, $ax^2+bx+c=0$, where a is not zero, has at most two real solutions. These solutions can be determined using the quadratic formula.

KEY CONCEPTS

- 3.1: Functions with a linear rate of change
- 3.2: The algebra and geometry of quadratic functions
- 3.3: Solving quadratic equations
- 3.4: Modeling with quadratic functions

KEY CONCEPT 3.1: FUNCTIONS WITH A LINEAR RATE OF CHANGE

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
3.1.1 Determine whether a relationship is quadratic or nonquadratic based on a numerical sequence whose indices increase by a constant amount.	**3.1.1a** In a table of values that represents a quadratic relationship and that has constant step sizes, the differences in the values of the relationship, called the first differences, exhibit a linear pattern. The second differences of a quadratic sequence are constant. **3.1.1b** Successive terms in a quadratic sequence can be obtained by adding corresponding successive terms of an arithmetic sequence.
3.1.2 Convert a given representation of a quadratic function to another representation of the quadratic function.	**3.1.2a** A graphical representation of a quadratic function displays ordered pairs that satisfy the relationship. The exact coordinates of the ordered pairs may or may not be evident from the graph of the function. **3.1.2b** A numerical representation of a quadratic function consists of only a subset of the ordered pairs that satisfy the relationship. **3.1.2c** An algebraic form of a quadratic function contains the complete information about the function because any output value can be determined from any given input value. **3.1.2d** A verbal representation of a quadratic function could refer to a symmetric scenario with a unique minimum or maximum.

Cross Connection: This key concept motivates the need for a nonlinear function to represent scenarios that cannot be adequately modeled by a linear function, such as gravity or area. Students should understand that any scenario, physical or otherwise, that involves the product of two linear expressions will be quadratic. Earlier in Algebra 1, students saw that physical scenarios that exhibit a constant rate of change can be modeled by a linear function. In AP Calculus and AP Physics, students will come to understand that particle motion problems that exhibit a constant acceleration can be modeled by a quadratic function.

KEY CONCEPT 3.2: THE ALGEBRA AND GEOMETRY OF QUADRATIC FUNCTIONS

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
3.2.1 Identify key characteristics of the graph of a quadratic function.	**3.2.1a** The graph of a quadratic function is a parabola. The parabola is symmetric about a vertical line that passes through the vertex of the parabola. **3.2.1b** The vertex of a parabola is the point on the curve where the outputs of the function change from increasing to decreasing or vice versa. The *y*-coordinate of the vertex of a parabola is the maximum or minimum value of the function. **3.2.1c** A parabola can have two *x*-intercepts, one *x*-intercept, or no *x*-intercepts. **3.2.1d** If the vertex of a parabola is a minimum value, then the parabola is said to be concave up. If the vertex is a maximum value, then the parabola is said to be concave down.
3.2.2 Translate between algebraic forms of a quadratic function using purposeful algebraic manipulation.	**3.2.2a** Common algebraic forms of a quadratic function include standard form, $f(x)=ax^2+bx+c$; factored form, $f(x)=a(x-r)(x-s)$; and vertex form, $f(x)=a(x-h)^2+k$, where a is not zero. **3.2.2b** Every quadratic function has a standard form and a vertex form, but not all quadratic functions have a factored form over the real numbers. **3.2.2c** The standard form can be purposefully manipulated into the vertex form of the same quadratic function by completing the square. **3.2.2d** The standard form and the factored form of a quadratic function can be translated into each other with purposeful use of the distributive property.

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
3.2.3 Describe key features of the graph of a quadratic function in reference to an algebraic form of the quadratic function.	**3.2.3a** The graph of a quadratic function whose standard form is $f(x)=ax^2+bx+c$ has a vertex with the x-coordinate at $x=-\frac{b}{2a}$. The y-coordinate of the vertex can be calculated by evaluating the function rule using the x-coordinate of the vertex. The graph is symmetric about the vertical line $x=-\frac{b}{2a}$. **3.2.3b** The graph of a quadratic function whose factored form is $f(x)=a(x-r)(x-s)$, where $a\neq 0$ and $r\neq s$, has two x-intercepts, at $(r,0)$ and $(s,0)$. **3.2.3c** The graph of a quadratic function whose vertex form is $f(x)=a(x-h)^2+k$ has a vertex at the coordinate (h,k). **3.2.3d** The graph of a quadratic function of the form $f(x)=a(x-r)^2$, where $a\neq 0$, can be interpreted as the factored form where the parabola has one x-intercept at $(r,0)$ or as the vertex form where the vertex of the parabola is at $(r,0)$.
3.2.4 Determine an algebraic rule for a quadratic function given a sufficient number of points from the graph.	**3.2.4a** There is a unique parabola that includes any three distinct noncollinear points. A quadratic function whose graph is the parabola that contains these points can be determined using their coordinates. **3.2.4b** If two distinct inputs of a quadratic function are associated with equal outputs, then the x-coordinate of the vertex of the parabola is located halfway between these inputs. **3.2.4c** The x-intercepts of a quadratic function are two convenient points that, with a third point, can be used to determine an algebraic rule for the quadratic function. **3.2.4d** An algebraic rule for a quadratic function can be determined from the vertex and one other point on the graph of the parabola.

Content Boundary: Quadratic functions can be written in many forms, and each form reveals or obscures certain features of the quadratic. Students should be able to translate between these forms and choose the representation that best serves the problem at hand. Fluent skill in transforming one algebraic form into another will help students in Algebra 2 and beyond in investigating the structure of polynomial and rational functions. Rote exercises in transforming expressions are less effective at producing fluency than exercises in which the student purposefully transforms expressions for a specific reason.

It is possible to orient a parabola so it opens to the left or right. However, the associated equation would not be a function of x and is beyond the scope of this course.

KEY CONCEPT 3.3: SOLVING QUADRATIC EQUATIONS

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
3.3.1 Describe the relationship between the algebraic and graphical representations of a quadratic equation.	**3.3.1a** The solutions to the quadratic equation $ax^2+bx+c=d$ can be interpreted graphically as the point(s) of intersection between the parabola defined by $y=ax^2+bx+c$ and the horizontal line defined by $y=d$. **3.3.1b** If a quadratic equation $ax^2+bx+c=0$ has real solutions $x=r$ and $x=s$, then the parabola defined by $y=ax^2+bx+c$ has x-intercepts at $(r,0)$ and $(s,0)$.
3.3.2 Solve quadratic equations by taking a square root.	**3.3.2a** For any positive real number *a*, there are two real numbers that satisfy the equation $x^2=a$, one positive and one negative. **3.3.2b** The notation "$\sqrt{a}$" represents the square root of *a* and refers only to the principal square root, or non-negative, number whose square equals *a*. **3.3.2c** There is no real number that will satisfy the equation $x^2=a$, when *a* is a negative real number.
3.3.3 Solve quadratic equations by factoring.	**3.3.3a** Factoring a quadratic expression yields an equivalent form of the expression that can be used to determine the roots of the associated quadratic equation. **3.3.3b** There can be multiple correct factorizations of a quadratic expression. **3.3.3c** The solutions of a quadratic equation in the form $a(x-r)(x-s)=0$ are $x=r$ and $x=s$ when $a\neq 0$.
3.3.4 Solve quadratic equations by completing the square.	**3.3.4a** Completing the square is an algebraic process of transforming a quadratic expression into a form that can be solved by adding, multiplying, and taking the square root. **3.3.4b** Every quadratic equation can be solved by completing the square, but the solutions may not be real numbers.

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
3.3.5 Solve quadratic equations using the quadratic formula.	**3.3.5a** Any quadratic equation can be written in the form $mx^2+nx+p=q$, which can be purposefully manipulated into the standard form of a quadratic equation, $ax^2+bx+c=0$. **3.3.5b** The quadratic formula, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, can be used to solve any quadratic equation of the form $ax^2+bx+c=0$, but the solutions may not be real numbers. **3.3.5c** The quadratic formula can be derived by completing the square on the standard form of a quadratic equation. **3.3.5d** The quadratic formula can be written as the sum of two terms, $x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$, which shows that the x-intercepts are each a horizontal distance of $\frac{\sqrt{b^2-4ac}}{2a}$ from the x-coordinate of the vertex.
3.3.6 Determine the number of real solutions to a quadratic equation.	**3.3.6a** If a quadratic equation has two rational solutions, then it can be factored into linear factors with integer coefficients. **3.3.6b** Given a quadratic equation of the form $ax^2+bx+c=0$, the value of the discriminant of the quadratic equation ($D=b^2-4ac$) can be used to determine whether the quadratic equation has two distinct real solutions ($D>0$), one real solution ($D=0$), or no real solutions ($D<0$).

Content Boundary: A focus of this key concept is having students make explicit connections between the real number solutions of a quadratic equation and the x-intercept(s) of the associated graph. Students should understand that all quadratic equations can be solved, but some quadratic equations require a new number system to adequately express the solution set. However, imaginary numbers are beyond the scope of this course.

KEY CONCEPT 3.4: MODELING WITH QUADRATIC FUNCTIONS

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
3.4.1 Model a contextual scenario with a quadratic function.	**3.4.1a** A contextual scenario where the output quantity increases and then decreases (or vice versa), such as accelerated motion, can be effectively modeled by a quadratic function. **3.4.1b** A contextual scenario whose physical manifestation resembles a parabola, such as a satellite dish or solar collector, can be effectively modeled by a quadratic function. **3.4.1c** A contextual scenario that can be expressed as a product of two linear expressions, such as profit or area, can be effectively modeled by a quadratic function.
3.4.2 Interpret solutions to quadratic equations derived from contextual scenarios.	**3.4.2a** A quadratic equation derived from a contextual scenario can be solved free of context, but the solution must be interpreted in context to be correctly understood. **3.4.2b** The solution to a quadratic equation derived from a context should involve the same units as the variables in the contextual scenario.
3.4.3 Interpret the vertex and roots of a quadratic model in context.	**3.4.3a** If the values of the vertex are included in the contextual domain and range of the problem, the x-value of the vertex of the parabola represents the input value that corresponds to either the minimum or maximum output value, and the y-value of the vertex of the parabola represents the minimum or maximum output value. **3.4.3b** The x-value(s) of the root(s) of a parabola often represent the extreme values (of the input variable) in a contextual scenario.

Unit 4: Exponent Properties and Exponential Functions

Suggested Timing: Approximately 5 weeks

Students explore exponent rules as an extension of geometric sequences and the properties of multiplication and division for real numbers. Students should make sense of exponent rules and not simply memorize them without understanding how they arise. The unit culminates in students investigating how exponential functions can model physical phenomena that exhibit a constant multiplicative growth. Exponential functions are framed as multiplicative analogues of linear functions. Thus, a tight connection should be drawn between these two classes of functions and their shared properties.

ENDURING UNDERSTANDINGS

Students will understand that ...

- Properties of exponents are derived from the properties of multiplication and division.
- An exponential function has constant multiplicative growth or decay.
- Exponential functions can be used to model contextual scenarios that involve constant multiplicative growth or decay.
- Graphs and tables can be used to estimate the solution to an equation that involves exponential expressions.

KEY CONCEPTS

- 4.1: Exponent rules and properties
- 4.2: Roots of real numbers
- 4.3: Sequences with multiplicative patterns
- 4.4: Exponential growth and decay

KEY CONCEPT 4.1: EXPONENT RULES AND PROPERTIES

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
4.1.1 Use exponent rules to express products and quotients of exponential expressions in equivalent forms.	**4.1.1a** Exponential expressions involving multiplication can be rewritten by invoking the rule $n^a \bullet n^b = n^{a+b}$, where $n > 0$. **4.1.1b** Exponential expressions involving division can be rewritten by invoking the rule $\frac{n^a}{n^b} = n^{a-b}$, where $n > 0$. **4.1.1c** Exponential expressions involving powers of powers can be rewritten by invoking the rule $\left(n^a\right)^b = n^{ab}$, where $n > 0$.
4.1.2 Use exponent rules to express numerical and variable expressions that involve negative exponents using positive exponents, and vice versa.	**4.1.2a** Any non-zero real number raised to the zero power is equal to 1. That is, $n^0 = 1$, where $n \neq 0$. **4.1.2b** Zero raised to the zero power is not defined in the real number system. **4.1.2c** A negative exponent of −1 can be used to represent a reciprocal. That is, $n^{-k} = \frac{1}{n^k}$, where $n \neq 0$. **4.1.2d** The properties of negative integer exponents, and those of an exponent of zero, are extensions of the properties of positive integer exponents.

Content Boundary: Students coming to Algebra 1 may be familiar with negative exponents from working with scientific notation in earlier courses, but their skills could be limited to superficial knowledge like "moving the decimal point." The focus here is for students to develop these formulas through extending the properties of multiplication and division. Students should not be expected to simplify excessively complicated quotients, such as $\frac{36x^3y^{-4}z^7}{72x^{-5}y^6z^8}$, because these expressions have limited usefulness outside of rote skill acquisition.

Cross Connection: Students should be able to flexibly and fluently translate among different forms of expressions involving exponents. For example, often in AP Calculus it is advantageous to rewrite the function $f\left(x\right) = \frac{1}{x^2}$ as $f\left(x\right) = x^{-2}$.

KEY CONCEPT 4.2: ROOTS OF REAL NUMBERS

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
4.2.1 Perform operations with rational and irrational numbers.	**4.2.1a** The value of an irrational number cannot be expressed exactly as a ratio of integers or as a nonrepeating, nonterminating decimal, and is often represented exactly by a symbol such as π or $\sqrt{2}$. **4.2.1b** An irrational number can be approximated to any specified degree of precision by a rational number.
4.2.2 Represent square roots of real numbers in equivalent forms.	**4.2.2a** The square root of a squared real number *a* is equivalent to the absolute value of the number. That is, $\sqrt{a^2} = \lvert a \rvert$. **4.2.2b** For any two nonnegative real numbers *a* and *b*, the product of their square roots is equal to the square root of their product. That is, $\sqrt{a} \bullet \sqrt{b} = \sqrt{ab}$. **4.2.2c** If a positive real number *a* can be written as the product of the square of a positive number *b* and another positive number *c*—that is, if $a = b^2c$—then the square root of *a* is equal to the square root of the product of b^2 and *c*. That is, $\sqrt{a} = \sqrt{b^2c} = \sqrt{b^2} \bullet \sqrt{c} = b\sqrt{c}$ where *a, b,* and *c* are all positive.
4.2.3 Use the laws of exponents to represent roots of real numbers in terms of rational number powers.	**4.2.3a** The square root of a nonnegative real number can be expressed with the exponent $\frac{1}{2}$. That is, $\sqrt{n} = n^{\frac{1}{2}}$ where $n \geq 0$. **4.2.3b** The cube root of any real number can be expressed with the exponent $\frac{1}{3}$. That is, $\sqrt[3]{n} = n^{\frac{1}{3}}$. **4.2.3c** The properties of rational exponents are extensions of the properties of integer exponents.

Content Boundary: The focus in this key concept is on having students understand that "simplifying" a square root is similar to reducing a fraction to lowest terms. That is, $\sqrt{8}$ expresses the same quantity as $2\sqrt{2}$, just as $\frac{6}{8}$ expresses the same quantity as $\frac{3}{4}$. It is not absolutely necessary to reduce a fraction to lowest terms and it is not absolutely necessary to simplify a square root to "lowest terms." Different equivalent forms of numbers exist, and the context of the problem may suggest when one form could be more useful than another.

Cross Connection: Students should start to identify circumstances where a particular form of a number provides an insight or advantage that other forms do not. In AP Calculus, students will often find it helpful to rewrite the function $f(x) = \sqrt{x}$ as $f(x) = x^{\frac{1}{2}}$.

KEY CONCEPT 4.3: SEQUENCES WITH MULTIPLICATIVE PATTERNS

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
4.3.1 Determine whether a relationship is exponential or nonexponential based on a numerical sequence whose indices increase by a constant amount.	**4.3.1a** A geometric sequence is an exponential relationship whose domain consists of consecutive integers. **4.3.1b** The ratios of successive terms of a geometric sequence are equivalent. **4.3.1c** A geometric sequence can be determined from the common ratio and any term in the sequence.
4.3.2 Convert a given representation of a geometric sequence to another representation of the geometric sequence.	**4.3.2a** The graph of a geometric sequence is a set of discrete points that lie on a curve. **4.3.2b** Successive terms in a geometric sequence are obtained by multiplying the previous term by the common ratio. To find the value of the term that occurs *n* terms after a specified term, multiply the specified term by the common ratio *n* times. **4.3.2c** A geometric sequence can be algebraically expressed with the formula $a_n = a_k \bullet c^{(n-k)}$, where a_n is the *n*th term, a_k is the *k*th term, and *c* is the common ratio between successive terms. **4.3.2d** A verbal representation of a geometric sequence describes a discrete domain and a constant multiplicative growth or decay.
4.3.3 Determine whether a relationship is exponential by analyzing a graphical, numerical, algebraic, or verbal representation.	**4.3.3a** A graph of an exponential function is a curve that exhibits asymptotic behavior to the left or right. **4.3.3b** The numerical representation of an exponential function will have ordered pairs where, if the inputs differ by a constant amount, then the ratios of corresponding outputs are equivalent. **4.3.3c** An algebraic representation of an exponential function often takes the form $f(x) = a \bullet c^x$, $c > 0$, where *c* is the constant growth or decay factor. **4.3.3d** A verbal representation of an exponential function describes a constant multiplicative growth or decay and known values from the relationship.

Cross Connection: The goal of introducing students to geometric sequences is to have them investigate multiplicative patterns as a counterpoint to the additive patterns of linear and quadratic sequences. Knowing formulas associated with geometric sequences is beyond the scope of this course.

KEY CONCEPT 4.4: EXPONENTIAL GROWTH AND DECAY

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
4.4.1 Calculate a growth or decay factor of an exponential relationship.	**4.4.1a** In an exponential relationship, the output values grow by equal factors over equal intervals. **4.4.1b** Given two input–output pairs in an exponential relationship, $(a, f(a))$ and $(b, f(b))$, where b is n units more than a, the n-unit growth (or decay) factor is the quotient of the corresponding outputs, $\frac{f(b)}{f(a)}$. **4.4.1c** Given two input–output pairs in an exponential relationship, $(a, f(a))$ and $(b, f(b))$, if the quotient of any two outputs, $\frac{f(b)}{f(a)}$, where $b > a$, is greater than 1, then the value is called a growth factor. **4.4.1d** Given two input–output pairs in an exponential relationship, $(a, f(a))$ and $(b, f(b))$, if the quotient of any two outputs, $\frac{f(b)}{f(a)}$, where $b > a$, is between 0 and 1, then the value is called a decay factor.
4.4.2 Create graphical or numerical representations of an exponential function using the common growth or decay factor.	**4.4.2a** Given any point on the graph of an exponential function, the common growth or decay factor can be used to generate all points on the graph of the curve that contains the point. **4.4.2b** Given any input–output pair from an exponential function, the common growth or decay factor can be used to generate all other pairs of values that satisfy the relationship. **4.4.2c** If the relationship represented in a table of values has a common ratio in outputs over equal differences of inputs, then the points on the associated graph will lie on a curve that is asymptotic to the left or the right. **4.4.2d** If the output values change by a factor of c when the input values differ by 1, then the output values change by a factor of c^k when the input values differ by k, where k is a real number.

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
4.4.3 Convert a given representation of an exponential function to another representation of the exponential function.	**4.4.3a** The graphical representation of an exponential function displays ordered pairs that satisfy the relationship. The exact coordinates of the ordered pairs may or may not be evident from the graph of the function. **4.4.3b** A numerical representation of an exponential function consists of only a subset of the ordered pairs that satisfy the relationship, but can be used to compute the common ratio of growth or decay and determine all other aspects of the exponential function. **4.4.3c** An algebraic representation of an exponential function contains the complete information about the function because any output value can be determined from any given input value. **4.4.3d** A verbal representation of an exponential function describes the common ratio of growth or decay and/or known input–output pairs from the function.
4.4.4 Approximate input and output values of an exponential function using representations of the exponential function.	**4.4.4a** An algebraic rule for an exponential function can be used to determine an exact output for a specified input and can be used to approximate an input for a specified output. **4.4.4b** A table of values for an exponential function can often be used to determine an exact or approximate output for a specified input or an exact or approximate input for a specified output. **4.4.4c** A graph of an exponential function can be used to approximate an output for a specified input and to approximate an input for a specified output.
4.4.5 Model a contextual scenario with an exponential function.	**4.4.5a** A contextual scenario that exhibits constant multiplicative growth or decay in its outputs over equal differences in the corresponding inputs can be modeled by an exponential function. **4.4.5b** Estimated inputs and outputs for exponential functions derived from a contextual scenario can be determined free of context, but the values must be interpreted in context to be correctly understood. **4.4.5c** The estimated inputs and outputs for an exponential function derived from a context should involve the same units as the variables in the contextual scenario.

Content Boundary: The focus in this key concept should be on having students use their knowledge of multiplicative patterns to explore simple exponential growth and decay relationships. Students will be expected to generate a table of values, construct a graph, and write an algebraic representation of an exponential function. Students should not solve problems involving formulas relating to geometric sequences, compound interest, or logarithms as these topics are beyond the scope of this course.

Pre-AP Algebra 1 Model Lessons

Model lessons in Pre-AP Algebra 1 are developed in collaboration with Algebra 1 educators across the country and are rooted in the course framework, shared principles, and areas of focus. Model lessons are carefully designed to illustrate on-grade-level instruction. Pre-AP strongly encourages teachers to internalize the lessons and then offer the supports, extensions, and adaptations necessary to help all students reach these on-grade-level goals.

The purpose of these model lessons is twofold:

- **Robust instructional support for teachers:** Pre-AP Algebra 1 model lessons are comprehensive lesson plans that, along with accompanying student resources, embody the Pre-AP approach to teaching and learning. Model lessons provide clear and substantial instructional guidance to support teachers as they engage students in the shared principles and areas of focus.
- **Key instructional strategies:** Commentary and analysis embedded in each lesson highlight not just what students and teachers do in the lesson, but also how and why they do it. This educative approach provides a way for teachers to gain unique insight into key instructional moves that are powerfully aligned with the Pre-AP approach to teaching and learning. In this way, each model lesson works to support teachers in the moment of use with students in their classroom.

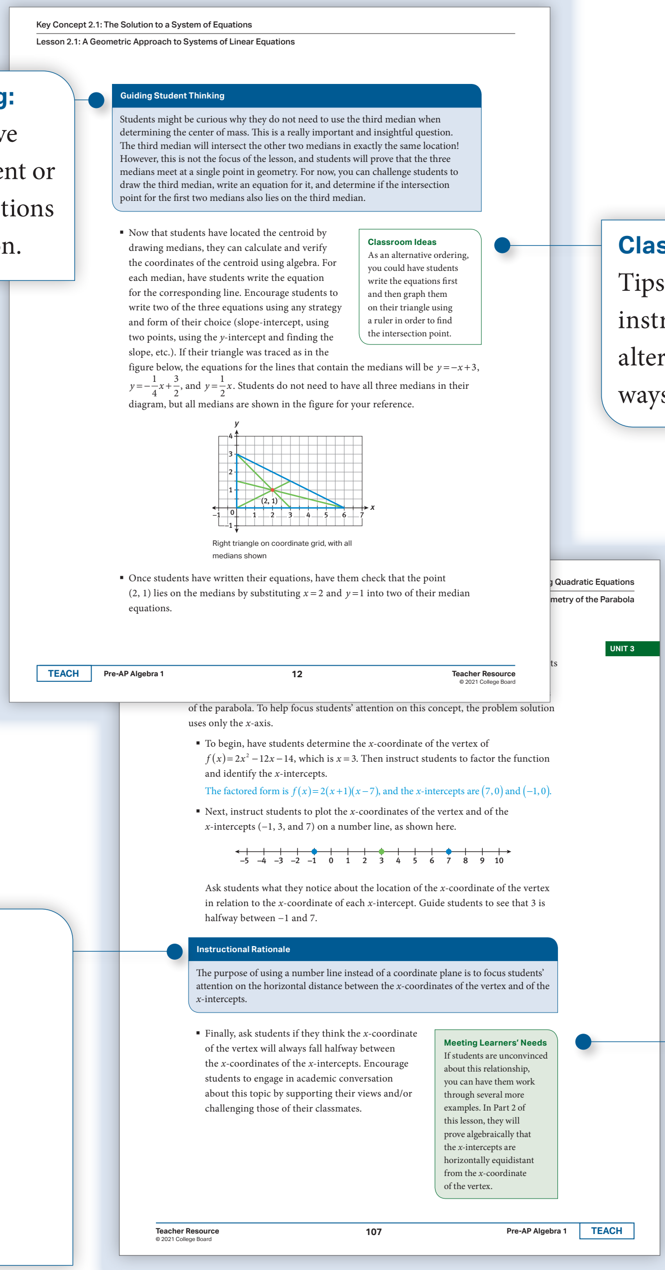
Teachers have the option to use any or all model lessons alongside their own locally developed instructional resources. Model lessons target content areas that tend to be challenging for teachers and students. While the lessons are distributed throughout all four units, they are concentrated more heavily in the beginning of the course to support teachers and students in establishing a strong foundation in the Pre-AP approach to teaching and learning.

SUPPORT FEATURES IN MODEL LESSONS

The following support features recur throughout the Pre-AP Algebra 1 lessons, to promote teacher understanding of the lesson design and provide direct-to-teacher strategies for adapting lessons to meet their students' needs:

- Instructional Rationale
- Guiding Student Thinking
- Meeting Learners' Needs
- Classroom Ideas

Guiding Student Thinking: Ways to facilitate productive student thinking and prevent or address student misconceptions in critical areas of the lesson.



Classroom Ideas: Tips related to the logistics of the instruction, such as suggestions for alternative presentation methods or ways to alleviate pacing concerns.

Instructional Rationale: Insight into the strategic design and purpose of the instructional choices, flow, and scaffolding within the model lesson. Rationales often describe how a concept is continued later in the lesson or unit.

Meeting Learners' Needs: Optional differentiation strategies to address diverse learning needs, such as ideas for just-in-time skill building during a lesson or ways to break a task into smaller tasks, if needed, to make it more accessible.

Pre-AP Algebra 1 Assessments for Learning

Pre-AP Algebra 1 assessments function as a component of the teaching and learning cycle. Progress is not measured by performance on any single assessment. Rather, Pre-AP Algebra 1 offers a place to practice, to grow, and to recognize that learning takes time. The assessments are updated and refreshed periodically.

LEARNING CHECKPOINTS

Based on the Pre-AP Algebra 1 Course Framework, the learning checkpoints require students to examine data, models, diagrams, and short texts—set in authentic contexts—in order to respond to a targeted set of questions that measure students' application of the key concepts and skills from the unit. All eight learning checkpoints are automatically scored, with results provided through feedback reports that contain explanations of all questions and answers as well as individual and class views for educators. Teachers also have access to assessment summaries on Pre-AP Classroom, which provide more insight into the question sets and targeted learning objectives for each assessment event.

The following tables provide a synopsis of key elements of the Pre-AP Algebra 1 learning checkpoints.

Format	Two learning checkpoints per unit Digitally administered with automated scoring and reporting Questions target both concepts and skills from the course framework
Time Allocated	Designed for one 45-minute class period per assessment
Number of Questions	10–12 questions per assessment ▪ 7–9 four-option multiple choice ▪ 3–5 technology-enhanced questions

Domains Assessed	
Learning Objectives	Learning objectives within each key concept from the course framework
Skills	Three skill categories aligned to the Pre-AP mathematics areas of focus assessed regularly across all eight learning checkpoints: ▪ greater authenticity of applications and modeling ▪ engagement in mathematical argumentation ▪ connections among multiple representations

Question Styles	Question sets consist of two to three questions that focus on a single stimulus or group of related stimuli, such as diagrams, graphs, or tables. Questions embed mathematical concepts in real-world contexts. *Please see page 60 for a sample question set that illustrates the types of questions included in Pre-AP learning checkpoints and the Pre-AP final exam.*

PERFORMANCE TASKS

Each unit includes one performance-based assessment designed to evaluate the depth of student understanding of key concepts and skills that are not easily assessed in a multiple-choice format.

These tasks, developed for students across a broad range of readiness levels, are accessible while still providing sufficient challenge and the opportunity to practice the analytical skills that will be required in AP mathematics courses and for college and career readiness. Teachers participating in the official Pre-AP Program will receive access to online learning modules to support them in evaluating student work for each performance task.

Format	One performance task per unit Administered in print Educator scored using scoring guidelines
Time Allocated	Approximately 45 minutes or as indicated
Number of Questions	An open-response task with multiple parts

Domains Assessed	
Key Concepts	Key concepts and prioritized learning objectives from the course framework
Skills	Three skill categories aligned to the Pre-AP mathematics areas of focus: ▪ greater authenticity of applications and modeling ▪ engagement in mathematical argumentation ▪ connections among multiple representations

PRACTICE PERFORMANCE TASKS

One or more practice performance tasks in each unit provide students with the opportunity to practice applying skills and knowledge in a context similar to a performance task, but in a more scaffolded environment. These tasks include strategies for adapting instruction based on student performance and ideas for modifying or extending tasks based on students' needs.

Performance Assessments At-a-Glance

Unit	Performance Assessment	Title	Teacher Access	Student Access
Unit 3 Quadratic Functions	Practice Performance Task	The Catapult	Teacher Resources: Units 3 & 4	Student Resources: Unit 3
	Practice Performance Task	Weaving a Rug		Student Resources: Unit 3
	Performance Task	The Path of a Football		Teacher-distributed handout
Unit 4 Exponent Properties and Exponential Functions	Practice Performance Task	Exponent Properties	Teacher Resources: Units 3 & 4	Student Resources: Unit 4
	Performance Task	Computer-Aided Drawing		Teacher-distributed handout

FINAL EXAM

Pre-AP Algebra 1 includes a final exam featuring multiple-choice and technology-enhanced questions as well as an open-response question. The final exam is a summative assessment designed to measure students' success in learning and applying the knowledge and skills articulated in the Pre-AP Algebra 1 Course Framework. The final exam's development follows best practices such as multiple levels of review by educators and experts in the field for content accuracy, fairness, and sensitivity. The questions on the final exam have been pretested, and the resulting data are collected and analyzed to ensure that the final exam is fair and represents an appropriate range of the knowledge and skills of the course.

The final exam is designed to be delivered on a secure digital platform in a classroom setting. Educators have the option of administering the final exam in a single extended session or two shorter consecutive sessions to accommodate a range of final exam schedules.

Multiple-choice and technology-enhanced questions are delivered digitally and scored automatically with detailed score reports available to educators. This portion of the final exam is designed to build on the question styles and formats of the learning checkpoints; thus, in addition to their formative purpose, the learning checkpoints provide practice and familiarity with the final exam. The open-response question, modeled after the performance tasks, is delivered as part of the digital final exam but is designed to be scored separately by educators using scoring guidelines that are designed and vetted with the question.

The tables on the following page provide a synopsis of key elements of the Pre-AP Algebra 1 Final Exam.

Format	Digitally administered Questions target both concepts and skills from the course framework Specific questions are graphing calculator enabled
Time Allocated	One 105-minute session or two sessions of 60 minutes and 45 minutes
Number of Questions	35–40 questions ▪ four-option multiple-choice questions ▪ technology-enhanced questions ▪ one multipart open-response question
Scoring	▪ automatic scoring for multiple-choice and technology-enhanced questions ▪ educator scoring for open-response questions ▪ comprehensive score reports with individual student and class views for educators

Domains Assessed	
Content	Key concepts and prioritized learning objectives from the course framework
Skills	Three skill categories aligned to the Pre-AP mathematics areas of focus assessed in the final exam: ▪ greater authenticity of applications and modeling ▪ engagement in mathematical argumentation ▪ connections among multiple representations

Question Styles	Question sets consist of two to three questions that focus on a single stimulus or group of related stimuli, such as diagrams, graphs, or tables. Questions embed mathematical concepts in real-world contexts. *Please see page 60 for a sample question set that illustrates the types of questions included in Pre-AP learning checkpoints and the Pre-AP final exam.*

SAMPLE ASSESSMENT QUESTIONS

The following questions are representative of what students and educators will encounter on the learning checkpoints and final exam.

1. Jazmin runs a lemonade stand every Saturday. She sells each cup of lemonade for the same price. Which of the following statements describes how Jazmin can most accurately model her daily revenue?

   (A) She can use a linear function to model her revenue because the rate of change, in terms of dollars earned per cup of lemonade sold, is constant.

   (B) She can use a quadratic function to model her revenue because the rate of change, in terms of dollars earned per cup of lemonade sold, is linear.

   (C) She can use an exponential function to model her revenue because the growth factor, in terms of dollars earned per cup of lemonade sold, is constant.

   (D) She can use a piecewise linear function to model her revenue because the rate of change, in terms of dollars earned per cup of lemonade sold, varies over the course of the day.

Assessment Focus

Question 1 assesses whether a student can identify an appropriate function to use for modeling a rate of change when given a real-world scenario.

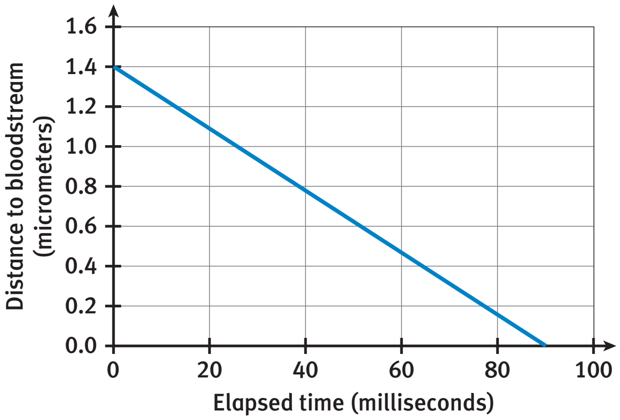
Correct Answer: A

Learning Objective:

1.1.4 Determine whether a relationship presented graphically or numerically is linear by examining the rate of change.

Area of Focus: Engagement in Mathematical Argumentation

2. In humans, oxygen molecules move from the lungs into the bloodstream, where they are carried throughout the body. The graph below shows the distance, in micrometers, an oxygen molecule is from the bloodstream and the elapsed time, in milliseconds, it took to move from the lungs into the bloodstream.



What is the value of the vertical axis intercept and what does it mean?

(A) 1.4 milliseconds is the time it took the oxygen molecule to move from the lungs into the bloodstream.

(B) 90 milliseconds is the time it took the oxygen molecule to move from the lungs into the bloodstream.

(C) 90 micrometers is the distance the oxygen molecule moved between the lungs and the bloodstream.

(D) 1.4 micrometers is the distance the oxygen molecule moved between the lungs and the bloodstream.

Assessment Focus

Question 2 assesses whether or not students can interpret the vertical axis intercept in context. They must apply the skill of interpreting information presented in a graph in order to solve the problem.

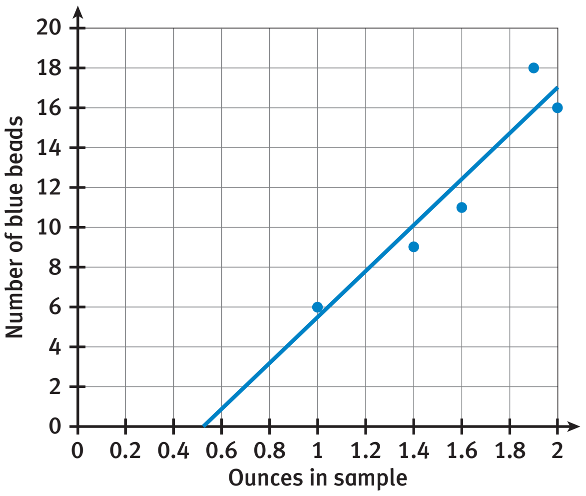
Correct Answer: D

Learning Objective:

1.2.6 Model a contextual scenario with a linear function.

Area of Focus: Greater Authenticity of Applications and Modeling

3. Eli is purchasing beads for an art project. The supply store sells the beads in bulk in multicolor assortments. Eli only wants to use blue beads for his project, but the store won't allow him to pick through the beads to select only blue ones. From the large bin of assorted beads, he takes five small samples of varying weight and counts the number of blue beads in each sample. The scatterplot represents his findings.



To determine an equation for a trend line, Eli uses two points that he thinks lie on the line: the number of blue beads for a sample that measures 0.6 ounces and the number of blue beads for a sample that measures 2 ounces. Which of the following is the correct equation of the trend line that Eli determined?

(A) $y = \left(\frac{16}{1.4}\right)x - \frac{41}{7}$

(B) $y = \left(\frac{15}{1.4}\right)x - \frac{38}{7}$

(C) $y = \left(\frac{1.4}{16}\right)x + \frac{673}{400}$

(D) $y = \left(\frac{1.4}{15}\right)x - \frac{1186}{75}$

Assessment Focus

Question 3 assesses whether or not students can determine the equation of the line of best fit. Students must also translate between information provided in the stimulus text and the scatterplot.

Correct Answer: A

Learning Objective:

1.4.4 Determine an equation for a trend line that describes trends in a scatterplot.

Area of Focus: Connections Among Multiple Representations

4. Laura is a general contractor who needs to order at least 110 sheets of plywood for a house she is building. There are two supply companies, Menureds and HouseCo, and neither one has enough plywood in stock for Laura's project. Menureds sells plywood in bundles of 3 sheets and they have 12 bundles available at $72 per bundle. HouseCo sells plywood in bundles of 4 sheets and they have 25 bundles available at $76 per bundle.

   Laura has budgeted up to $2,200 for plywood. Which combination of bundles gets Laura at least 110 sheets and and stays under budget?

   (A) 12 bundles from Menureds and 18 bundles from HouseCo

   (B) 10 bundles from Menureds and 20 bundles from HouseCo

   (C) 12 bundles from Menureds and 25 bundles from HouseCo

   (D) 5 bundles from Menureds and 24 bundles from HouseCo

Assessment Focus

Question 4 assesses whether or not students can represent and model a scenario involving a simple system of inequalities and interpret potential solutions.

Correct Answer: D

Learning Objective:

2.4.3 Model a contextual scenario with a system of linear inequalities.

Area of Focus: Greater Authenticity of Applications and Modeling

Pre-AP Algebra 1 Course Designation

Schools can earn an official Pre-AP Algebra 1 course designation by meeting the requirements summarized below. Pre-AP Course Audit Administrators and teachers will complete a Pre-AP Course Audit process to attest to these requirements. All schools offering courses that have received a Pre-AP Course Designation will be listed in the Pre-AP Course Ledger, in a process similar to that used for listing authorized AP courses.

PROGRAM REQUIREMENTS

- The school ensures that Pre-AP frameworks and assessments serve as the foundation for all sections of the course at the school. This means that the school must not establish any barriers (e.g., test scores, grades in prior coursework, teacher or counselor recommendation) to student access and participation in the Pre-AP Algebra 1 coursework.
- Teachers have read the most recent *Pre-AP Algebra 1 Course Guide*.
- Teachers administer each performance task and at least one of two learning checkpoints per unit.
- Teachers and at least one administrator per site complete a Pre-AP Summer Institute or the Online Foundational Module Series. Teachers complete at least one Online Performance Task Scoring Module.
- Teachers align instruction to the Pre-AP Algebra 1 Course Framework and ensure their course meets the curricular requirements summarized below.
- The school ensures that the resource requirements summarized below are met.

CURRICULAR REQUIREMENTS

- The course provides opportunities for students to develop understanding of the Pre-AP Algebra 1 key concepts and skills articulated in the course framework through the four units of study.
- The course provides opportunities for students to engage in the Pre-AP shared instructional principles.
    - close observation and analysis
    - evidence-based writing
    - higher-order questioning
    - academic conversation

- The course provides opportunities for students to engage in the three Pre-AP Algebra 1 areas of focus. The areas of focus are:
  - greater authenticity of applications and modeling
  - engagement in mathematical argumentation
  - connections among multiple representations
- The instructional plan for the course includes opportunities for students to continue to practice and develop disciplinary skills.
- The instructional plan reflects time and instructional methods for engaging students in reflection and feedback based on their progress.
- The instructional plan reflects making responsive adjustments to instruction based on student performance.

RESOURCE REQUIREMENTS

- The school ensures that participating teachers and students are provided computer and internet access for completion of course and assessment requirements.
- Teachers should have consistent access to a video projector for sharing web-based instructional content and short web videos.

Accessing the Digital Materials

Pre-AP Classroom is the online application through which teachers and students can access Pre-AP instructional resources and assessments. The digital platform is similar to AP Classroom, the online system used for AP courses.

Pre-AP coordinators receive access to Pre-AP Classroom via an access code delivered after orders are processed. Teachers receive access after the Pre-AP Course Audit process has been completed.

Once teachers have created course sections, student can enroll in them via access code. When both teachers and students have access, teachers can share instructional resources with students, assign and score assessments, and complete online learning modules; students can view resources shared by the teacher, take assessments, and receive feedback reports to understand progress and growth.

Unit 3

Unit 3
Quadratic Functions

Overview

SUGGESTED TIMING: APPROXIMATELY 9 WEEKS

In this unit, students develop a strong foundation in the important concept of quadratic functions. Students should understand that quadratic functions have a linear rate of change and are often formed by multiplying two linear expressions, and therefore are not linear. Quadratic functions are useful for modeling phenomena that have a linear rate of change and symmetry around a unique minimum or maximum. This foundational understanding of quadratics helps students build their conceptual knowledge of nonlinear functions and prepares them for further study of polynomial and rational functions in Algebra 2.

ENDURING UNDERSTANDINGS

This unit focuses on the following enduring understandings:

- Quadratic functions have a linear rate of change.
- Quadratic functions can be expressed as a product of linear factors.
- Quadratic functions can be used to model scenarios that involve a linear rate of change and symmetry around a unique minimum or maximum.
- Every quadratic equation, $ax^2 + bx + c = 0$, where a is not zero, has at most two real solutions. These solutions can be determined using the quadratic formula.

KEY CONCEPTS

This unit addresses the following key concepts:

- 3.1: Functions with a Linear Rate of Change
- 3.2: The Algebra and Geometry of Quadratic Functions
- 3.3: Solving Quadratic Equations
- 3.4: Modeling with Quadratic Functions

UNIT 3

UNIT RESOURCES

The tables below outline the resources provided by Pre-AP for this unit.

Lessons for Key Concept 3.1: Functions with a Linear Rate of Change				
Lesson Title	**Learning Objectives Addressed**	**Essential Knowledge Addressed**	**Suggested Timing**	**Areas of Focus**
3.1: Introducing Quadratic Functions	3.1.1, 3.1.2	3.1.1a, 3.1.2a, 3.1.2b	~90 minutes	Engagement in Mathematical Argumentation, Connections Among Multiple Representations
3.2: Area Models for Quadratic Functions	3.1.1, 3.1.2, 3.4.1	3.1.1a, 3.1.1b, 3.1.2a, 3.1.2c, 3.1.2d, 3.4.1a, 3.4.1c	~90 minutes	Engagement in Mathematical Argumentation, Connections Among Multiple Representations
3.3: Revenue and Profit	3.1.2, 3.4.1, 3.4.3	3.1.2c, 3.1.2d, 3.4.1c, 3.4.3a, 3.4.3b	~135 minutes	Greater Authenticity of Applications and Modeling, Engagement in Mathematical Argumentation, Connections Among Multiple Representations
All learning objectives and essential knowledge statements for this key concept are addressed with the provided materials.				

Lessons for Key Concept 3.2: The Algebra and Geometry of Quadratic Functions				
Lesson Title	**Learning Objectives Addressed**	**Essential Knowledge Addressed**	**Suggested Timing**	**Areas of Focus**
3.4: The Factored Form of a Quadratic	3.2.1, 3.2.2, 3.2.3	3.2.1a, 3.2.1b, 3.2.1c, 3.2.2a, 3.2.2d, 3.2.3b	~90 minutes	Connections Among Multiple Representations
3.5: Graphs and the Factored Form of a Quadratic	3.2.2, 3.2.3, 3.2.4	3.2.2a, 3.2.3b, 3.2.4a, 3.2.4c	~60 minutes	Connections Among Multiple Representations

The following Key Concept 3.2 essential knowledge statement is not addressed in Pre-AP lessons. Address this in teacher-developed materials.

- Essential Knowledge Statement: 3.2.3d

Practice Performance Task: The Catapult (~45 minutes)
This practice performance task assesses learning objectives and essential knowledge statements addressed up to this point in the unit.

Learning Checkpoint 1: Key Concepts 3.1 and 3.2 (~45 minutes)
This learning checkpoint assesses learning objectives and essential knowledge statements from Key Concepts 3.1 and 3.2. For sample items and learning checkpoint details, visit Pre-AP Classroom.

Lessons for Key Concept 3.3: Solving Quadratic Equations				
Lesson Title	**Learning Objectives Addressed**	**Essential Knowledge Addressed**	**Suggested Timing**	**Areas of Focus**
3.6: Connecting Standard Form to Vertex Form	3.2.2, 3.2.3	3.2.2a, 3.2.2b, 3.2.2c, 3.2.3a, 3.2.3c	~55 minutes	Engagement in Mathematical Argumentation, Connections Among Multiple Representations

UNIT 3

3.7: The Quadratic Formula	3.2.2, 3.3.5	3.2.2c, 3.3.5a, 3.3.5b, 3.3.5c, 3.3.5d	~75 minutes	Engagement in Mathematical Argumentation
3.8: The Symmetry of the Parabola	3.2.3, 3.3.1, 3.4.3	3.2.3a, 3.3.1b, 3.4.3a, 3.4.3b	~45 minutes	Connections Among Multiple Representations
3.9: Interpreting the Discriminant	3.3.2, 3.3.5, 3.3.6	3.3.2c, 3.3.5b, 3.3.6b	~60 minutes	Engagement in Mathematical Argumentation, Connections Among Multiple Representations

The following Key Concept 3.3 learning objectives and essential knowledge statements are not addressed in Pre-AP lessons. Address these in teacher-developed materials.

- Learning Objectives: 3.3.3, 3.3.4
- Essential Knowledge Statements: 3.3.1a, 3.3.2a, 3.3.2b, 3.3.3a, 3.3.3b, 3.3.3c, 3.3.4a, 3.3.4b, 3.3.6a

Lessons for Key Concept 3.4: Modeling with Quadratic Functions

Lesson Title	Learning Objectives Addressed	Essential Knowledge Addressed	Suggested Timing	Areas of Focus
3.10: Pursuit Problems	3.3.5, 3.4.1, 3.4.2	3.3.5b, 3.4.1a, 3.4.2a, 3.4.2b	~60 minutes	Greater Authenticity of Applications and Modeling, Engagement in Mathematical Argumentation
3.11: Gravity and Free-Fall Investigations	3.2.4, 3.4.1, 3.4.2	3.2.4c, 3.2.4d, 3.4.1a, 3.4.2a, 3.2.4b	~45 minutes	Greater Authenticity of Applications and Modeling, Engagement in Mathematical Argumentation

3.12: The Golden Ratio	3.3.5, 3.4.1	3.3.5b, 3.4.1c	~60 minutes	Engagement in Mathematical Argumentation
3.13: Finding a Formula for Triangular Numbers	3.1.1, 3.2.4, 3.3.5	3.1.1a, 3.1.1b, 3.2.4a, 3.3.5b	~60 minutes	Engagement in Mathematical Argumentation, Connections Among Multiple Representations

The following Key Concept 3.4 essential knowledge statement is not addressed in Pre-AP lessons. Address this in teacher-developed materials.

- Essential Knowledge Statement: 3.4.1b

Practice Performance Task: Weaving a Rug (~45 minutes)

This practice performance task assesses learning objectives and essential knowledge statements addressed up to this point in the unit.

Performance Task for Unit 3 (~45 minutes)

This performance task draws on learning objectives and essential knowledge statements addressed throughout the unit.

Learning Checkpoint 2: Key Concepts 3.3 and 3.4 (~45 minutes)

This learning checkpoint assesses learning objectives and essential knowledge statements from Key Concepts 3.3 and 3.4. For sample items and learning checkpoint details, visit Pre-AP Classroom.

LESSON 3.1

Introducing Quadratic Functions

OVERVIEW

LESSON DESCRIPTION

Part 1: Exploring a Nonlinear Sequence
Students examine a staircase built from LEGO® bricks and explore the resulting quadratic sequence.

Part 2: Playing *Snake*
Students explore another nonlinear sequence using a popular mobile game.

Part 3: Summary and Practice
Students have an opportunity to explore several more sequences to determine if they are arithmetic, quadratic, or neither.

CONTENT FOCUS

This lesson is students' first exposure to nonlinear functions. They are introduced to quadratic functions through several different contexts, all of which involve nonlinear but predictable sequences. By the end of the lesson, students should understand that quadratic sequences differ from arithmetic sequences because quadratic sequences do not have a constant rate of change. However, quadratic sequences do have a consistent pattern: The rate of change is linear, so the rate of change of the rate of change is constant. In this lesson, students explore some quadratic sequences and distinguish between sequences that are linear, quadratic, or neither based on the rate of change. By focusing on the rate of change of different types of sequences, this lesson provides a very rudimentary introduction to differential calculus, which is the study of rates of change.

AREAS OF FOCUS

- Engagement in Mathematical Argumentation
- Connections Among Multiple Representations

SUGGESTED TIMING

~90 minutes

HANDOUTS

Lesson

- 3.1.A: Playing Snake

Practice

- 3.1.B: Practice with Quadratic Sequences

MATERIALS

- graph paper

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ Quadratic functions have a linear rate of change.	
Learning Objectives	**Essential Knowledge**
3.1.1 Determine whether a relationship is quadratic or nonquadratic based on a numerical sequence whose indices increase by a constant amount.	**3.1.1a** In a table of values that represents a quadratic relationship and that has constant step sizes, the differences in the values of the relationship, called the first differences, exhibit a linear pattern. The second differences of a quadratic sequence are constant.
3.1.2 Convert a given representation of a quadratic function to another representation of the quadratic function.	**3.1.2a** A graphical representation of a quadratic function displays ordered pairs that satisfy the relationship. The exact coordinates of the ordered pairs may or may not be evident from the graph of the function. **3.1.2b** A numerical representation of a quadratic function consists of only a subset of the ordered pairs that satisfy the relationship.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

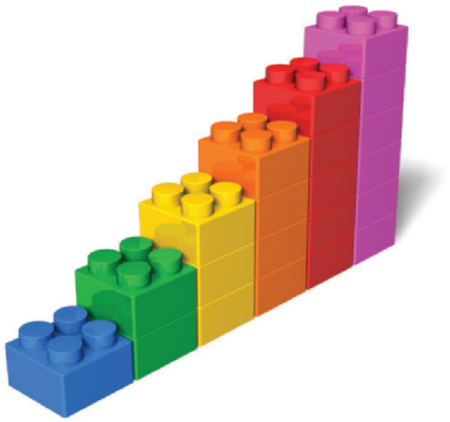
> Are the following sequences of numbers arithmetic, quadratic, or neither? Explain your reasoning.
>
> (a) 2, 5, 10, 17, 26, …
>
> (b) 2, 5, 8, 11, 14, …
>
> (c) 2, 5, 2, 5, 2, 5, …

PART 1: EXPLORING A NONLINEAR SEQUENCE

In the first part of this lesson, students examine an image of a staircase built from LEGO bricks and explore the resulting quadratic sequence. This part of the lesson establishes for students the definition of a quadratic sequence.

- To begin, display the following figure for the whole class, and ask students to observe the image and look for patterns. Then, present them with the following scenario:

> Suppose that you are building with LEGO bricks and want to continue the staircase pattern, using only single bricks, as shown in the picture. If you want to build a staircase that is 10 bricks tall, how many total bricks would you need to build it?



- First, ask students to produce both a low and a high estimate for the answer before working on the problem. Ask them to share their estimates with the rest of the class. Record the estimates where everyone can see them.
- Next, have students work on the problem individually or in pairs.

Guiding Student Thinking

Most students will look for a pattern in the number of bricks needed to create a staircase of increasing height and then count until they reach a staircase that is 10 bricks tall. Once they reach 10, they will sum up the total number of stacked bricks needed for each of the 10 steps.

A staircase that is 10 bricks tall will require a total of 55 bricks to build.

UNIT 3

- Once most students have determined an answer, lead them through a discussion about the scenario. Highlight the relationship between the height of the staircase and the number of bricks needed, as well as how this relationship is a function. Consider using the following questions to guide student thinking:
  - What are the two quantities in this problem?

    The height of the staircase and the number of bricks needed.
  - Can we think about the relationship between the two quantities like a function? Why or why not?

    Yes, we can think about this like a function because a staircase of a given height will be associated with only one amount of bricks.
  - What quantity could we use as the input? What quantity could we use as the output? Why?

    We could use the height of the staircase as the input and the number of bricks as the output. If we know how tall we want the staircase, we will be able to determine how many bricks we need to build it.
  - What variables could we use to stand for those quantities?

    Use h for the height of the staircase and $b(h)$ for the total number bricks needed to build the staircase.
- After the discussion, ask students to provide values for the total number of bricks needed for different heights of a brick staircase. Then, construct a table of values for the staircase pattern that can be displayed for the whole class to see.

h	$b(h)$
1	1
2	3
3	6
4	10
5	15
6	21
7	28
8	36
9	45
10	55

- Encourage students to share their observations about the table of values. You can use these questions to help guide the conversation:
    - What patterns do you notice? Is this sequence of numbers in the $b(h)$ column a linear function? Why or why not?

      The $b(h)$ column is increasing by a different amount each time: first by 2, then by 3, then by 4, and so on. It is not a linear function, because the $b(h)$ values do not increase by a constant amount even though the h values do.

Meeting Learners' Needs

This is an opportunity to review arithmetic sequences and constant rates of change for students who would benefit from further discussion of these ideas.

- Now, add a third column to the table to help students see how the values in the $b(h)$ column are calculated.

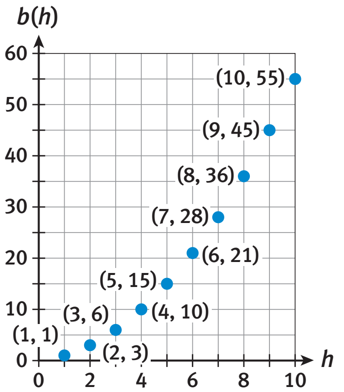
h	$b(h)$	Calculations
1	1	—
2	3	1 + 2
3	6	3 + 3
4	10	6 + 4
5	15	10 + 5
6	21	15 + 6
7	28	21 + 7
8	36	28 + 8
9	45	36 + 9
10	55	45 + 10
Height	Total Bricks for Staircase of Height h	Previous Total Bricks + Height of Staircase

Instructional Rationale

Although many students might recognize the pattern, the expanded calculation in the third column will support students who do not see how the total bricks in each subsequent staircase relies on how many more bricks are added to the staircase.

UNIT 3

- Next, have students work independently to plot the values from the table. They should plot the coordinates by hand on graph paper (first quadrant only), using the following scales: horizontal axis: 0–10 units; vertical axis: 0–60 units.



Graph of the staircase sequence

- After the students have graphed the values, ask the following:
  - Does the graph appear to be linear? Why or why not?

    No, because the points do not lie on a line; the vertical distances between the points increase even though the horizontal distances between the points are a constant 1 unit.

- Tell students that the total number of bricks needed to build a staircase is an example of a quadratic sequence. Use the table that the class created to show students that the difference between consecutive values in the $b(h)$ column (first differences) is increasing, not constant. Then show them that the difference between consecutive differences (second differences) is 1. Students may also notice that in the calculation column, the amount being added each time is increasing by one. This is why the second difference is 1.

- To summarize, you may want to have students record this definition of a quadratic sequence:

  A quadratic sequence is a sequence of numbers where the difference between consecutive differences (the second difference) is constant.

Meeting Learners' Needs

If students need another example, consider showing this one:

–2, –5, –10, –17, –26, …

The first differences are –3, –5, –7, –9. The second differences are a constant –2.

PART 2: PLAYING *SNAKE*

In this part of the lesson, students use what they learned through the staircase problem to explore a new scenario involving the video game *Snake*. The snake's length as it eats more food can be expressed as a quadratic sequence.

- To begin, ask students if they are familiar with *Snake*. If they are not, explain what it is and how it is played. You could show students a video of the game being played, or find a free online version to play, briefly, as a class.
- Next, invite students to work with a partner on the *Snake* problem. See **Handout 3.1.A: Playing *Snake*** for the student task. The handout with answers is shown here.

Classroom Ideas

As an alternative to completing the *Snake* problem in class, it could be assigned to be done out of class with an in-class discussion the next day.

The video game *Snake* has been around for decades. In fact, Nokia, the manufacturer of one of the first affordable cell phones, included the game on every phone.

Snake is simple. You control a snake that never stops moving around the screen, and you must prevent the snake from hitting itself or the edges of the screen. In one version of the game, the snake starts off being five units long. As you move around eating small pieces of food at various places on the screen, the length of the snake grows.

(a) In round 1, the snake grows by two units for every one piece of food. How long will it be after it has eaten three pieces of food? Seven pieces of food? Fifteen pieces of food? How would you describe the snake's growth?
After three pieces of food, the snake will be 11 units long; after seven pieces of food, 19 units long; and after 15 pieces of food, 35 units long. The snake gets two units longer with each piece of food.

(b) In round 2, the length of the snake grows by one unit after one piece of food; after the second piece of food, the snake grows by two units; after the third piece, the snake grows by three units; and so on. If the snake is five units long to start, how long will it be after it has eaten three pieces of food? Seven pieces of food? Fifteen pieces of food? How would you describe the snake's growth?
After three pieces of food, the snake will be 11 units long; after seven pieces of food, 33 units long; and after 15 pieces of food, 125 units long. The snake's length grows by an increasing amount (1, 2, 3, …).

Handout 3.1.A

(c) People who played this game often found that it started off easy but became difficult rather quickly. Which do you think would be more difficult to play, round 1 or round 2? Justify your answer.
Round 1 would be easier because the snake's growth is pretty slow. In round 2, the snake is going to get very long, very quickly.

(d) You have been asked to design round 3. To make this the most challenging round, what would you tell the snake to do every time it eats? Explain why this round would be the most difficult.
Answers will vary. Allow students to present their approaches to round 3 and have them explain what makes it a challenging round. For example, the snake could double in length each time it eats a piece of food, which means it will get very long even more quickly.

Handout 3.1.A

- You may want to guide student work by discussing each question as a whole class after students have had ample time to work on it. During the class discussion of questions *a* and *b*, consider following up with these questions, to check students' understanding:
  - Can this situation be modeled by a function? What are the quantities involved? What would be the input and output? What kind of function would it be?

    Yes, this could be modeled by a function. The quantities are the number of pieces of food eaten and the length of the snake. The input could be the number of pieces of food eaten, and the output could be the length of the snake. It would be a linear function because the rate of change is constant.

    Yes, this could be modeled by a function. The quantities are the numbers of pieces of food eaten and the length of the snake. The input could be the number of pieces of food eaten, and the output could be the length of the snake. It would be a quadratic function because the rate of change is linear.

PART 3: SUMMARY AND PRACTICE

The last part of the lesson gives students an opportunity to explore more sequences. They practice extending sequences using the constant second difference, plotting the points of a sequence, and determining if sequences are linear, quadratic, or neither. See **Handout 3.1.B: Practice with Quadratic Sequences** for practice problems.

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

Are the following sequences of numbers arithmetic, quadratic, or neither? Explain your reasoning.

(a) 2, 5, 10, 17, 26, …

This is a quadratic sequence because the difference in consecutive differences (or second differences) is constant. We can also see that the sequence of differences is an arithmetic sequence.

(b) 2, 5, 8, 11, 14, …

This is an arithmetic sequence because the first difference is constant.

(c) 2, 5, 2, 5, 2, 5, …

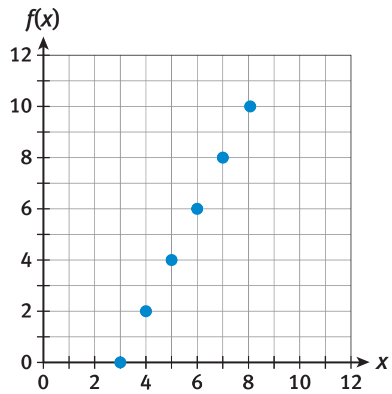
This sequence is neither arithmetic nor quadratic because neither the first nor second difference is constant.

HANDOUT ANSWERS AND GUIDANCE

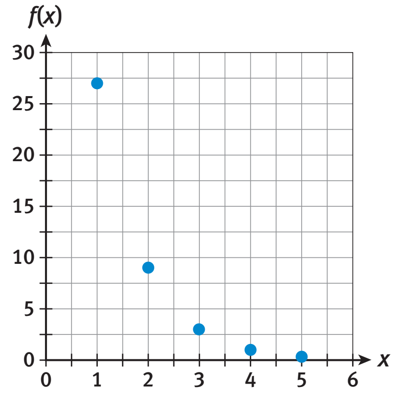
To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 3.1.B: Practice with Quadratic Sequences

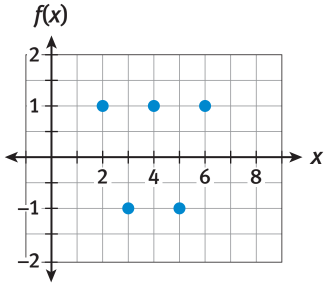
1. The next four numbers are –54, –70, –88, –98. This is a quadratic sequence.
2. The next four numbers are 19, 22, 25, 28. This is an arithmetic sequence.
3. The next four numbers are 36, 49, 64, 81. This is a quadratic sequence.
4. The next four numbers are 64, –128, 256, –512. This sequence is neither arithmetic nor quadratic.
5. The table represents a linear function because the output values change by a constant difference of 2. The graph should look like the one shown here.



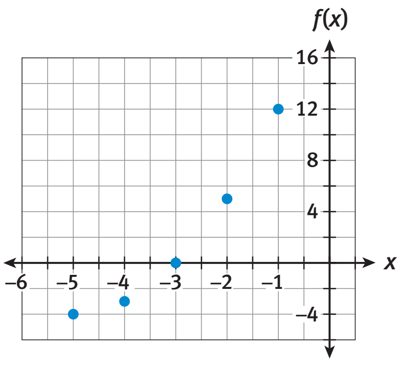
6. This is neither linear nor quadratic because the output values do not have constant first or second differences. The graph should look like the one shown here.



7. This is neither linear nor quadratic because the output values do not have constant first or second differences. The graph should look like the one shown here.



8. The graph is quadratic because the second difference is constant (+2). The graph should look like the one shown here.



9. (a) 12, 20, 28, 36
   (b) The sequence is arithmetic because 8 line segments are added in each successive figure.
   (c) The next figure would have 44 line segments because $36 + 8 = 44$.
10. (a) This is a quadratic sequence because there is a constant second difference of 1.
   (b) The 13th term in the sequence would be 24.5.
   (c) The 10th term in the sequence should be around 15.

LESSON 3.2

UNIT 3

Area Models for Quadratic Functions

OVERVIEW

LESSON DESCRIPTION

Part 1: Warming Up with Triangular Numbers
Students revisit the sequence of numbers from Lesson 3.1 in a new format.

Part 2: Exploring Area
Students learn how to write one algebraic form of a quadratic function.

Part 3: Maximizing Area
Students maximize the area of a garden plot and explore the graph of a quadratic function.

Part 4: Summary and Practice
Students define quadratic functions in terms of sequences, graphs, and algebraic form.

CONTENT FOCUS

This lesson introduces students to another model of quadratic growth and develops an expanded definition of quadratic functions. Students explore how the area of a square increases by nonconstant amounts as the side length increases by one unit. Students also practice multiplying linear expressions to express quadratic functions in standard form.

AREAS OF FOCUS

- Engagement in Mathematical Argumentation
- Connections Among Multiple Representations

SUGGESTED TIMING

~90 minutes

HANDOUT

Practice

- 3.2: Practice with Identifying Quadratic Functions

MATERIALS

- graphing utility or graph paper
- toothpicks for students (at least 16 per pair of students)

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ Quadratic functions have a linear rate of change.	
Learning Objectives	**Essential Knowledge**
3.1.1 Determine whether a relationship is quadratic or nonquadratic based on a numerical sequence whose indices increase by a constant amount.	**3.1.1a** In a table of values that represents a quadratic relationship and that has constant step sizes, the differences in the values of the relationship, called the first differences, exhibit a linear pattern. The second differences of a quadratic sequence are constant. **3.1.1b** Successive terms in a quadratic sequence can be obtained by adding corresponding successive terms of an arithmetic sequence.
3.1.2 Convert a given representation of a quadratic function to another representation of the quadratic function.	**3.1.2a** A graphical representation of a quadratic function displays ordered pairs that satisfy the relationship. The exact coordinates of the ordered pairs may or may not be evident from the graph of the function. **3.1.2c** An algebraic form of a quadratic function contains the complete information about the function because any output value can be determined from any given input value. **3.1.2d** A verbal representation of a quadratic function could refer to a symmetric scenario with a unique minimum or maximum.

3.4.1 Model a contextual scenario with a quadratic function.	**3.4.1a** A contextual scenario where the output quantity increases and then decreases (or vice versa), such as accelerated motion, can be effectively modeled by a quadratic function. **3.4.1c** A contextual scenario that can be expressed as a product of two linear expressions, such as profit or area, can be effectively modeled by a quadratic function.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

For each of the functions given, make a table of values for $x = 0, 1, 2, 3, 4$. Then graph the function and perform any necessary multiplication. Explain which function is linear and which is quadratic using the differences, the shape of the graph, and the algebraic form of the function.

(a) $h(x) = (x+4)(x-3)$

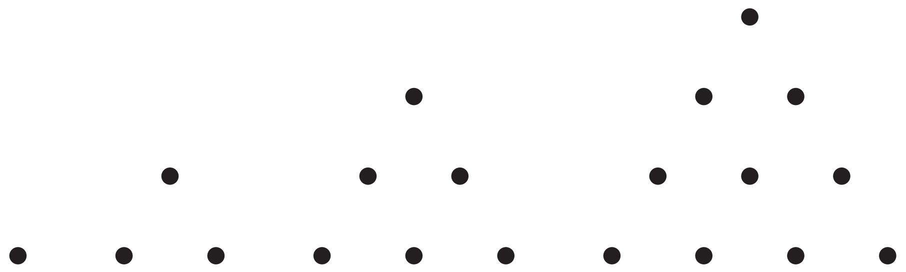
(b) $g(x) = \frac{1}{2}(5x-6)$

UNIT 3

PART 1: WARMING UP WITH TRIANGULAR NUMBERS

This lesson begins by revisiting the sequence of numbers associated with the stair pattern from Lesson 3.1 in a slightly different form. Instead of working with a staircase pattern, students explore a bowling pin pattern. The goal is for students to continue thinking about nonlinear patterns and to start connecting these nonlinear patterns to area.

- To begin, display the bowling pin pattern shown in the following figure for the whole class to see. Ask students to observe the image and look for patterns in the sequence of pin formations.

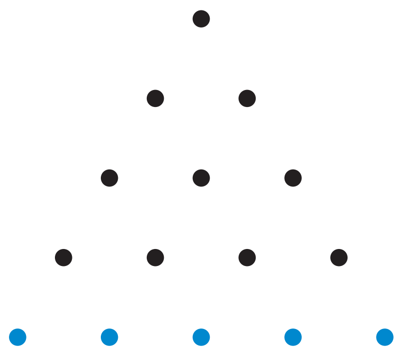


- Once students have had some time to observe the image, engage them in a whole-class discussion of their initial observations and record observations for the class to see. Continue to guide student thinking by asking questions like the following:
  - How many dots are there in each stage of the sequence?

    There are 1, 3, 6, and 10 dots.
  - How would you build the next triangle?

    Add a row of five dots to the bottom of the fourth triangle.
  - Draw the fifth triangle in the sequence. How many dots does it have?

    There are 15 dots in the fifth triangle.



- Describe the sequence. Is this sequence of numbers linear, quadratic, or neither? How do you know?

  The sequence is quadratic, because the amount of change in the number of dots in subsequent formations forms an arithmetic sequence. The sequence isn't linear, because we don't add the same number of dots each time. Each picture includes the previous one, with a new row of dots along the bottom. That row of dots is longer each time. So the rate of change is increasing.

- Explain to students that sequences of numbers such as 1, 3, 6, 10, 15, 21,... are called *triangular numbers* because of the overall shape the dots form. Then, ask the following question.
  - Is 55 a triangular number? Justify your reasoning.

    Fifty-five is a triangular number. It is possible to continue the pattern and get 55 dots to form a triangle on the 10th stage.

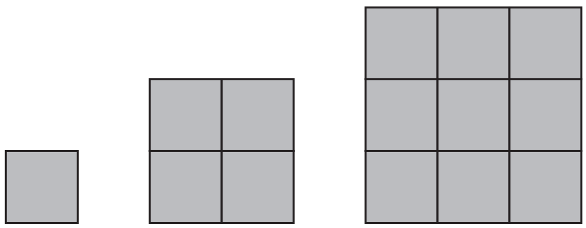
Meeting Learners' Needs

Some students may notice that the triangles' area fills up faster than the side lengths grow. You can highlight this by contrasting the image of the triangle interior with the square perimeter problem from Lesson 1.2, in which only the perimeter grew by the same amount each time.

PART 2: EXPLORING AREA

In this part of the lesson, students explore and compare differences in how the perimeter and the area of a square change as the side length of the square increases. They then work to create a formula that can predict the area of the next square in the sequence. This activity is designed to help students make meaningful connections between calculations and symbols by looking for patterns in repeated calculations and then writing those calculations using algebraic notation.

- To begin, display the sequence of successively larger squares shown in the following figure for the whole class to see.



- Next, when students have had time to observe the figure, lead a guided, whole-class discussion comparing and contrasting how the perimeter of the squares change as the side length increases with how the area of the squares change as the side length increases, using the following prompts.

- Make a table of values that relates the side length n to the perimeter $P(n)$.

n	$P(n)$
2	8
3	12
4	16
…	…

- Using a graphing utility, or by hand, plot the points from the table of values. Do the plotted points appear linear, quadratic, or neither? How do you know?

  The plotted points appear linear because the points seem to lie on a line. Students could also reason that the rate of change is constant because as n increases by 1, $P(n)$ increases by 4.

- Now make a table of values that relates the side length n to the area $A(n)$.

n	$A(n)$
1	1
2	4
3	9
4	16
…	…

- Using a graphing utility, or by hand, plot the points from the table of values. Do the plotted points appear linear, quadratic, or neither? How do you know?

  The plotted points appear quadratic because the graph has a curved shape.

- Which increases more as the side length increases: the perimeter or the area?

  The area increases more, because its rate of change is increasing, not constant.

▪ Next, display the tables of values and the graphs of both $P(n)$ and $A(n)$ for the whole class. Lead a brief review of linear functions, using the following prompts.

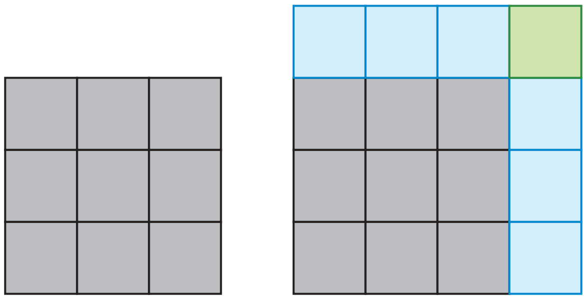
  - How do we write a function rule for the perimeter of the square in terms of the side length?

    $P(n) = 4n$

- Use your function rule to determine the perimeter of a square that has a side length of 10 units.

  $P(10) = 4(10) = 40$ units

- Now have students draw a 3×3 square on their paper if they haven't already done so. Then, tell them to add squares to the top and side of the 3×3 square to make it a 4×4 square. Refer to the figure below for a model of the process.



Transforming a 3×3 square into a 4×4 square

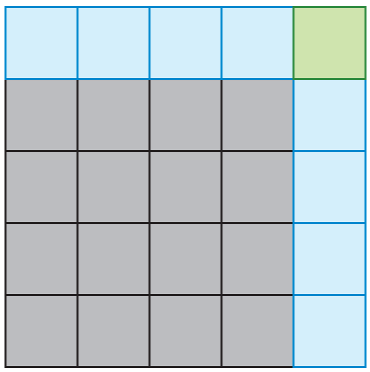
Classroom Ideas
To support students in drawing the square patterns, you could provide graph paper, or you could provide students with the first three squares in the sequence already drawn.

- Ask students to reflect on how many squares they added to the 3×3 square to create the 4×4 square and where they added the squares. Collect the ideas and record them for the class to see.

  Sample responses:

  - The 4×4 square is the 3×3 square + 3 squares on top + 3 squares on side + 1 corner square.
  - The 4×4 square is the 3×3 square + 4 squares on top + 3 squares on side.

- Next, have students add squares to the top and side of the 4×4 square to make a 5×5 square. See the following figure for a model of the result.



Transforming a 4×4 square into a 5×5 square

Classroom Ideas
Using colored pencils to draw the squares and highlight important features for students can help make the patterns emerge more quickly.

UNIT 3

- Then, ask students to reflect on how many squares they added to the 4×4 square to create the 5×5 square and where they added the squares. Collect the ideas and record them for the class to see.

  Sample responses:

  - The 5×5 square is the 4×4 square + 4 squares on top + 4 squares on side + 1 corner square.
  - The 5×5 square is the 4×4 square + 5 squares on top + 4 squares on side.

- Lead students in a guided, whole-class discussion to help them identify patterns in the square sequence. Add a column to the table for n and $A(n)$ to record calculation strategies for each row (i.e., how to get from one row to the next one). As a class, fill in the appropriate strategies to calculate the area of squares with side lengths $n=1$ through $n=6$. Then, challenge students to complete the last row using $n+1$ as the length of the side and to explain their reasoning.

Calculating the Area of a Square

n	*A*(*n*)	Sample Strategy 1	Sample Strategy 2
1	1		
2	4	$1+1+1+1$	$1+2+1$
3	9	$4+2+2+1$	$4+3+2$
4	16	$9+3+3+1$	$9+4+3$
5	25	$16+4+4+1$	$16+5+4$
6	36	$25+5+5+1$	$25+6+5$
New Length	New Area	*Previous Area + Previous Side Length + Previous Side Length + 1*	*Previous Area + New Side Length + Previous Side Length*
$n+1$	$A(n+1)=(n+1)^2$	$A(n)+n+n+1$ n^2+2n+1	$A(n)+(n+1)+n$ n^2+2n+1

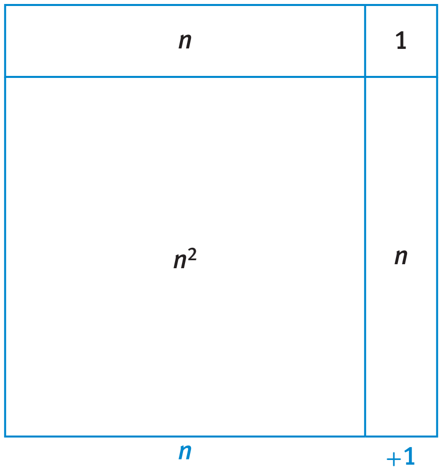
Guiding Student Thinking

Students may have difficulty understanding that the parentheses in $A(n+1)$ and the parentheses in $(n+1)^2$ or $(n+1)(n+1)$ have different meanings. Explain to students that the parentheses in $A(n+1)$ tell us that $n+1$ is being used as the *input* of function A, whereas the parentheses in $(n+1)^2$ are grouping symbols.

- Once students have completed the last row of the table, explain it shows an important algebraic identity:

  $$(n+1)^2 = (n+1)(n+1) = n^2 + 2n + 1$$

  Students will often mistakenly "distribute" the exponent over a sum: $(a+b)^2 \neq a^2 + b^2$. You can display the area model shown here to help students connect the algebra with the geometry of a square with side length $(n+1)$.



The geometry of $(n+1)^2$

- At this point, lead a whole-class discussion or demonstration to show how the distributive property applies to the main problem. Start by asking the following questions:
  - What does an exponent of 2 mean? How can we rewrite $(n+1)^2$ without using an exponent?

    An exponent of 2 means to multiply a term by itself; therefore, $(n+1)^2$ is the same as $(n+1)(n+1)$.

- For students who have not multiplied binomials previously, there is a lot to figure out in the expression $(n+1)(n+1)$. To make the expression a little easier to understand, suggest to students that you replace one of the $(n+1)$ with another variable, such as Q, so $(n+1)(n+1)$ becomes $Q \bullet (n+1)$. Lead students through some questions to help them see how the distributive property works with two binomials.
  - If we have an expression like $Q \bullet (n+1)$, how do we perform the distribution?

    $Q \bullet (n+1)$ is the same as $Q \bullet n + Q \bullet 1$

  - We know that $Q \bullet (n+1) = Q \bullet n + Q \bullet 1$. But, we replaced $(n+1)$ with Q, so let's substitute the $(n+1)$ back in place of Q. What would that look like?

    Substituting $(n+1)$ for all the Qs would give:

Meeting Learners' Needs

Some students may need additional support with understanding the distributive property. Explain that whenever addition and multiplication are performed together, the distributive property comes into play. Students have been using the distributive property ever since elementary school, even though they might not realize it. For example, the distributive property allows you to think about $7 \bullet 31$ as $7 \bullet 30 + 7 \bullet 1$ or to multiply $2\frac{1}{4} \bullet 3$ as $2 \bullet 3 + \frac{1}{4} \bullet 3$.

$$Q \bullet (n+1) = Q \bullet n + Q \bullet 1$$
$$(n+1) \bullet (n+1) = (n+1) \bullet n + (n+1) \bullet 1$$

- How do we perform the distribution for $(n+1) \bullet n + (n+1) \bullet 1$?

  $(n+1) \bullet n + (n+1) \bullet 1 = n \bullet n + 1 \bullet n + n \bullet 1 + 1 \bullet 1$

- How can we simplify the final expression?

  $n^2 + n + n + 1 = n^2 + 2n + 1$

- After these questions, make sure that students connect all the calculations they've done back to the original identity. It is easy to get lost in the variables. They should see that $(n+1)^2$ can be rewritten as $(n+1)(n+1)$, which can be expanded through the distributive property to be $n^2 + 2n + 1$. This algebraic calculation helps confirm the geometric observations from above.

- Finally, display the sequence of numbers for the staircase pattern from Lesson 3.1 and the sequence of numbers for area of the squares in this lesson. Then ask students:

  - We saw that the staircase pattern has a sequence of numbers $b(h)$: 1, 3, 6, 10, 15, 21, ... and the area pattern has a sequence of numbers $A(n)$: 1, 4, 9, 12, 25, 36, ... What is the same and what is different about these sequences?

    Neither sequence is arithmetic, because they do not have a constant first difference. Both sequences are quadratic because the second differences are constant. The second difference for $b(h)$ is 1, but the second difference for $A(n)$ is 2.

- Summarize the important conclusions about quadratic sequences, and have students record them:

  (1) The first differences are an arithmetic sequence, (2) the second differences are constant, and (3) the formula for a quadratic sequence will have a term with an exponent of 2.

PART 3: MAXIMIZING AREA

In this part of the lesson, students explore how to determine the maximum area of a rectangular region where the perimeter of the region is limited. The numbers in this problem were chosen so the maximum area occurs at an integer value for the side length, which happens when the region is a square.

Meeting Learners' Needs

The maximum area of a rectangular region where the perimeter is limited will always occur when the region is a square. You can vary the perimeter available to be any integer value, but the side length may not be a rational number.

- Have students work in pairs to complete the Garden Area Design Challenge. Provide each pair of students with 16 toothpicks so they can model the fencing for the garden. Share the challenge with students:

> You want to fence in a rectangular region for your garden, and you have 16 sections of fencing, each 1 foot long. You want to make the area that you enclose as large as possible. What are the dimensions of the largest garden you can fence in, and what is the area? (Note: Assume that the dimensions must be whole numbers.)

- Circulate around the room while students are working and make sure they understand the directions. The gardens need to be fully enclosed by toothpicks. Encourage students to record their findings as they try different rectangular configurations.
- Invite student pairs to share their findings with the whole class, and record their findings for the class to see. Most pairs of students will have figured out that the maximum area is 16 square feet when the dimensions of the garden are 4 feet $\times$ 4 feet. The remainder of the lesson is devoted to figuring why the maximum area happens at the side length where it occurs.

UNIT 3

- Next, ask the students to identify which quantities in the problem should be used as the input and as the output of an area function. Then, have students decide whether to write the area function in terms of the length, $A(l)$, or in terms of the width, $A(w)$.
- With student input, generate a table of values that shows the length and width of the garden as well as the area of the garden. See the following table for a model.

w	l	$A(w)$
0	8	0
1	7	7
2	6	12
3	5	15
4	4	16
5	3	15
6	2	12
7	1	7
8	0	0

Meeting Learners' Needs

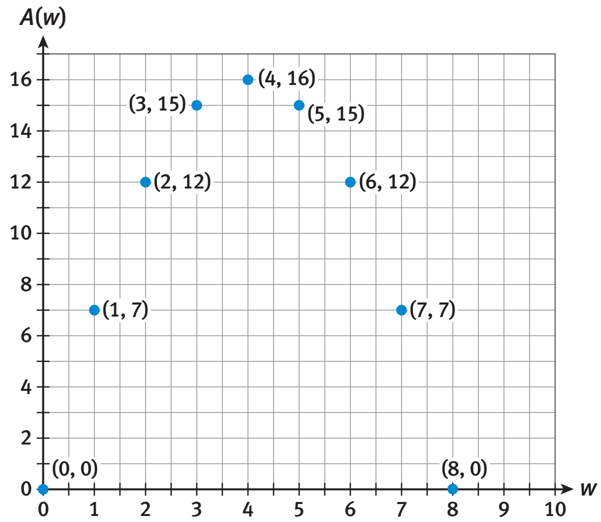
Some students might not be sure about whether to write the area in terms of $A(l)$ or $A(w)$. Encourage them to just pick one and then to partner with a student who picked the other and observe how, or whether, the area functions differ. This will help them see that the exercise can be done either way.

- Now ask students to look at the sequence of numbers in the $A(w)$ column and reflect about the following questions:
  - Is the sequence arithmetic, quadratic, or neither?

    The consecutive differences are 7, 5, 3, 1, –1, –3, –5, –7. The second difference is a constant –2. Therefore the sequence is quadratic.
  - What do you think the graph of function A would look like? Will it be linear or curved? Will the second difference have any impact on the shape of the graph?

    The graph will not be a line; it will have some kind of curve. Students should make predictions about what impact the negative second difference will have on the behavior of the graph. At this point, these are just predictions.

- Let students know that they can ignore the l column for now, because they will be graphing the input–output relationship between w and $A(w)$. Some students might realize that w and l have the same values, just in different order. If a student does point this out, let them know that it is an important observation and that you will come back to it later in the class.
- Now, have students plot the points $(w, A(w))$ on a graphing utility. Display the following graph of the points.



Lead a whole-class discussion, using these prompts:

- Ask students to identify which axis should be labeled "width of garden" and which should be labeled "area of garden."
- Ask students what they notice about the graph. Try to elicit the following information:
  - The graph is symmetrical. For example, two inputs have the same output: 0 and 8; 1 and 7; 2 and 6; 3 and 5.
  - This symmetry in the graph of $A(w)$ with respect to w happens because our region is a rectangle, so we get the same area when the width is 2 and the length is 6 as we do when the width is 6 and the length is 2.
  - The maximum area you can create with 16 foot-long sections of fencing is 16 square feet, since this is the maximum "height" of the curve.
  - The dimensions of the garden that produce the maximum area are 4 feet by 4 feet, which is a square.
  - This curve opens downward, which may have something to do with the negative value of the second difference.

- Next, lead a whole-class discussion to show students how to develop a formula for $A(w)$ in terms of w.
  - How can we calculate the length of the garden if we know the width of the garden?

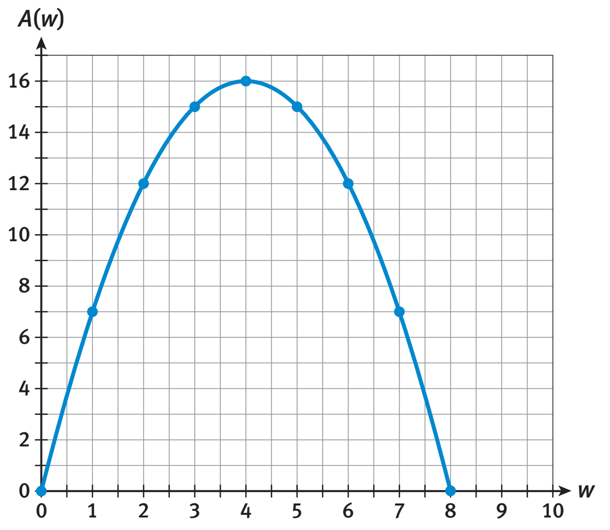
    We subtract twice the width from the perimeter and then divide the result by 2 to get the length. The length has to be $l = \dfrac{16 - 2w}{2}$, or $l = 8 - w$.
  - How do you figure out the area of the garden using w and $8 - w$ as the width and length?

    Multiply length by width, so $A = w(8 - w)$.

Meeting Learners' Needs

To support students in constructing this formula, encourage them to draw a picture of an arbitrary rectangle and label opposite sides with w and l. Then, students can write $w + l + w + l = 16$ and solve for l in terms of w.

- Finally, have students use a graphing utility to graph the function $A(w) = w(8 - w)$ on the same graph where they plotted the points earlier. See the figure for a model.



Graph of the area function

Then ask the following questions:

- What do you notice about the shape of this graph?

  The graph is curved; the graph has a peak; the graph connects all the points we graphed earlier; the graph looks the same on the right side and the left side of the peak.

- How do you perform the distribution in $A(w)=w(8-w)$?

  $A(w)=w(8-w)=8w-w^2$.

- What do you notice about the expanded formula for the area?

  The new formula for area has a term with an exponent of 2.

- What does this mean about the formula for area?

  The formula must be quadratic.

PART 4: SUMMARY AND PRACTICE

In this part of the lesson, students should take some time to formally define quadratic functions in terms of their second differences, the shape of the graph, and the algebraic form of the function. You can use this opportunity to create a Frayer vocabulary organizer for "quadratic functions." Students should also have time to practice determining whether a function in algebraic form is a quadratic function. See **Handout 3.2: Practice with Identifying Quadratic Functions**.

- Introduce students to the term *parabola* to describe the graph of a quadratic function, along with the terms *line of symmetry* and *maximum value*. Show students examples of parabolas with a variety of widths, vertex placements, and x-intercepts.
- Define *quadratic function,* and have students record the following definition:

  A function f is quadratic if and only if: (1) a sequence of the output values whose input values differ by a fixed amount, such as 1, have a constant second difference, (2) the graph has the shape of a parabola, and (3) the algebraic expression for the function has the form $f(x)=ax^2+bx+c$, where a, b, and c are constants, and a is not zero.

Guiding Student Thinking

As students complete problems 3 and 4 on Handout 3.2, it is essential that they use inputs that change by a fixed amount. Otherwise, they will not be able to draw conclusions from the patterns in the outputs.

UNIT 3

ASSESS AND REFLECT ON THE LESSON

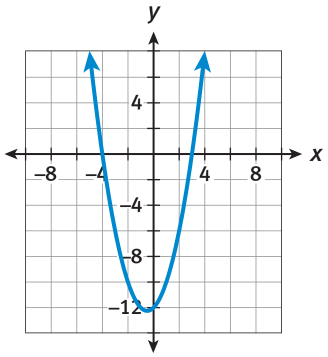
FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

For each of the functions shown, make a table of values for $x = 0, 1, 2, 3, 4$. Then graph the function and perform any necessary multiplication. Explain which function is linear and which is quadratic using the differences, the shape of the graph, and the algebraic form of the function.

(a) $h(x) = (x+4)(x-3)$

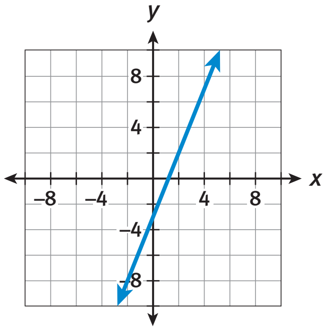
x	$h(x)$
0	−12
1	−10
2	−6
3	0
4	8



The second differences are constant (equal to 2), which is a property of quadratic functions. The graph is a parabola, which is a property of quadratic functions. The algebraic form is $h(x) = x^2 + x - 12$. It has an exponent of 2, which is a property of quadratic functions. Therefore, the function is quadratic.

(b) $g(x) = \frac{1}{2}(5x - 6)$

x	$h(x)$
0	−3
1	−0.5
2	2
3	4.5
4	7



The first differences are constant (equal to 2.5), which is a property of linear functions.

The graph is a line, which is a property of linear functions. The algebraic form is $g(x) = \frac{5}{2}x - 3$. It does not have an exponent of 2 so it cannot be a quadratic function. Therefore, the function is linear.

UNIT 3

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 3.2: Practice with Identifying Quadratic Functions

1. (a) x^2+2x

   (b) $3y^2-4y$

   (c) $43-8w$

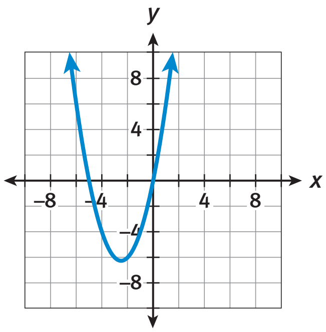
   (d) $x^2-3x-18$

   (e) $-3z^2+3z$

2. The table of values represents a quadratic function. As the x-value increases by 1, the output values have a constant second difference of 1. The parabolic shape of the graph also confirms that this is a quadratic function.

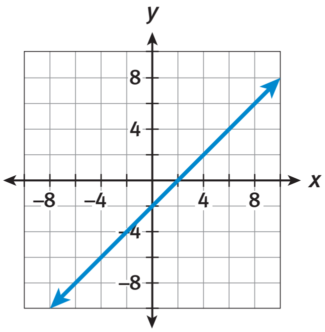
3. (a) $p(x)=x^2+5x$; This is quadratic, because for a fixed change in the input, the outputs of the table of values have a constant second difference, the graph is a parabola, and the algebraic expression has the form $ax^2 + bx + c$.

x	$p(x)$
–5	0
–4	–4
–3	–6
–2	–6
–1	–4
0	0



   (b) $g(x)=x-2$; This is linear, because for a fixed change in the input, the outputs of the table of values have a constant difference, the graph is a line, and the algebraic expression has the form $mx + b$.

x	$g(x)$
–2	–4
–1	–3
0	–2
1	–1
2	0



UNIT 3

LESSON 3.3

Revenue and Profit

OVERVIEW

LESSON DESCRIPTION

Part 1: Warming Up with Economic Principles
Students do a problem to introduce them to the important vocabulary and concepts of economics.

Part 2: Finding the Best Price
Students engage in an extended modeling task about revenue and profit.

Part 3: Defining Quadratic Functions
Students algebraically manipulate the profit function to learn about standard form of quadratic functions.

CONTENT FOCUS

This lesson has students create and analyze a problem involving two economic phenomena: (1) the existence of a best price and (2) maximum revenue. It is designed to span several class periods and provide students with an opportunity to model a context with a quadratic function. Students use their knowledge of linear functions to build components of the model and use a quadratic function to maximize the profit from selling pies at a bake sale. The modeling process unfolds in this order: They create a cost function, then a price function, then a revenue function, and finally a profit function. The numbers in this scenario were chosen for relative ease of computation so students can invest their cognitive effort in understanding the process of creating, using, and interpreting a mathematical model. Students also learn the standard form of a quadratic function.

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Engagement in Mathematical Argumentation
- Connections Among Multiple Representations

SUGGESTED TIMING

~135 minutes

HANDOUTS

Practice

- 3.3.A: Pie-Selling Scenario
- 3.3.B: Graphing Revenue Function
- 3.3.C: Practice with Quadratics

MATERIALS

- graphing utility

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ Quadratic functions have a linear rate of change. ▪ Quadratic functions can be used to model scenarios that involve a linear rate of change and symmetry around a unique minimum or maximum.	
Learning Objectives	**Essential Knowledge**
3.1.2 Convert a given representation of a quadratic function to another representation of the quadratic function.	**3.1.2c** An algebraic form of a quadratic function contains the complete information about the function because any output value can be determined from any given input value. **3.1.2d** A verbal representation of a quadratic function could refer to a symmetric scenario with a unique minimum or maximum.
3.4.1 Model a contextual scenario with a quadratic function.	**3.4.1c** A contextual scenario that can be expressed as a product of two linear expressions, such as profit or area, can be effectively modeled by a quadratic function.
3.4.3 Interpret the vertex and roots of a quadratic model in context.	**3.4.3a** If the values of the vertex are included in the contextual domain and range of the problem, the x-value of the vertex of the parabola represents the input value that corresponds to either the minimum or maximum output value, and the y-value of the vertex of the parabola represents the minimum or maximum output value. **3.4.3b** The x-value(s) of the root(s) of a parabola often represent the extreme values (of the input variable) in a contextual scenario.

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FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

Because of your amazing work maximizing the revenue for the math club, the basketball team wants you to help them. They want to know how much admission they should charge for their games to make the most money. The athletic director tells you that when they charged \$5 per ticket they sold 600 tickets, but when they tried to charge \$20 per ticket they sold no tickets.

(a) You can assume the relationship between the price of the ticket and the number of tickets sold is linear. Write a linear function n that models the number of tickets sold, $n(p)$, as a function of the price of the ticket, p.

(b) To calculate revenue, you multiply the price of the ticket by the number of tickets sold. Write a function r that models the revenue generated by selling tickets, $r(p)$, as a function of the price of the ticket, p.

(c) Use your function r from part (b) to recommend a ticket price to the basketball team. Support your recommendation by explaining how much revenue they could generate from your ticket price versus a higher or lower price.

PART 1: WARMING UP WITH ECONOMIC PRINCIPLES

The goal of this part of the lesson is for students to become familiar with the terms *cost, price, revenue*, and *profit*. Understanding these concepts, at least at a basic level, is necessary for success in the latter parts of the lesson.

- To begin, display for students this scenario involving selling lemonade:

> Miles and Lucy decide to sell lemonade to earn some money to go to the movies. They buy supplies such as pitchers, ice, and lemonade mix for a cost of $13.50. The lemonade mix makes 8 quarts of lemonade. They set up shop, and decide to sell one cup of lemonade for a price of $0.75. If they sell all 8 quarts of lemonade, how much revenue will they make, and will they make enough profit to buy movie tickets for $12 each? (There are 4 cups in 1 quart.)

Ask students to read through the scenario and identify the quantities and terms in the problem. Then, lead a guided discussion of the key terms with the whole class:

- What are the important terms (vocabulary) in the problem?

  The important terms are *cost*, *price*, *revenue*, and *profit*.

- What does *cost* mean, how do you determine business costs in general, and what are the costs for this lemonade business?

  Cost is the amount of money you spend on supplies and equipment and other business expenses. In general, you figure it out by determining how much you need to spend to run your business. For selling lemonade, the initial costs are given as $13.50 for supplies.

- What does *price* mean, how do you determine price in general, and what are the prices in this scenario?

  Price is the amount you will charge for the product or service. Price is set by the business owner, usually with some consideration of how much money they will make. For their lemonade business, Miles and Lucy have set the price as $0.75 per cup.

- What does *revenue* mean, how do you determine revenue in general, and what is the revenue for the lemonade business?

  Revenue is the amount of money you generate from selling your product or service before you deduct cost (business expenses). To calculate revenue, you multiply the number of items sold by the price of each item. For the lemonade business, the revenue is 0.75 times the number of cups of lemonade sold. If they sell all 8 quarts, or 32 cups, of lemonade, their revenue will be $24.

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- What does *profit* mean, how do you determine profit in general, and what is the profit for the lemonade business?

  Profit is the amount of money you have left after costs are subtracted from the revenue. To calculate profit, you subtract the cost from the revenue. For selling lemonade, the revenue is \$24 and the cost is \$13.50, so the profit is \$10.50.

- Can Miles and Lucy both go to the movies if they sell all 8 quarts of lemonade?

  They cannot go to the movies because they have not earned enough profit.
  Profit = \$10.50. Cost of movie tickets = 2(\$12) = \$24.
  The profit is less than the cost of movie tickets.

- Summarize important terms for the whole class, and instruct students to record the following information:

Profit = Revenue − Cost = \$24.00 − \$13.50 = \$10.50
Price = \$0.75
Revenue (amount of money generated) = \$0.75(32)
= \$24

Meeting Learners' Needs
You could extend the problem by challenging students to determine a price at which Miles and Lucy should sell each cup of lemonade so that they earn enough to go to the movies.

They would have to sell each cup of lemonade for more than \$1.17 to earn enough profit to go to the movies.

PART 2: FINDING THE BEST PRICE

In this part of the lesson, students engage in an extended modeling task about profit and revenue. The task is designed to take several class periods to complete. It is much more important for students to engage in the modeling process and understand what they are doing than for them to get through the task quickly. This part of the lesson provides students with an opportunity to use quadratic functions to model real-world situations for which a minimum or a maximum value is desired. In higher-level math classes, such as calculus, the process of determining a maximum or minimum given several constraints is called *optimization*.

- Have the whole class read the scenario on **Handout 3.3.A: Pie-Selling Scenario** also shown here.

> Every year, the math club at your school sells pies on Pi Day as a fundraiser. Since you have done this for several years, you have data to use in setting the price. When you made 160 pies, you sold them all at \$6 per pie. When you made 80 pies, you sold them all for \$10 each. Your principal has allowed you to use the school cafeteria kitchen for a one-time, nonrefundable fee of \$300, and the ingredients for each pie cost \$2. How many pies should you make to maximize profit? How much should you sell the pies for? How much money will you make?

Handout 3.3.A

Guiding Student Thinking

Students might think that you can set whatever price you want, but economics does not work that simply (especially if you are trying to make a lot of money). The next set of questions is designed to help students understand that the price of the pies depends on the production output.

- Before getting into the specifics of the problem, help students begin to think about the general economic principles involved by asking them the following questions:
  - If the price of the pies is really high, do you think we will sell a lot of pies or not that many pies? Why?

    Sample response: If we have a high price, we will probably not sell that many pies because people don't like to spend a lot of money.
  - If we know we're not going to sell a lot of pies, will our costs be high or low? Why?

    Sample response: If we know we're going to sell just a few pies, we might be able to keep costs low by only buying the supplies we need.
  - If the price of the pies is really low, do you think we will sell a lot of pies or not that many pies? Why?

    Sample response: If we have a low price, we will probably sell a lot of pies because lower prices typically appeal to consumers.
  - If we know we are going to sell a lot of pies, will our costs be high or low? Why?

    Sample response: If we know we're going to sell a lot of pies, our costs will probably be high because we will need a lot of supplies.

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- If the price is high, we don't sell that many pies, but the costs are low. If the price is low, we sell a lot of pies but the costs are high. What might happen to our revenue and profit in each case?

  Sample response: If the price is high and sales are low, the revenue might not be that high. But the costs are low, so profit might be acceptable. If the price is low and the sales are high, the revenue could be high. But because the costs are high, profit might be low.

- Before moving on, summarize for students the primary challenge of the problem (indicated in the final question above):

  There should be a pie price that will make the costs low enough and the revenue high enough to make a profit.

  Now, help students to identify the quantities in the task using the following discussion questions, and record student responses for the whole class to see.

  - What quantities will we need to keep track of in the problem?

    We should keep track of the number of pies, the cost of the pies, the price at which the pies are sold, the revenue generated by selling the pies, and the profit we make.

  - Is there one quantity on which all the rest depend? Which quantity would make sense to assign as the input for the problem?

    It makes sense to use the number of pies as the independent variable because all the other quantities will use that number.

  - Can we think about the other quantities as functions of the number of pies?

    Yes, because the given number of pies will yield a specific cost, price, profit, and revenue.

Guiding Student Thinking

Students should be able to figure out the important quantities by recalling the discussion during the warm up and thinking through the general principles. As you lead the class through writing the rules to determine each of these quantities, you may want to encourage students to first write out their thinking in words and then find the appropriate symbols. Letting students explain in their own words what the function is supposed to do can help them figure out how to mathematize their thoughts.

- Next, display the following definitions to the whole class, and instruct students to record what each of their function outputs will represent before moving on to writing the function rules:

  Input variable = x number of pies

  $C(x)$ = cost (the total money spent)

  $E(x)$ = price (how much people will pay per pie)

  $R(x)$ = revenue (total money generated by sale)

  $P(x)$ = profit (money remaining after costs)

BUILDING THE COST FUNCTION

- Once students have recorded what their function outputs will be, let them know that the first function they will be writing is the cost function. Help students identify the important information about the cost in the problem by asking the following questions:
  - What do we know about the costs for making pies?

    The use of the kitchen is going to cost $300 and the ingredients for each pie will be $2.
  - What do these quantities mean?

    There is a constant cost per pie and an initial cost.
  - What kinds of functions can you write with this information?

    A linear function usually uses that information.
  - What would be a linear function that models the cost in terms of the number of pies?

    Slope-intercept form is the most efficient form to use with the information given, so $C(x) = 2x + 300$.
- Display the cost function so all students can see it. You will come back to it in a little while.

Guiding Student Thinking

Students might have created cost functions with negative values, anticipating that cost needs to be subtracted. Tell students that they are correctly thinking about how to use the cost eventually, but that we are going to think about cost as a positive value.

BUILDING THE PRICE FUNCTION

- Let students know that they will be writing the price function next. Use the following questions to help students figure out that they can write a linear function for the price of the pies with the information they know:
  - What information are we given about the price of the pies?

    When you made 160 pies, you sold all the pies at $6 per pie. When you made 80 pies, you sold all the pies for $10 each.
  - What are the quantities involved?

    The number of pies sold and the price of the pies sold.
  - In general, does it look like the number of pies sold increases or decreases as the price gets higher?

    The number of pies sold will decrease as the price of the pies increases.
  - What kind of function can we write if we have two sets of values for a pair of quantities?

    With two data points, we can use point-slope form to write a linear function.

Instructional Rationale

The relationship between the price and the number of pies sold does not have to be linear, but it is the best we can do with two data points and a decreasing trend. Linear models that are used in real life are rarely perfect. Authentic modeling often involves making simplifying assumptions, recognizing that the mathematical model is not perfect, and testing whether the model is useful for understanding the situation. Good modeling lessons involve identifying assumptions and investigating the impact of changing any of the assumptions. In this case, it is appropriate to use a linear function because it is the simplest mathematical model that can capture the phenomenon we see, which is "more pies, lower price."

- Ask students the following questions. Then, display the price function so all students can see it. You will come back to it in a little while.
  - Which of the quantities in the problem should we use as the input? Which should we use as the output?

    The input is the number of pies, and the output is the price of each pie.
  - How can we represent the information as two ordered pairs of numbers?

    $(160, 6)$ and $(80, 10)$

- What would be a linear function that models the price in terms of the number of pies?
  Using the two points, we find the rate of change to be $-\frac{1}{20}$ dollars per pie. Then, using point slope form, we find that $E(x)=-\frac{1}{20}(x-160)+6$. This simplifies to $E(x)=-\frac{1}{20}x+14$.

BUILDING THE REVENUE FUNCTION

- Next, let students know that they will be writing the revenue function as a whole class. This will transition the class out of the linear functions they constructed for cost and price and into quadratic functions. Help students see how multiplying the price of a pie by the number of pies defines the revenue by having them answer the following questions:
  - If we were to sell 15 pies at \$6 each, how much revenue would we make?
    We would make $15 \bullet \$6 = \$90$.
  - If we were to sell one pie at \$6, how much revenue would we make?
    We would make $1 \bullet \$6 = \$6$.
  - If we were to sell 378 pies at \$6 each, how much revenue would we make?
    We would make $378 \bullet \$6 = \$2{,}268$.
  - If we were to sell x pies at \$6 each, how much revenue would we make?
    We would make $6x$ dollars.
  - How do you calculate revenue if you know the price per item and the number of items you sell?
    Multiply the price by the number of items.
  - If we were to sell x pies at E dollars each, how should we calculate our revenue?
    Multiply x by E.
  - Since we know that the price of the pies can be modeled by $E(x)$ and the number of pies is x, how can we use these known quantities to construct the revenue function?
    Multiply them together: $R(x)=x \bullet E(x)=x\left(14-\frac{1}{20}x\right)$.
  - What kind of function is $R(x)$, and how do you know?
    $R(x)$ is a quadratic function. You can tell because if you distributed the x, you would get a term with an exponent of 2. Also, it is the product of two linear expressions, so it will be a quadratic expression.

Classroom Ideas
The numbers in this discussion can be arbitrarily chosen because the purpose of this exchange is to get students reasoning inductively, using specific examples, and then generalizing to variables.

Instructional Rationale

Although it is important for students to recognize this function as quadratic, it is not necessary to expand it before graphing. Students will benefit from learning that a graphing utility is capable of graphing non simplified equations.

- Next, have students work in groups of two or three on the questions on **Handout 3.3.B: Graphing Revenue Function** to predict what the graph of the revenue function might look like. The handout with answers is shown here.

- What will the revenue be if we sell 10 pies?

  The revenue will be \$135: $R(10)=10\left(14-\frac{1}{20}\bullet 10\right)=10(13.5)=135.$

- If we don't sell any pies, what will our revenue be?

  If we don't sell anything, we bring in zero dollars: $R(0)=0\left(14-\frac{1}{20}\bullet 0\right)=0(14)=0.$

- Which of the quantities should we display on the horizontal and vertical axes?

  The horizontal axis should display the input, so the number of pies sold. The vertical axis should display the revenue generated by selling pies.

- What quadrants will be used for the graph of the revenue? Why?

  We will use the first quadrant, because the number of pies sold will be either zero or positive (not negative). Therefore, the revenue will be either zero or positive.

- Could the number of pies (the input) be a decimal or fraction? Could the revenue generated (the output) be a decimal or fraction?

  The number of pies will not be a decimal or a fraction, because we cannot sell partial pies. The revenue could be a decimal or a fraction, because the price of a pie could be a decimal.

- Looking at the function, would you predict that the graph will be a line or another shape? What is your justification? What might be the shape of the graph?

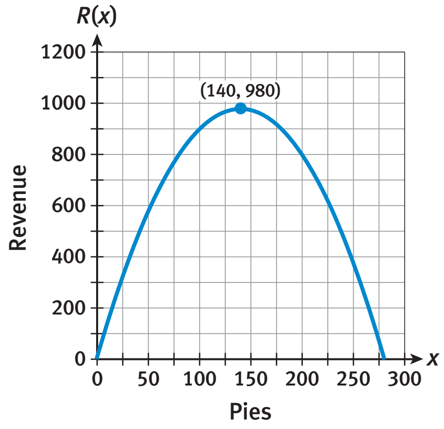
  The graph will not be linear, because there is not a constant rate of change. The graph will look like a curve, because the function is quadratic.

Handout 3.3.B

Guiding Student Thinking

Students might anticipate that the graph will be a parabola because that shape was introduced in the previous lesson. Encourage students to explain the shape in terms of the function's rate of change. Allow them to make conjectures, but make sure they test them by using the graphing utility to graph the function.

- Next, have students work independently to graph the revenue equation using a graphing utility. The graph is shown here for reference:



Graph of the revenue function

Guiding Student Thinking

Earlier, students explained that the number of pies must be a whole number since we are assuming we cannot sell a partial pie. As a result, students might wonder why the graphing utility plots a smooth curve. Help them understand that the graphing utility shows all the x-values and we have to interpret the model for inputs that make sense in the context of the situation.

- Once all students have finished their graphs, engage them in a whole-class discussion of their predictions and the resulting graph.
  - Does this graph match your prediction?

    Yes, the shape is a parabola.
  - Which part of the graph will indicate the maximum revenue?

    The top of the curve (the vertex).

- Determine the coordinates of the vertex.

  The vertex is $(140,\ 980)$.

- What does that ordered pair of numbers mean in the context of the problem?

  That 140 pies will bring in the most revenue: \$980.

- Is our goal to maximize revenue?

  No, the goal is to maximize profit.

- The last step in writing the revenue function is to have students plot the point $(140, 980)$ with their graphing utility (if possible). Make sure they understand and record what the point symbolizes: 140 pies will produce a revenue of \$980.

Guiding Student Thinking

Students may expect that the maximum revenue will also provide the maximum profit. Explain to students that this will not be the case, because the costs of making 140 pies are too high, so too much of the revenue will be lost to the costs. In the final part of the problem, students write the profit function and determine how many pies to make and sell to earn the most profit.

BUILDING THE PROFIT FUNCTION

- Now let students know that they will be writing the profit function as a class. Ask them the following questions to help guide them through finding the function and the maximum profit.
  - How do we calculate profit?

    You should subtract the cost from the revenue.

  - What is the profit function?

    $P(x) = R(x) - C(x) = x\left(14 - \frac{1}{20}x\right) - (2x + 300)$. (Note: At this point, it is not necessary to perform the distribution.)

- Next, before actually graphing the function, ask students to make predictions about what the graph of the profit function might look like:
  - Which of the quantities should we display on the horizontal and vertical axes?

    The horizontal axis should display the input—the number of pies sold. The vertical axis should display the profit generated by selling pies.

  - What would it mean if we had a negative profit?

    We would lose money.

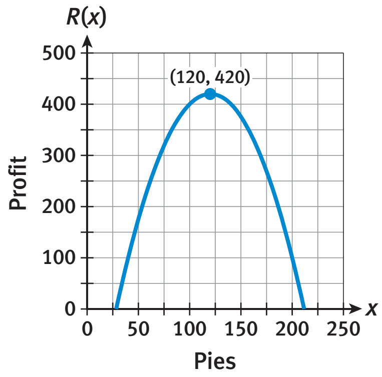
- What quadrants will we use for the graph?

  We need the positive horizontal axis, because the number of pies cannot be negative. We need the positive and negative vertical axis, because profit could be positive or negative. Therefore, we will use quadrants 1 and 4.

- Looking at the equation, would you predict that this will be a line or another shape? Please explain your reasoning.

  It might be a parabola, because the revenue function made a parabola and the revenue function is part of the profit function.

▪ Now, have students test their predictions by graphing the profit function with a graphing utility. The graph that they should get is shown in the figure to the right.



After students have finished graphing, ask them these questions:

- Which part of the graph will reveal the maximum profit?

  The top of the parabola, the vertex.

- What are the coordinates of that point?

  The vertex is $(120,\ 420)$.

- What does it mean in the context of the problem?

  If we sell 120 pies, we make \$420 profit.

- Is the number of pies we should sell to maximize revenue the same as the number of pies we should sell to maximize profit? Please explain your reasoning.

  No, it is not. Making 140 pies costs too much, and we will not generate enough revenue to justify the expense.

▪ Finally, lead a whole-class discussion summarizing the important features of the problem. You can ask the following questions to make sure students understand what they've done:

- How many pies should you make to maximize profit?

  120 pies

- How much should you sell the pies for?

  \$8

- How much money will you make?

  $420

Students may need clarification about why they should make 120 pies and sell them at $8. Review and explain the model with students. If we make 121 pies instead of 120, we have to spend more money producing pies. This means a higher total cost. At the same time, the model indicates that the price we can get is a little lower than $8 per pie. As a result, our profit will be lower. On the other hand, if we only make 119 pies, we will spend less on producing pies (so we have a lower total cost), and we would have to sell them at a price higher than $8. This would make up for the one fewer pie sold. However, our profit will be also lower in this scenario. Making 120 pies for $8 is the right tradeoff between setting a price that is low enough to appeal to consumers but not so low that we lose money on costs. Explain to students that *optimization* often involves making a tradeoff between competing factors or constraints.

PART 3: DEFINING QUADRATIC FUNCTIONS

In this part of the lesson, students focus on algebraically manipulating the profit function. By the end of the lesson, students will know the standard form of a quadratic function and be able to define quadratic functions in terms of standard form.

- First, lead the class through a demonstration of how to manipulate the profit function algebraically, so that it has a different form but the same function.
  - Have students refer to the profit function from earlier in the lesson. If they have not recorded it yet, have them do so: $P(x) = x\left(14 - \frac{1}{20}x\right) - (2x + 300)$.
  - Ask students to distribute the x in the first part of the function:

    $14x - \frac{1}{20}x^2$.
  - Next, ask students to distribute the negative in the second part of the function:

    $-2x - 300$.
  - Then, have them combine both parts of the function:

    $P(x) = 14x - \frac{1}{20}x^2 - 2x - 300$.
  - Finally, have students simplify the function by combining like terms:

    $P(x) = -\frac{1}{20}x^2 + 12x - 300$.

- At this point, you may want to ask students to explain how the profit function is an example of the standard form of a quadratic function. Have students record the standard form in their notes if they do not have it already: $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers, and a is not zero.
- Explain to students that every quadratic function can be written in standard form; therefore, if a function cannot be written in this form, it isn't a quadratic function.
- Next, have students test whether this form of the profit function produces the same graph as the profit function from Part 2 by graphing it on a graphing utility.
- Explain to students that writing an equation in standard form is helpful for identifying a function as quadratic (because it will have an x^2 term), and for identifying the vertical intercept (because when $x = 0$, $y = c$). Finally, ask students a few questions about the graph and the equation:
  - What does the vertical intercept represent for the pie business scenario?

    Before we make any pies $(x = 0)$, there is a \$300 nonrefundable cost to use the kitchen.
  - What is the smallest number of pies we need to sell to break even?

    Looking at the graph, it is about 28 pies. You have to sell at least 29 pies to make a profit.
  - If we sell too many pies, we will start to lose money again. Why is this?

    The additional cost exceeds the additional revenue we would generate.
  - What is the largest number of pies we can sell and still make a profit, even if it's not the maximum profit?

    From the graph, this value is about 211 pies.

Guiding Student Thinking

Students might focus on the decreasing function as evidence of losing money. However, help students differentiate between making less money (x-values greater than that of the vertex) and x-values that would cause a negative profit—losing money.

If students need more practice exploring the economic and/or mathematical concepts in this lesson, you may want to assign the Formative Assessment Goal, which is a simpler version of the pie-selling problem. Students may also benefit from some additional practice with expanding quadratic functions and graphing functions with a graphing utility. See **Handout 3.3.C: Practice with Quadratics**.

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

Because of your amazing work maximizing the revenue for the math club, the basketball team wants you to help them. They want to know how much admission they should charge for their games to make the most money. The team manager tells you that when they charged \$5 per ticket, they sold 600 tickets, but when they tried to charge \$20 per ticket, they sold no tickets.

(a) You can assume the relationship between the price of the ticket and the number of tickets sold is linear. Write a linear function n that models the number of tickets sold, $n(p)$, as a function of the price of the ticket, p.

A linear function that models the number of tickets sold in terms of the price of the ticket is $n(p) = -40p + 800$.

(b) To calculate revenue, you multiply the price of the ticket by the number of tickets sold. Write a function r that models the revenue generated by selling tickets, $r(p)$, as a function of the price of the ticket, p.

Multiply the price by the number of tickets sold, and put it into standard form:

$$\begin{aligned} r(p) &= p \bullet n(p) \\ &= p(-40p + 800) \\ &= -40p^2 + 800p \end{aligned}$$

(c) Use your function r from part (b) to recommend a ticket price to the basketball team. Support your recommendation by explaining how much revenue they could generate from your ticket price versus a higher or lower price.

Because $r(p)$ is a quadratic function whose graph is concave down, there will be a maximum value. The maximum value is \$4,000, which will be realized when the ticket price is \$10 per ticket.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 3.3.C: Practice with Quadratics

Note: Students can either use a graphing utility or make tables to confirm answers for questions 5–8.

1. $f(x) = -x^2 + 8x - 15$
2. $g(x) = x^2 + 10x + 24$
3. $h(x) = 2x^2 + 3x - 14$
4. $m(x) = 3x^2 - 5x - 8$
5. Graph d
6. Graph a
7. Graph b
8. Graph c

UNIT 3

LESSON 3.4

The Factored Form of a Quadratic

OVERVIEW

LESSON DESCRIPTION

Part 1: Warming Up with *x*- and *y*-intercepts
Students briefly review how to calculate intercepts of a line.

Part 2: Exploring Factored Form
Students explore the algebraic form of a function they wrote previously and discover that a quadratic function in factored form can yield a lot of information.

Part 3: Extend and Practice
Students explore a quadratic function with an *x*-intercept that is difficult to discern from the graph alone. Students also complete additional practice problems.

CONTENT FOCUS

Students do not factor quadratic trinomials in this lesson. Rather, quadratic functions are given in factored form, and students work to make a clear connection between the factors in the algebraic expression and the *x*-intercepts of the graph. This is intended to illustrate for students why factoring a quadratic is a useful skill to master. That is, writing the quadratic expression in factored form directly reveals the *x*-intercepts of the graph. Students determine the *x*-intercepts of the parabola associated with a particular quadratic function. When working with students to expand binomials, avoid relying on memorization tricks that do not apply to cubic or higher polynomials; instead, characterize their work as repeatedly applying the distributive property. This reduces the amount of new knowledge they have to assimilate and better prepares them for more advanced algebra.

AREA OF FOCUS

- Connections Among Multiple Representations

SUGGESTED TIMING

~90 minutes

HANDOUTS

Lesson

- 3.4.A: Intercept Review

Practice

- 3.4.B: Practice with Factored Form

MATERIALS

- graphing utility, one per student

COURSE FRAMEWORK CONNECTIONS

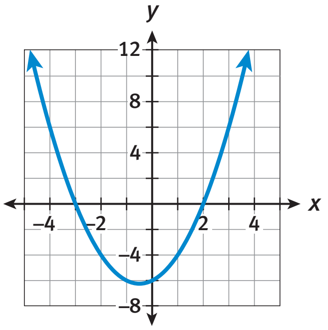
Enduring Understandings	
▪ Quadratic functions can be expressed as a product of linear factors.	
Learning Objectives	**Essential Knowledge**
3.2.1 Identify key characteristics of the graph of a quadratic function.	**3.2.1a** The graph of a quadratic function is a parabola. The parabola is symmetric about a vertical line that passes through the vertex of the parabola. **3.2.1b** The vertex of a parabola is the point on the curve where the outputs of the function change from increasing to decreasing or vice versa. The y-coordinate of the vertex of a parabola is the maximum or minimum value of the function. **3.2.1c** A parabola can have two x-intercepts, one x-intercept, or no x-intercepts.
3.2.2 Translate between algebraic forms of a quadratic function using purposeful algebraic manipulation.	**3.2.2a** Common algebraic forms of a quadratic function include standard form, $f(x)=ax^2+bx+c$; factored form, $f(x)=a(x-r)(x-s)$; and vertex form, $f(x)=a(x-h)^2+k$, where a is not zero. **3.2.2d** The standard form and the factored form of a quadratic function can be translated into each other with purposeful use of the distributive property.
3.2.3 Describe key features of the graph of a quadratic function in reference to an algebraic form of the quadratic function.	**3.2.3b** The graph of a quadratic function whose factored form is $f(x)=a(x-r)(x-s)$, where $a \neq 0$ and $r \neq s$, has two x-intercepts, at $(r,0)$ and $(s,0)$.

UNIT 3

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

1. Consider the quadratic function $f(x) = (x-4)(x+5)$. Predict what the x-intercepts of the graph will be. Then verify your prediction using a graphing utility.
2. Consider the graph of the quadratic function shown here:



What are the x-intercepts? What could be the factored form of the quadratic function?

PART 1: WARMING UP WITH *x*- AND *y*-INTERCEPTS

At the beginning of this lesson, students briefly review x- and y-intercepts with linear functions. A main goal of this part of the lesson is for students to remember that the x-coordinate of the y-intercept is 0 and the y-coordinate of the x-intercept is 0.

- Have students work in pairs on problems reviewing intercepts with linear functions. See **Handout 3.4.A: Intercept Review**.
- Once student pairs have finished with their problems, have them combine to create groups of four. Ask each group of four to compare and contrast their answers. They should work as a team to come to a consensus on their solution. As you circulate around the room, listening to students in working groups, make sure that they understand how to find intercepts algebraically and graphically and can appropriately write them using function notation.
- Once all groups are finished, lead a whole-class debrief to help students synthesize their thinking. You could use questions such as the following:
  - Can a linear function have exactly two x-intercepts? Why or why not?

    No, because there is only one x-value that makes the y-value equal to zero. Because it is a line, it cannot double back on itself.
  - Can a linear function have exactly two y-intercepts? Why or why not?

    No. Because it is a function, there is only one output value for every input, and because it is a line, it cannot double back on itself.

Meeting Learners' Needs

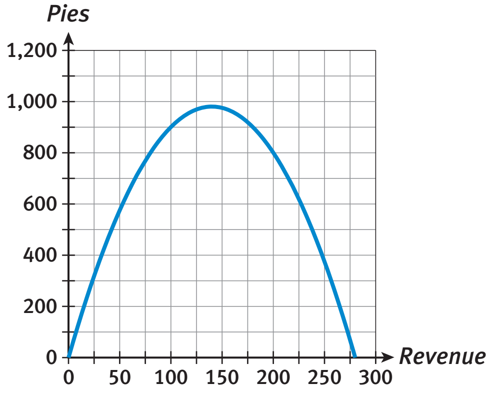
You may need to support some groups' discussions more deeply. You can have students consider whether every linear function has a y-intercept and an x-intercept.

- The linear function $y = b$, where b is a constant, does not have any x-intercepts.
- There are physical scenarios that have a contextual domain that does not include $x = 0$. For example, in an equilateral triangle, the perimeter P is a linear function of the side length s. The domain includes only $s > 0$, because the side lengths of a triangle cannot be negative or zero. Therefore, $P(s) = 3s$ has no y-intercept in context.

PART 2: EXPLORING FACTORED FORM

In this part of the lesson, students more deeply explore the algebraic form of the revenue function they wrote in the pie-selling problem in Lesson 3.3 and connect it to its graph. Specifically, students learn how a quadratic function in factored form can yield a lot of information about the x-intercepts of the parabola.

- Start by reminding students about the pie-selling problem, and have them find the functions they wrote down for the problem. Ask students what they can recall about the problem to elicit the following information: The price function $E(x)$ depended on x, the number of pies made, and was given by the function $E(x)=14-\frac{1}{20}x$. Because the revenue is the total amount of money collected (before cost), the revenue function was determined by multiplying the price, $E(x)$, by the number of pies sold, x: $R(x)=x \bullet E(x)=x\left(14-\frac{1}{20}x\right)$.
- Next, display the graph of the revenue function, shown here, for the class to see.



- In order to have students inspect the graph more carefully, ask them the following series of questions:
  - Is this function linear, quadratic, or neither? How do you know?

    The revenue function is quadratic because its standard form will look like $f(x)=ax^2+bx+c$ and its graph is a parabola. The function is a product of two linear expressions.

  - How many x-intercepts does this graph have? What are the values of the x-intercepts?

    This graph has two x-intercepts at $x=0$ and $x=280$, or $f(0)=0$ and $f(280)=0$.

  - How could you check that you have the correct x-intercepts?

    Substitute the values of $x=0$ and $x=280$ back into the revenue function to check that $R(x)=0$.

  - What is the x-value at the vertex? How is it related to the x-intercepts?

    The x-value of the vertex is 140. This should be halfway between the two x-intercepts.

- How many y-intercepts does the graph have? What is the value of the y-intercept?
  This graph has one y-intercept at $y = 0$.
- Do quadratic functions and linear functions have the same number of intercepts?
  Linear functions typically have only one x-intercept, and quadratic functions can have two. Both linear and quadratic functions typically have one y-intercept.

Instructional Rationale

In order for students to master different functions and their properties, they need practice making predictions without actually calculating or graphing anything. This is a great way to formatively assess students' understanding about what information they can and cannot obtain from the graph of a function.

- Now, build on the previous question sequence to help students see that the x-intercepts are not just the places where the parabola crosses the x-axis; they are also the values that make each factor in a quadratic function equal zero. Have students walk you through the steps on the board to algebraically verify that the x-intercepts are actually $x = 0$ and $x = 280$:

$$R(0) = 0\left(14 - \frac{1}{20} \bullet 0\right) = 0 \bullet 14 = 0$$

$$R(280) = 280\left(14 - \frac{1}{20} \bullet 280\right) = 280 \bullet (14 - 14)$$
$$= 280 \bullet 0 = 0$$

Then, point out that $R(x) = x\left(14 - \frac{1}{20}x\right)$ is the product of two linear factors, x and $14 - \frac{1}{20}x$, and that the x-intercepts are the values for x that make each linear factor zero. Since there are two linear factors in a quadratic function, it makes sense that there should be two x-intercepts.

Meeting Learners' Needs

Some students may need a "factors" review. You can ask them questions such as, "What is a factor?" or "What are five examples of factors?" or "What is a linear factor?" For students who are having difficulty, you may want to provide a list that includes different kinds of factors and have students identify which are linear factors and why.

Guiding Student Thinking

Students may be unconvinced that x and $14-\frac{1}{20}x$ are linear factors. You can have students graph $y=x$ and $y=14-\frac{1}{20}x$ independently to see that the graphs are lines. It could also be worth pointing out that $y=x$ has an x-intercept of $x=0$ and $y=14-\frac{1}{20}x$ has an x-intercept of $x=280$ just like the parabola. This shows that when you multiply linear expressions, the x-intercepts of each linear expression become x-intercepts for the new function.

- Next, display for students the algebraic form of a different function: $f(x)=(x-2)(x+1)$. Give them a few minutes to make predictions about this function without graphing or making a table of values. Then ask some guiding questions:
  - Is this function linear or quadratic? How do you know?

    The function is not linear, because there are two linear factors. Students might conclude that the function is quadratic because it is the product of two linear expressions, just like the previous example.
  - How many x-intercepts do you think this function will have? Can you guess what they might be?

    Sample response: There will probably be two x-intercepts. They will be the two values that make the factors equal to zero.
  - Can you predict the shape of the graph?

    Sample response: It will probably look like a parabola.
  - How many y-intercepts does this function have? Can you guess what it might be?

    Sample response: There will only be one y-intercept.

Guiding Student Thinking

In this question sequence, students might not be able to come up with the x-intercepts at $x=-1$ and $x=2$. That is okay. The goal is for them to recognize that there will be two x-intercepts.

Students also may not be able to figure out the y-intercept. Because determining the y-intercept is easier than finding the x-intercept, you can take the time now to make sure they can do it. Ask them what the x-value of the y-intercept should be. When they correctly identify that $x=0$, then show them that they can substitute $x=0$ into the function to determine that $f(0)=(0-2)(0+1)=-2$.

- Next, as a class, complete the table of values for the quadratic $f(x)=(x-2)(x+1)$. It is important for students to show all their work, as in the table here, so that they can identify patterns. You want them to see the symmetry of the calculations that is reflected in the graph.

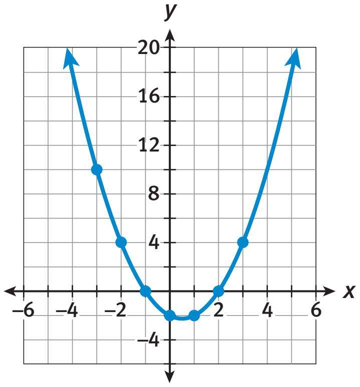
x	$f(x)=(x-2)(x+1)$
-3	$f(-3)=(-3-2)(-3+1)=(-5)(-2)=10$
-2	$f(-2)=(-2-2)(-2+1)=(-4)(-1)=4$
-1	$f(-1)=(-1-2)(-1+1)=(-3)(0)=0$
0	$f(0)=(0-2)(0+1)=(-2)(1)=-2$
1	$f(1)=(1-2)(1+1)=(-1)(2)=-2$
2	$f(2)=(2-2)(2+1)=(0)(3)=0$
3	$f(3)=(3-2)(3+1)=(1)(4)=4$
4	$f(4)=(4-2)(4+1)=(2)(5)=10$

- Once the table has been completed, ask students to identify any patterns they see. They should identify that there are only two rows that give a y-value of zero, and that y-values around the zeros are symmetric. Ask students:
  - What do you notice about the two values of x that resulted in a zero?

    Sample response: The x-values made one of the factors equal to zero, which made the product zero.
  - What do you now know about the graph of this function?

    Sample response: The graph crosses the x-axis at $x=-1$ and $x=2$ and crosses the y-axis at $y=-2$.

    Students might also reason that there is a minimum value that must be halfway between the two x-intercepts (at $x=0.5$).
  - Next, have students use a graphing utility to plot the points from the table of values and then graph the function $f(x)=(x-2)(x+1)$ on the same graph. They should see that the parabola goes through all the points, as shown on the next page.



- At this point, pause and ask students whether the function is a quadratic and if they can justify their reasoning. They should be able to use the criteria from Lesson 3.3:
  1. The table of values (in which the x-values have a constant difference) has consecutive differences in the y-values that increase or decrease linearly.
  2. The graph has the shape of a parabola.
  3. The expanded form of the algebraic rule has an exponent of 2.
- Finally, have students perform the multiplication to write this function in standard form and check by graphing.
  - What is the standard form of this quadratic function?

    $f(x) = x^2 - x - 2$
  - How can you verify that this is the same equation?

    Graph this equation on the same grid as the points and the function in factored form to see that they produce the same graph.

Meeting Learners' Needs

If students need some elaboration on this concept, you can reinforce the connection between the x-intercepts and the factored form by showing students a linear function with one intercept, a quadratic function in factored form with two intercepts, and a cubic function in factored form with three intercepts.

PART 3: EXTEND AND PRACTICE

In the final part of this lesson, students explore a quadratic function with an x-intercept that is difficult to discern from the graph because it is not an integer value. This example helps students see the value of factored form in finding the x-intercepts: given the factored form of the quadratic, it is easy to determine the x-values that make each factor zero.

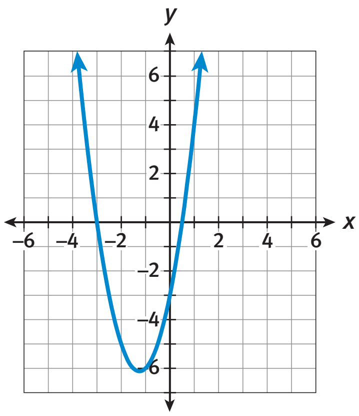
- First, have students conjecture what the x-intercepts of the function $f(x)=(2x-1)(x+3)$ will be. Then ask:
  - How could we determine algebraically what values of x will make each factor equal zero?

    Set each factor equal to zero, and determine where the x-intercepts will be.

    $2x-1=0 \qquad x+3=0$

    $x=\frac{1}{2} \qquad x=-3$

- Then have students verify their calculations by graphing the function, as shown here.



Also have students multiply the binomials to see that the standard form of the function is $f(x)=2x^2+5x-3$.

To conclude, give students an opportunity to practice writing the factored form of a quadratic with known x-intercepts. Have them reflect on the advantages of each representation method.

- From the graph, you can see there is a minimum value.
- The factored form is most efficient for determining roots.
- The standard form is most efficient for determining the vertical intercept.

See **Handout 3.4.B: Practice with Factored Form** for some additional problems.

Instructional Rationale

In this lesson, students should be concerned only with the shape of the graph and the x-intercepts. But, infinitely many quadratic functions can have the same two x-intercepts. A third point on the graph is necessary to “lock in” the parabola. In future lessons students will learn how to determine the equation of a quadratic function given three points on the graph.

UNIT 3

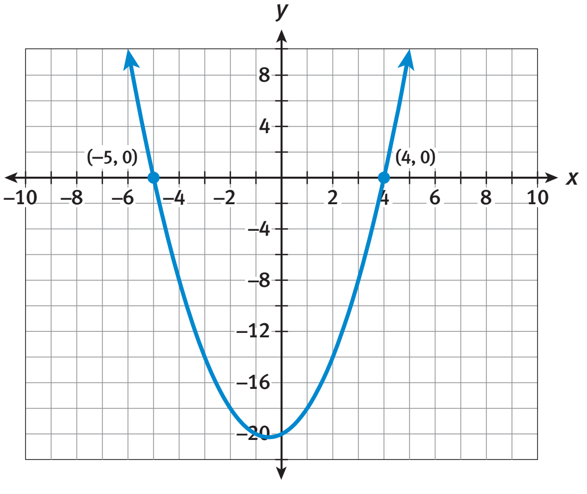
ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

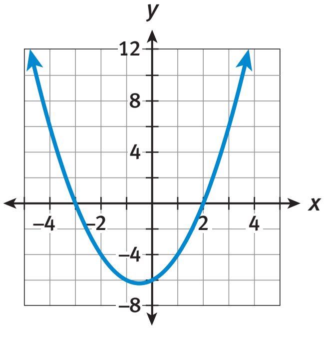
When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

1. Consider the quadratic function $f(x) = (x-4)(x+5)$. Predict what the x-intercepts of the graph will be. Then verify your prediction using a graphing utility.

   The x-intercepts will be $x = 4$ and $x = -5$. The graph is shown here.



2. Consider the graph of the quadratic function shown here:



   What are the x-intercepts? What could be the factored form of the quadratic function?

   The x-intercepts will be $x = -3$ and $x = 2$. The factored form could be $f(x) = (x+3)(x-2)$.

HANDOUT ANSWERS AND GUIDANCE

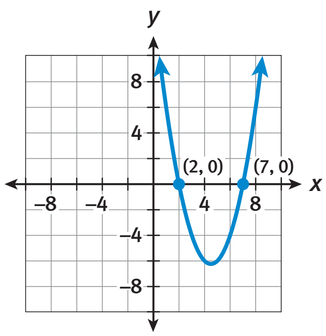
To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 3.4.A: Intercept Review

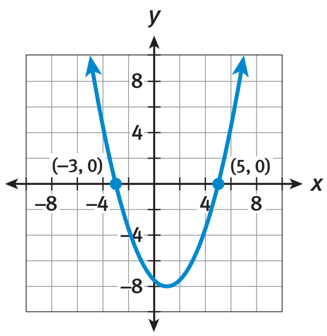
1. $f(0)=14$; $f(280)=0$
2. $g(0)=-6$; $g(9)=0$
3. $h(0)=8$; $h(16)=0$
4. $C(0)=\frac{7}{4}$; $C\left(-\frac{7}{3}\right)=0$
5. $w(0)=-8$; $w\left(\frac{8}{3}\right)=0$
6. $d(0)=-\frac{4}{3}$; $d(2)=0$

Handout 3.4.B: Practice with Factored Form

1. A graph could be:



2. A graph could be:



3. A function could be $f(x)=(x-4)(x-10)$.
4. A function could be $f(x)=(x+5)(x-3)$.
5. A function could be $f(x)=(x+2)(x+4)$.
6. A function could be $f(x)=-1(x+2)(x-5)$.
7. A function could be $f(x)=(x-1)(x-7)$.
8. A function could be $f(x)=(x+5)(x-1)$.

UNIT 3

LESSON 3.5

Graphs and the Factored Form of a Quadratic

OVERVIEW

LESSON DESCRIPTION

Part 1: Warming Up with Graphs and Factors
Students match the factored form of a quadratic function with its graph.

Part 2: Determining the Scale Factor
Students learn how to write the equation of a parabola using the x-intercepts and another point.

Part 3: The Scale Factor and Matching Graphs and Functions
Students complete a matching activity similar to the one earlier in this lesson, but now they must figure out the a-value to match the graphs correctly.

CONTENT FOCUS

In this lesson, students learn how to determine the scale factor of a quadratic equation in factored form and strengthen their conceptual understanding of the connection between the algebraic form of a quadratic function and its graph. Students should conclude that three points (the x-intercepts and a third point) will uniquely determine a parabola and its associated equation; they do not have to use the y-intercept as the third point, even though it is convenient for some calculations. It is possible to write a quadratic equation given any three points, but that is beyond the scope of this lesson and will be addressed later in this unit, in Lesson 3.13.

AREA OF FOCUS

- Connections Among Multiple Representations

SUGGESTED TIMING

~60 minutes

HANDOUTS

Lesson

- 3.5.A: Matching Graphs and Functions, with cards cut out
- 3.5.B: Matching Graphs, Functions, and Scale Factors, with cards cut out

MATERIALS

- graphing utility, one per student

COURSE FRAMEWORK CONNECTIONS

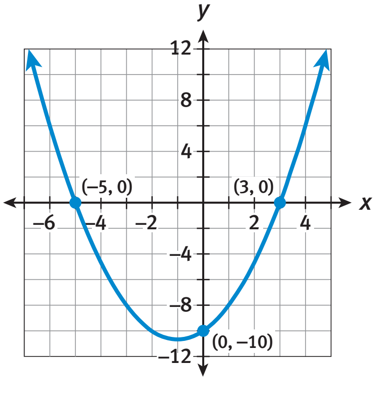
Enduring Understandings	
▪ Quadratic functions can be expressed as a product of linear factors.	
Learning Objectives	**Essential Knowledge**
3.2.2 Translate between algebraic forms of a quadratic function using purposeful algebraic manipulation.	**3.2.2a** Common algebraic forms of a quadratic function include standard form, $f(x)=ax^2+bx+c$; factored form, $f(x)=a(x-r)(x-s)$; and vertex form, $f(x)=a(x-h)^2+k$, where a is not zero.
3.2.3 Describe key features of the graph of a quadratic function in reference to an algebraic form of the quadratic function.	**3.2.3b** The graph of a quadratic function whose factored form is $f(x)=a(x-r)(x-s)$, where $a \neq 0$ and $r \neq s$, has two x-intercepts, at $(r,0)$ and $(s,0)$.
3.2.4 Determine an algebraic rule for a quadratic function given a sufficient number of points from the graph.	**3.2.4a** There is a unique parabola that includes any three distinct noncollinear points. A quadratic function whose graph is the parabola that contains these points can be determined using their coordinates. **3.2.4c** The x-intercepts of a quadratic function are two convenient points that, with a third point, can be used to determine an algebraic rule for the quadratic function.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

1. We know that two points determine a line. How many points determine a parabola?
2. **Version A (text only):** Suppose that a parabola has x-intercepts $(3, 0)$ and $(-5, 0)$ and the y-intercept is $(0, -10)$. What is the factored form of the quadratic function that passes through these points?

   Version B (with graph): What is the factored form of the quadratic function that will produce the parabola shown here?



PART 1: WARMING UP WITH GRAPHS AND FACTORS

In this part of the lesson, students match graphs of parabolas with quadratic functions in factored form. All the quadratic functions in this lesson are in factored form. The matching activity in this part of the lesson is designed to give students practice making the connection between the factor $(x-r)$ and the coordinate $(r,0)$ as the corresponding x-intercept of the graph.

- To begin, have students work in pairs to match several functions with their corresponding graphs. See **Handout 3.5.A: Matching Graphs and Functions**. After students have matched the graphs and functions, check their solutions as a class.
- Follow up with questions to help all students see the relationship between the factor $(x-r)$ and the coordinate $(r,0)$. You might ask:
  - If a quadratic function has a factor of $(x-3)$, what would be an x-intercept for the graph of the function?

    The parabola would intersect the x-axis at the coordinate $(3,0)$.
  - If a quadratic function has a factor of $(x-3)$ and you evaluate the function at $x=3$, what happens?

    When you evaluate the factor $(x-3)$ at $x = 3$, the value of the function is 0 because $(3-3)=0$.
  - Does it always happen that when you substitute an x-coordinate of an x-intercept into a quadratic function, it makes the value of one of the factors zero? Why or why not?

    Yes, it has to happen this way because the y-coordinate of the x-intercept has to be zero.
  - Suppose a quadratic function has a factor of $(2x-1)$. What x-value will make the value of the factor zero? What would be the x-intercept?

    The x-value that will make the value of the factor zero is $x=\frac{1}{2}$, so the x-intercept should be $\left(\frac{1}{2},0\right)$.

Meeting Learners' Needs

This is an opportunity to review the zero-product property: If the product of two factors is zero, then at least one of the factors must be zero.

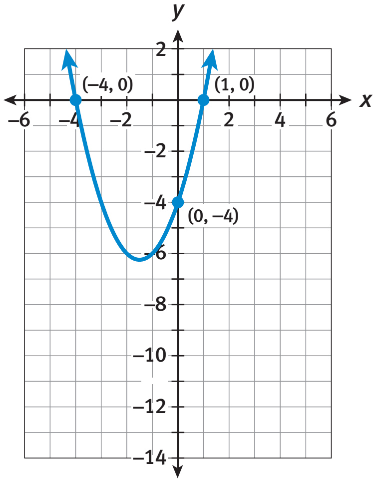
PART 2: DETERMINING THE SCALE FACTOR

In this part of the lesson, students learn that the x-intercepts of a parabola are two important points, but by themselves they are not sufficient to write the equation of the parabola; you need three points to determine the parabola's shape and algebraic rule. Students should see that this feature further distinguishes quadratic functions from linear functions, because two points are sufficient for writing a linear function. Students learn how to use the x-intercepts and another point on the parabola, often the y-intercept, to completely define a quadratic function in the form $f(x)=a(x-r)(x-s)$.

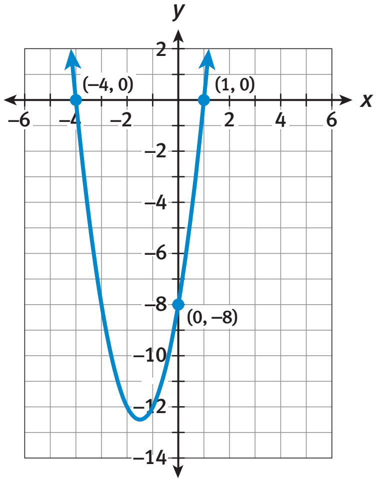
- You can start this part of the lesson by asking students the following questions:
  - Can you draw two different parabolas that will have x-intercepts at $(-4,0)$ and $(1,0)$? Try to do it.

    Yes. (Drawings will vary.)
  - How many parabolas do you think you could draw that will have x-intercepts at $(-4,0)$ and $(1,0)$? Why?

    There will be infinitely many, because the parabola could be stretched vertically and reflected over the horizontal axis.
- Next, display the following two graphs, Graph A and Graph B, so all students can see them.



Graph A



Graph B

Classroom Ideas

Instead of using the two static graphs provided here, you could show a video of a quadratic function of the form $f(x)=a(x-1)(x+4)$ where a varies. This will help students see very clearly that there are infinitely many parabolas with the same x-intercepts. A graphing utility with a slider that varies the value of a would also work.

- Invite students to compare and contrast the graphs. You could ask:
  - How are these graphs similar? How are they different?

    Sample response: They have the same x-intercepts but have different y-intercepts. They are both parabolas that open up. They have different minimum values, or vertices. Graph B appears to be narrower.

Encourage students to use their own language ("Graph A is shorter, Graph B looks stretched out") to describe the parabolas, and then connect their words to the actual vocabulary to create meaning. This is a good time to elicit responses from all students since the questions "How are the graphs similar? How are they different?" are low-risk.

- Once students have described the parabolas, ask the following questions:
  - Which graph represents the function $f(x)=(x+4)(x-1)$? How do you know?

  Since both graphs have the same x-intercepts, this question will probably be challenging for students. They may start a debate about how to determine which one is the right graph. Encourage students to use mathematical reasoning to back up their responses. Simply stating "Graph A has the correct x-intercepts" is not enough without more justification.
- Students may arrive at the idea of using the y-intercept as a test point, leading them to conclude Graph A is the right answer. If they do not, guide them with questions such as the following:
  - Can we use any points other than the x-intercepts to find out which graph represents the function? What other points do we know for sure?

    We can use the y-intercepts: $(0,-4)$ and $(0,-8)$.
  - Test the y-intercepts to see which graph represents the function.

    Graph A represents the function because $f(0)=(0+4)(0-1)=(4)(-1)=-4$, so the point $(0,-4)$ should lie on the parabola. The point $(0,-8)$ does not lie on the parabola.
- Explain to students that their next challenge is to determine an equation for Graph B. To begin, have students make observations about the y-intercepts of the graphs. You might ask:
  - What is the y-intercept of Graph B?

    It is $(0,-8)$.
  - How is it different from the y-intercept of Graph A?

    It is 4 units further down than $(0,-4)$. It is twice as far away from the origin as $(0,-4)$.
- Suggest that students work together in small groups to brainstorm how they could determine the equation. After a few minutes of brainstorming, invite groups to offer their suggestions.

Guiding Student Thinking

Students are likely to suggest subtracting a constant from the function f to determine the function for Graph B. Doing so will change the y-intercept, but it will also change the x-intercepts. It is important to help students understand that multiplying f by a constant does not change the x-intercepts, but will change the y-intercept, and all the other non–x-intercept points. This means that you can make the coefficient a any value you want (as long as it isn't zero) to scale the parabola to the correct shape.

- As time allows, you can test out the variety of approaches suggested by students. However, be sure to test out the approach of subtracting a constant. Use questions such as the following:
  - Suppose we wanted to translate the y-intercept down 4 units. What operation do you think we could do that would move the parabola down?

    Sample response: We could subtract 4 from the function f.
  - If we subtracted 4 from the output of the function f, will that move only the y-intercept? Does the new graph have the same x-intercepts as the old graph? Check it with your graphing utility.

    It moves the entire parabola down four units. The new graph has different x-intercepts.
  - What can you conclude about subtracting from the output of the function f? Will subtracting (or adding) only adjust the y-intercept and leave the x-intercepts alone?

    Subtracting changes all the outputs and moves the entire parabola, not only the y-intercept.
- Next, test out the multiplication approach with the class, using questions such as the following:
  - Suppose we wanted the y-intercept to be twice as far away. What operation could we try to make something twice as much?

    Sample response: We could multiply by 2.
  - Try to multiply the factored form by 2. What happens to the graph of the function? Does the y-intercept change? Do the x-intercepts change?

    The new graph is a stretched out version of the old graph. The y-intercept did change to $(0,-8)$ but the x-intercepts stayed the same.

- What is the new function rule? Let's call it g.

  The new function is $g(x)=2f(x)=2(x+4)(x-1)$.

- Why do you think the x-intercepts stayed the same?

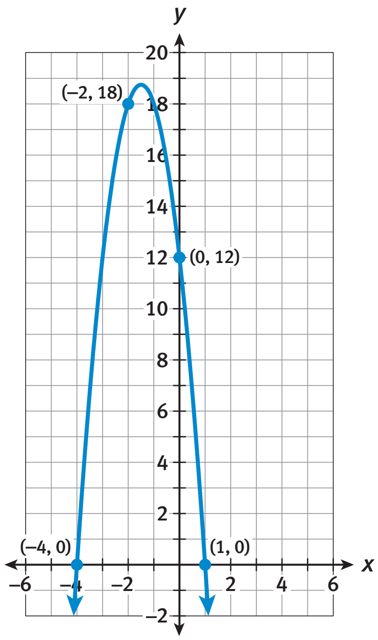
  The x-intercepts stay the same because when you substitute $x=-4$ or $x=1$ into the function g, you still get zero. So the points $(-4,0)$ and $(1,0)$ lie on the parabola.

Encourage students to try different values instead of 2 and observe how the graph of the function changes with different scale factors.

- Have students record this definition for factored form:

  A quadratic function written in factored form is $f(x)=a(x-r)(x-s)$, *where* a *is not zero and* r *and* s *are real numbers.*

- Display for the class Graph C, shown here. Point out that this is another graph that has the same x-intercepts, but a different y-intercept.



Graph C

Challenge students to determine the factored form of the function for this graph. Encourage students to make predictions about the value of a before performing any computations. This is a good way to have students utilize their number sense. Students

UNIT 3

might predict that the a-value has to be negative since this parabola opens downward. They also might predict that the a-value is "greater" than 2 since the parabola in Graph C looks skinnier than the one in Graph B. Have students use a graphing utility to test their conjectures.

- Next, have students work through a method to determine the a-value. You can help students develop a method for determining the a-value by showing them how to strategically use the factored form and a third point. The third point does not have to be the y-intercept. Shown below are examples that each use a different point.

Classroom Ideas

If you give students other points on the parabola, you could assign different students different (x, y) pairs to use to determine the a-value. This will help students see that any third point, along with the x-intercepts, can be used to find the factored form of a quadratic function.

Verbal Explanation	Symbolic Notation	
	With the y-intercept $(0, 12)$	With the point $(-2, 18)$
Start with the factored form of the function.	$f(x) = a(x+4)(x-1)$	$f(x) = a(x+4)(x-1)$
Substitute the x- and y-values into the function.	$f(0) = a(0+4)(0-1) = 12$	$f(-2) = a(-2+4)(-2-1) = 18$
Simplify as much as possible.	$a(4)(-1) = 12$ $-4a = 12$	$a(2)(-3) = 18$ $-6a = 18$
Solve for a.	$-4a = 12$ $a = -3$	$-6a = 18$ $a = -3$
Write the new function.	$g(x) = -3(x+4)(x-1)$	$g(x) = -3(x+4)(x-1)$

- Finally, have students check with their graphing utility that the function $g(x) = -3(x+4)(x-1)$ matches Graph C.

PART 3: THE SCALE FACTOR AND MATCHING GRAPHS AND FUNCTIONS

In this part of the lesson, students complete **Handout 3.5.B: Matching Graphs, Functions, and Scale Factors**. This is different from the matching activity that began the lesson, because students have to figure out the a-value to match the graphs correctly. Several of the graphs have the same x-intercepts, so students need to use a third point to determine the correct function rule.

- Assign partners and have them work on matching the graphs and function rules. Some of the function rules do not have graph matches, and some of the graphs do not have algebraic matches. For graphs without a function rule match, have students write the function rule. For functions without a graph match, have students identify the x- and y-intercepts and then draw a graph.
- After they complete the handout, encourage students to reflect on their thinking.
  - What was your general approach to matching the graphs with the algebraic function rules? Did you have to change your methods based on the difficulty of the functions?

    I looked for the x-intercepts first.
  - When you had two graphs with the same x-intercepts, how did you decide which algebraic rule matched which graph?

    I checked a third point on the graph to figure out the correct a-value.

UNIT 3

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

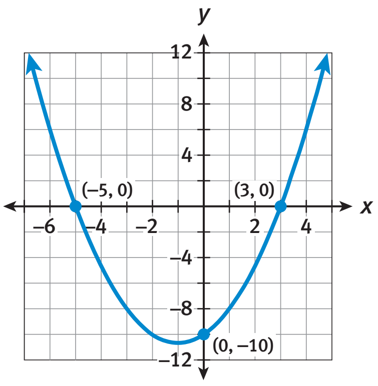
When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

1. We know that two points determine a line. How many points determine a parabola?

   You need three points to determine a parabola.

2. **Version A (text-only):** Suppose that a parabola has x-intercepts $(3, 0)$ and $(-5, 0)$ and the y-intercept is $(0, -10)$. What is the factored form of the quadratic function that passes through these points?

   Version B (with graph): What is the factored form of the quadratic function that will produce the parabola shown here?



   The factored form of the quadratic function is $f(x) = \frac{2}{3}(x-3)(x+5)$.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 3.5.A: Matching Graphs and Functions

F1, G5

F2, G7

F3, G6

F4, G9

F5, G2

F6, G4

F7, G3

F8, G10

F9, G1

F10, G8

Handout 3.5.B: Matching Graphs, Functions, and Scale Factors

G1, no function match. $f(x) = -4(x+2)(x-1)$

G2, no function match. $f(x) = \frac{4}{3}(x-3)(x-4)$

F6, no graph match. x-intercepts $(-3, 0)$ and $(-4, 0)$; y-intercept $(0, -3)$

F9, no graph match. x-intercepts $(-2, 0)$ and $(1, 0)$; y-intercept $(0, -2)$

F1, G4

F2, G3

F3, G7

F4, G6

F5, G10

F7, G8

F8, G9

UNIT 3

PRACTICE PERFORMANCE TASK

The Catapult

OVERVIEW

DESCRIPTION

Students apply their understanding of quadratic patterns to a novel context. Through this performance task they make arguments about the scenario by appealing to the algebraic rule for a quadratic function, its graph, and a representative table of values. It is appropriate to use this practice performance task after Key Concept 3.2.

CONTENT FOCUS

This practice performance task allows students an opportunity to transfer the knowledge they've developed in recent lessons to a new scenario based on a verbal description and data from a graph. Students use their understanding of quadratic relationships to make predictions about changes in the scenario. They also show that a sequence related to the scenario is best modeled by a quadratic function. This task is also a great transition to the next key concept of The Graph of a Quadratic Function.

AREAS OF FOCUS

- Engagement in Mathematical Argumentation
- Connections Among Multiple Representations

SUGGESTED TIMING

~45 minutes

HANDOUT

- Practice Performance Task: The Catapult

MATERIALS

- graphing utility
- copies of scoring guidelines for student use (optional)

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ Quadratic functions have a linear rate of change. ▪ Quadratic functions can be used to model scenarios that involve a linear rate of change and symmetry around a unique minimum or maximum.	
Learning Objectives	**Essential Knowledge**
3.2.1 Identify key characteristics of the graph of a quadratic function.	**3.2.1a** The graph of a quadratic function is a parabola. The parabola is symmetric about a vertical line that passes through the vertex of the parabola.
3.2.2 Translate between algebraic forms of a quadratic function using purposeful algebraic manipulation.	**3.2.2a** Common algebraic forms of a quadratic function include standard form, $f(x)=ax^2+bx+c$; factored form, $f(x)=a(x-r)(x-s)$; and vertex form, $f(x)=a(x-h)^2+k$, where a is not zero.
3.2.3 Describe key features of the graph of a quadratic function in reference to an algebraic form of the quadratic function.	**3.2.3b** The graph of a quadratic function whose factored form is $f(x)=a(x-r)(x-s)$, where $a \neq 0$ and $r \neq s$, has two x-intercepts, at $(r,0)$ and $(s,0)$. **3.2.3c** The graph of a quadratic function whose vertex form is $f(x)=a(x-h)^2+k$ has a vertex at the coordinate (h,k).
3.2.4 Determine an algebraic rule for a quadratic function given a sufficient number of points from the graph.	**3.2.4a** There is a unique parabola that includes any three distinct noncollinear points. A quadratic function whose graph is the parabola that contains these points can be determined using their coordinates. **3.2.4c** The x-intercepts of a quadratic function are two convenient points that, with a third point, can be used to determine an algebraic rule for the quadratic function. **3.2.4d** An algebraic rule for a quadratic function can be determined from the vertex and one other point on the graph of the parabola.

UNIT 3

SUPPORTING STUDENTS

BEFORE THE TASK

In this performance task, students are expected to use their understanding of quadratic patterns to analyze a contextual scenario involving projectile motion. Students do not need to know any physics formulas to engage with the problem; they only need to call on their understanding of quadratic functions.

- Introduce students to the practice performance task. The student handout for the task is included before the scoring guidelines for reference.
- To prepare students to engage in the task, you could begin by posing some questions to students.
    - What is a quadratic function? What characterizes a quadratic function?

      A quadratic function has a linear rate of change; a quadratic function can be expressed as the product of two linear expressions; a quadratic function can be written in the form $f(x) = ax^2 + bx + c$ where a is not 0.
    - What is special about the graph of a quadratic function?

      The graph of a quadratic function is a U-shaped curve called a parabola. Parabolas have a unique minimum or maximum value. Parabolas are symmetric about a line containing its vertex.
    - How many x-intercepts could a parabola have? Explain your answer.

      A parabola could have two x-intercepts if the vertex is below the x-axis and the parabola opens up or if the vertex is above the x-axis and the parabola opens down. A parabola could have one x-intercept if the vertex is on the x-axis. A parabola could have no x-intercepts if the vertex is below the x-axis and the parabola opens down or if the vertex is above the x-axis and the parabola opens up.
    - How are the x-intercepts of the parabola and the factored form of the quadratic function related to each other (if there are x-intercepts)?

      If the parabola has two x-intercepts, p and q, then the factored form of the quadratic function will be $f(x) = a(x - p)(x - q)$, where a is not zero. If the parabola has one x-intercept, r, then the factored form of the quadratic function will be $f(x) = a(x - r)^2$, where a is not zero.

DURING THE TASK

The goal of this task is for students to demonstrate their understanding about quadratic patterns in context. The type of problem in the task may be unfamiliar to students if they have not done many problems involving real-world scenarios.

- Students could work individually or in pairs to complete the task. There may be ample work and enough potential discussion areas for up to, but not more than, two students in each group. In larger groups, some students may not have an opportunity to legitimately engage in the task.
- By this time in the course, students should have some experience engaging with these types of questions. You could remove some of the previous supports you used, such as dividing the task into parts. Allow students time to engage with the entire task before discussing their solutions.

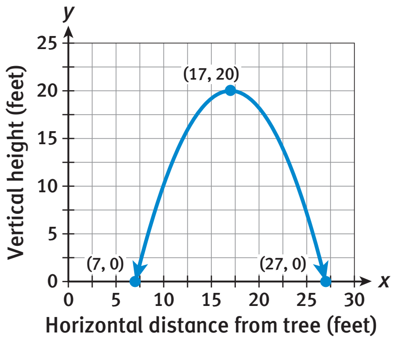
During the task, students might ask about which tools they can use to answer the questions. If it is your classroom norm that students can use calculators or other digital tools to complete assessments, then you should continue to allow students to use their familiar tools. If it is not in your classroom routine for students to use technology, then this would not be an appropriate time to introduce those tools. The implementation of this task should be adapted to your particular classroom and available materials.

AFTER THE TASK

- After students have worked on the practice performance task for 20–30 minutes, you may want to have students assess their own work using the scoring guidelines. By now, students should be more comfortable with performance tasks and allowing them to score their own work helps them take ownership over identifying areas where they may need additional practice.
- You could have students first work together in groups of 2–3 to engage in some peer-to-peer review and revision to their answers. Then you can give each group a copy of the scoring guidelines, or clearly display them for the whole class to use to evaluate and revise their group's suggested answers.
- Students should understand that converting their score into a percent does not provide a good measure of how they performed on the task. You can use the suggested scoring conversion guide located after the scoring guidelines.
- To conclude, you may want to lead a whole-class discussion of the task by inviting students to ask questions, review their solutions, share their struggles, and examine any new insights they formed.

The Catapult

You have been transported to a jungle-scape in the body of a mathematician. You build a catapult to launch messages to your team, who are on the other side of a deep and wide ravine. The graph below shows the trajectory of your message. You launch the message 7 feet east of a tree that is 5 feet west of the edge of the ravine. The ravine is 15 feet wide.



(a) The graph above shows the path of the message as it crosses the ravine. Use the information given to determine a function that defines the path of the message. Explain your answer.

(b) Suppose that you had launched the message 6.75 feet west from the edge of the ravine, instead of 5 feet west from the edge of the ravine. Would the message have made it across the ravine? Show your work and explain your answer.

(c) Your friends on the other side of the ravine don't know where you are standing when you launch the message or how far the message has to travel in order to reach their side. All they can measure is the height of the message at any given point in time since you launched the message. Here are their measurements:

Time (seconds)	Height (feet)
0.25	6
0.50	11
0.75	15
1.00	18

How would you determine the algebraic rule that describes the relationship between the length of time the message is in the air and the height of the message? (Note: It is not necessary to find this algebraic rule.)

SCORING GUIDELINES

There are 9 possible points for this practice performance task.

Part (a)

Sample Solutions	Points Possible
There are two likely approaches to determining the function: the vertex form and the factored form. For the vertex form, the student correctly identifies the vertex as $(17, 20)$ and writes the form without the parameter a: $y = a(x-17)^2 + 20.$ Then the student uses one of the known intercepts (or another point on the parabola) to solve for a: $0 = a(7-17)^2 + 20$ $0 = a(100) + 20$ $a = -\frac{1}{5}$	**3 points maximum** 1 point for writing a correct equation in terms of x, y, and a in any form 1 point for stating the correct value of the parameter a 1 point for using these values to obtain a correct equation in any form

Continues on next page.

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The correct vertex form is: $$y=-\frac{1}{5}(x-17)^2+20.$$ For the factored form, the student correctly identifies the intercepts as $(7,0)$ and $(27,0)$ and writes the form without the a parameter: $$y=a(x-7)(x-27).$$ Then the student uses the vertex to solve for a: $$20=a(17-7)(17-27)$$ $$20=a(-100)$$ $$a=-\frac{1}{5}$$ The correct factored form is: $$y=-\frac{1}{5}(x-7)(x-27).$$	

Targeted Feedback for Student Responses

If many students struggle with part (a), it is likely they are experiencing difficulty coordinating multiple pieces of information from the contextual situation. You can encourage students to begin by writing out the information they know and asking them which form of the quadratic rule will be most efficient.

TEACHER NOTES AND REFLECTIONS

Part (b)

Sample Solutions	Points Possible
A solution that uses the best conceptual understanding of quadratic relationships involves translating the curve left by 1.75 feet. This gives a new left intercept of $(5.25, 0)$. Students should recognize the symmetry of the parabola, and translate the right intercept to the left by 1.75 feet to $(25.25, 0)$. Therefore the distance between the starting and ending location of the message is still 20 feet. The ravine spans 15 feet, which means that the minimum distance the message has to travel is $6.75 + 15 = 21.75$ feet. Because the message will travel 20 feet, which is less than the 21.75 feet necessary to land on the other side of the ravine, it will not make it across the ravine.	**3 points maximum** 1 point for stating both x-intercepts after translating the parabola 1.75 feet to the left 1 point for stating the correct minimum distance the message has to travel 1 point for stating the correct distance that the message actually travels and that it will not make it across the ravine
Targeted Feedback for Student Responses	
If many students struggle with part (b), it is possible they are having a hard time interpreting the new launch point in terms of the situation. Alternatively, they may conclude that the first numerical value they find answers the question, and reason that since 25.25 > 15, the message crosses the ravine. Encourage students to label the location of the ravine on the graph to help them coordinate the distance from the tree with the location of the ravine.	

TEACHER NOTES AND REFLECTIONS

Part (c)

Sample Solutions	Points Possible
For a complete answer, the student should check to see if the second differences of the sequence are constant, to prove that the relationship is quadratic, OR appeal to their knowledge of physics, or both. The student could explain that they would extend the table of values, assert that the height is 0 at time $t = 0$, and find the other time when the height is 0. Then, they would explain that they will use the intercepts and a third point to determine a function in factored form. (See Lesson 3.5.) The student could explain that they would extend the table of values and try to establish the vertex. Once the vertex is found, one more point can be used to determine the vertex form of the quadratic function.	**3 points maximum** 1 point for stating that the algebraic rule must be quadratic with correct reasoning 2 points for stating a clear and valid method for determining the algebraic rule

Targeted Feedback for Student Responses
If many students struggle with part (c), they may be having a hard time thinking abstractly about a process to define an algebraic rule. Encourage them to write out a list of the information they would need to know, and how they would use that information to determine the algebraic rule.

TEACHER NOTES AND REFLECTIONS

Suggested point conversion, if assigning a grade to this problem:

Points Received	Appropriate Letter Grade (if Graded)	How Students Should Interpret Their Score
8 or 9 points	A	"I know all of this algebra really well."
6 to 7 points	B	"I know all of this algebra well, but I made a few mistakes."
4 to 5 points	C	"I know some of this algebra well, but not all of it."
2 to 3 points	D	"I only know a little bit of this algebra."
0 or 1 point	F	"I don't know much of this algebra at all."

UNIT 3

LESSON 3.6

Connecting Standard Form to Vertex Form

OVERVIEW

LESSON DESCRIPTION

Part 1: Warming Up with Vertex Form
Students review two different methods for finding the vertex of a parabola given its equation.

Part 2: Developing Formulas for the Vertex
Students develop formulas for h and k using the parameters a, b, and c from the standard form of a quadratic function.

Part 3: Summary and Practice
Students work in pairs to gain some familiarity with the newly derived formulas for h and k.

AREAS OF FOCUS

- Engagement in Mathematical Argumentation
- Connections Among Multiple Representations

SUGGESTED TIMING

~55 minutes

HANDOUT

Practice

- 3.6: Practice Using Formulas to Find the Vertex

MATERIALS

- graphing utility, one per student
- graph paper

CONTENT FOCUS

By this point in the unit, students should have familiarity and fluency with factoring quadratic functions written in standard form, expanding quadratic functions written in factored form or vertex form, and completing the square to express a standard-form quadratic function in vertex form. While completing the square is a useful technique, it can be cumbersome when the coefficients of a quadratic function are not integers. By comparing vertex form and standard form we can determine the coordinates of the vertex quickly. One major benefit of the formulas for the vertex values derived in the lesson is that they can be used to derive the quadratic formula without completing the square. That will be the focus of Lesson 3.7.

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings
▪ Quadratic functions can be expressed as a product of linear factors. ▪ Every quadratic equation, $ax^2 + bx + c = 0$, where a is not zero, has at most two real solutions. These solutions can be determined using the quadratic formula.

Learning Objectives	Essential Knowledge
3.2.2 Translate between algebraic forms of a quadratic function using purposeful algebraic manipulation.	**3.2.2a** Common algebraic forms of a quadratic function include standard form, $f(x)=ax^2+bx+c$; factored form, $f(x)=a(x-r)(x-s)$; and vertex form, $f(x)=a(x-h)^2+k$, where a is not zero. **3.2.2b** Every quadratic function has a standard form and a vertex form, but not all quadratic functions have a factored form over the real numbers. **3.2.2c** The standard form can be purposefully manipulated into the vertex form of the same quadratic function by completing the square.
3.2.3 Describe key features of the graph of a quadratic function in reference to an algebraic form of the quadratic function.	**3.2.3a** The graph of a quadratic function whose standard form is $f(x)=ax^2+bx+c$ has a vertex with the x-coordinate at $x=-\frac{b}{2a}$. The y-coordinate of the vertex can be calculated by evaluating the function rule using the x-coordinate of the vertex. The graph is symmetric about the vertical line $x=-\frac{b}{2a}$. **3.2.3c** The graph of a quadratic function whose vertex form is $f(x)=a(x-h)^2+k$ has a vertex at the coordinate (h,k).

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

(a) Write the quadratic function $f(x)=3x^2-9x+4$ in vertex form.

(b) Write the quadratic function $g(x)=-\frac{3}{4}(x+8)^2+7$ in standard form.

PART 1: WARMING UP WITH VERTEX FORM

A goal for this part of the lesson is to remind students about two different methods for finding the vertex of a parabola given its equation. One method is to graph the parabola and read the vertex from the graph. This method can be unreliable because the vertex may not be easily read from the graph. Alternatively, completing the square will transform a quadratic function from standard form into vertex form. However, that method can be computationally intense. In the next part of the lesson, students develop a formula to compute the *x*-coordinate of the vertex of a parabola.

- To begin, have students graph the quadratic function $f(x)=2x^2-4x+5$ using a graphing utility and find the vertex of the parabola. Students should identify the vertex as (1, 3).
- Next, have students write the equation in vertex form. Students might recognize from the standard form that the *a* parameter is 2, and then substitute the values for *a, h*, and *k* into the vertex form $f(x)=a(x-h)^2+k$ to arrive at the function $f(x)=2(x-1)^2+3$.
- Have students complete the square to verify algebraically that the function they wrote in the previous step is correct. They should show work like any of the three possibilities shown here.

  Sample responses:

  $$f(x)=2(x^2-2x)+5$$
  $$f(x)=2(x^2-2x+1)+5-2$$
  $$f(x)=2(x-1)^2+3$$

  $$f(x)=2(x^2-2x)+5$$
  $$f(x)=2(x^2-2x+1-1)+5$$
  $$f(x)=2(x^2-2x+1)-2+5$$
  $$f(x)=2(x^2-2x+1)+3$$
  $$f(x)=2(x-1)^2+3$$

  $$\frac{1}{2}f(x)=x^2-2x+\frac{5}{2}$$
  $$\frac{1}{2}f(x)=x^2-2x+1-1+\frac{5}{2}$$
  $$\frac{1}{2}f(x)=(x-1)^2+\frac{3}{2}$$
  $$f(x)=2(x-1)^2+3$$

- Once students have completed the square, pose this question to them: Is there another way to find the equation in vertex form *without graphing or completing the square*? Allow students time to make suggestions and conjectures about whether it is possible.

Instructional Rationale

Completing the square is an extremely useful tool, but it can be hard if the coefficients aren't integers. Having a formula to find the vertex of a parabola can simplify the task of converting a function from standard form to vertex form. In Part 2 of the lesson, students develop some helpful formulas.

PART 2: DEVELOPING FORMULAS FOR THE VERTEX

The goal of this portion of the lesson is to have students develop formulas for h and k using the parameters a, b, and c from the standard form of a quadratic function. The first formula is the familiar $h=-\frac{b}{2a}$. The second formula, $k=c-\frac{b^2}{4a}$, is less conventional and somewhat less easy to use, but it comes from determining the output to the quadratic when the input is $h=-\frac{b}{2a}$. If students already know the $\frac{-b}{2a}$ formula for the x-coordinate of the vertex, this lesson is an opportunity to revisit it and show them that the formula follows logically from the vertex form of a quadratic function. In the next lesson students use the formulas for h and k to develop the quadratic formula.

- First, provide students with these two quadratic functions, and ask them how they could determine whether they are the same function:

  $$f(x)=-3(x-2)^2+5 \qquad g(x)=-3x^2+12x-7$$

  Some students may suggest graphing both functions on the same coordinate plane to compare the parabolas. Other students may suggest rewriting the function rule for f in standard form to compare to the standard form of function g. Help students see that a major advantage of the algebraic approach over the graphical approach is that the algebra would reveal small differences that might be difficult to see in the graphs.

- Have students expand the function rule for f to verify that it is indeed equivalent to function g. Sample work is shown here:

  $$f(x)=-3(x-2)^2+5$$
  $$f(x)=-3(x-2)(x-2)+5$$
  $$f(x)=-3(x^2-4x+4)+5$$
  $$f(x)=-3x^2+12x-12+5$$
  $$f(x)=-3x^2+12x-7$$

Next, you will generalize the process of taking a function rule given in vertex form and writing it in standard form. Have students record the approach taken and give them space to ask clarifying questions along the way. The idea is to expand the vertex form of a quadratic function and then compare it to the standard form to see if there are any alternative methods to completing the square.

Instructional Rationale

There is an unusually high quantity of symbolic manipulation using only letters. Some students may be uncomfortable with the "alphabet soup" of mathematics at play here. The payoff is not in the manipulations themselves but in having students understand that many formulas, especially the quadratic formula, are developed through logical deductions.

- Ask students to explain the steps they took to rewrite the function rule for f in standard form. Then walk students through the same manipulations for the vertex form of a quadratic equation with its parameters. Showing the work side-by-side, as done here, can help students better understand the manipulations to the vertex form.

$f(x)=-3(x-2)^2+5$	$v(x)=a(x-h)^2+k$
$f(x)=-3(x-2)(x-2)+5$	$v(x)=a(x-h)(x-h)+k$
$f(x)=-3\left(x^2-4x+4\right)+5$	$v(x)=a\left(x^2-2xh+h^2\right)+k$
$f(x)=-3x^2+12x-12+5$	$v(x)=ax^2-2ahx+ah^2+k$
$f(x)=-3x^2+12x-7$	$v(x)=ax^2+(-2ah)x+\left(ah^2+k\right)$

- Once the vertex form function has been expanded, it is written in standard form. It looks strange, and students may need help seeing the structure. Display for students one function rule directly above the other to compare the parameters. You can draw arrows to show how the parameters match.

$$\begin{array}{l} v(x)=\underbrace{a}\,x^2+\underbrace{(-2ah)}\,x+\underbrace{\left(ah^2+k\right)} \\ \qquad\quad\ \downarrow \qquad\quad\ \downarrow \qquad\qquad \downarrow \\ v(x)=ax^2+\quad bx \quad+\qquad c \end{array}$$

- Ask students what they notice about the two forms of the function. Guide students to make the following three important observations:
  - The parameter a is the same for both functions. This is important because the value of a is always the same.
  - The b-value in standard form is in the same place as $-2ah$ from the expansion of the vertex form; they are the coefficients of x. Setting the two coefficients equal reveals this formula for h:

$$b=-2ah$$
$$\frac{b}{-2}=ah$$
$$h=\frac{-b}{2a}$$

- The constants c and $ah^2 + k$ can be set equal to each other. This may require some explaining for students. There is no x-variable with either the c or the $ah^2 + k$, so they must be constants. Solving the equation produces the formula for k:

$$k = c - ah^2$$
$$k = c - a\left(\frac{-b}{2a}\right)^2$$
$$k = c - a\left(\frac{-b}{2a}\right)\left(\frac{-b}{2a}\right)$$
$$k = c - a\left(\frac{b^2}{4a^2}\right)$$
$$k = c - \frac{b^2}{4a}$$

- An alternative method to determine a formula for k is to calculate $f(h)$ or $f\left(\frac{-b}{2a}\right)$ using the standard form $f(x) = ax^2 + bx + c$. That approach is valid and will yield the same result.

- Have students identify the vertex of the function $f(x) = -3(x-2)^2 + 5$ as $(2, 5)$. Then have them try out their new formulas for h and k using the function $g(x) = -3x^2 + 12x - 7$ from earlier in the class:

$$h = \frac{-b}{2a}$$
$$h = \frac{-12}{2(-3)}$$
$$h = \frac{-12}{-6}$$
$$h = 2$$

$$k = c - \frac{b^2}{4a}$$
$$k = -7 - \frac{12^2}{4(-3)}$$
$$k = -7 - \frac{144}{-12}$$
$$k = -7 + 12$$
$$k = 5$$

PART 3: SUMMARY AND PRACTICE

Have students work in pairs to complete **Handout 3.6: Practice Using Formulas to Find the Vertex**. In this practice assignment, students gain some familiarity with the newly derived formulas for h and k. Both students work with the same function. Student A finds the vertex using the formulas for h and k from this lesson, while Student B finds the vertex by graphing. After students have determined that their vertices match, they switch roles and repeat the process using a different quadratic equation. To conclude this assignment, student pairs apply the formulas to a word problem.

UNIT 3

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

(a) Write the quadratic function $f(x)=3x^2-9x+4$ in vertex form.

$f(x)=3(x-1.5)^2-2.75$

(b) Write the quadratic function $g(x)=-\frac{3}{4}(x+8)^2+7$ in standard form.

$g(x)=-\frac{3}{4}x^2-12x-41$

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 3.6: Practice Using Formulas to Find the Vertex

1. The vertex for $y=-3x^2+6x+2$ is $(1, 5)$.
2. The vertex for $y=x^2-2x+6$ is $(1, 5)$.

Word Problem: The maximum height of the pass is 16 feet. Students can solve either graphically or algebraically. Using the graph of the function, the vertex is located at $(30, 16)$. Using the formula for the k-value of the vertex will give $15-\dfrac{\left(\frac{1}{15}\right)^2}{\frac{4}{900}}=16$.

LESSON 3.7

The Quadratic Formula

OVERVIEW

LESSON DESCRIPTION

Part 1: Warming Up with the Vertex Formulas
Students begin the lesson using the formulas for h and k that they developed in the previous lesson.

Part 2: Deriving the Quadratic Formula
Students use the formulas they developed for h and k to derive the quadratic formula.

Part 3: Summary and Practice
Students use the quadratic formula to solve quadratic equations.

AREA OF FOCUS

- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~75 minutes

HANDOUT

Practice

- 3.7: Practice with the Quadratic Formula

MATERIALS

- graphing utility, one per student

CONTENT FOCUS

Typically, students derive the quadratic formula by completing the square on a quadratic equation written in standard form. The derivation in this lesson reaches the same result with less computational complexity. The two most important conclusions for students to draw from this lesson are:

- The quadratic formula can be used to solve any quadratic equation.
- The quadratic formula is a logical result of solving a quadratic equation that is written in vertex form.

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ Every quadratic equation, $ax^2 + bx + c = 0$, where a is not zero, has at most two real solutions. These solutions can be determined using the quadratic formula.	
Learning Objectives	**Essential Knowledge**
3.2.2 Translate between algebraic forms of a quadratic function using purposeful algebraic manipulation.	**3.2.2c** The standard form can be purposefully manipulated into the vertex form of the same quadratic function by completing the square.

UNIT 3

3.3.5 Solve quadratic equations using the quadratic formula.	**3.3.5a** Any quadratic equation can be written in the form $mx^2 + nx + p = q$, which can be purposefully manipulated into the standard form of a quadratic equation, $ax^2 + bx + c = 0$. **3.3.5b** The quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, can be used to solve any quadratic equation of the form $ax^2 + bx + c = 0$, but the solutions may not be real numbers. **3.3.5c** The quadratic formula can be derived by completing the square on the standard form of a quadratic equation. **3.3.5d** The quadratic formula can be written as the sum of two terms, $x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$, which shows that the x-intercepts are each a horizontal distance of $\frac{\sqrt{b^2 - 4ac}}{2a}$ from the x-coordinate of the vertex.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

Solve the following quadratic equations.

(a) $-\frac{2}{3}(x + 0.2)^2 + 5.8 = 0$

(b) $0.25x^2 + 0.65x - 1.67 = 0$

PART 1: WARMING UP WITH THE VERTEX FORMULAS

The beginning of the lesson reminds students about the utility of the two formulas that they developed in the previous lesson: $h = -\frac{b}{2a}$ and $k = c - \frac{b^2}{4a}$.

- To start, show students the following quadratic function written in standard form, with noninteger coefficients: $f(x) = 2.4x^2 - 7.2x + 1.3$. Have students work individually to find the vertex of the graph, $(1.5, -4.1)$, and then write the function rule in vertex form, $f(x) = 2.4(x - 1.5)^2 - 4.1$.
- Invite students to share their answers and talk about which methods they used. Guide the conversation to highlight the efficiency of using the formulas for h and k in this activity. For example, students who attempted to use the graph would have found it impossible to do so, and completing the square would have been computationally challenging. Point out that using the formulas for h and k makes finding the vertex very straightforward.
- Next, instruct students to verify graphically that the two versions of $f(x)$, the standard form and the vertex form, will produce the same graph. Then, have students use a graphing utility to find the x-intercepts of the graph: approximately $(0.193, 0)$ and $(2.807, 0)$.
- Hold a class vote in response to the following question: "Are the x-intercepts you found with the graphing utility estimates or exact values?" Invite students to discuss their positions and challenge the opposing view. Guide the discussion to help students recognize that these are only numerical approximations of the x-intercepts. Then ask:
  - How could we find the exact x-intercepts? Which form of the equation would be easier to use?

    We can solve the equation $f(x) = 0$ in vertex form. Vertex form would be easier to use than standard form.

Meeting Learners' Needs
If your students are already familiar with solving a quadratic equation from vertex form, use this opportunity to make sure they recall the correct steps to take.

- Now have students set the vertex form of the function equal to zero and solve the equation. When students have finished, invite them to describe the steps they took to solve the equation. Sample work is shown here.

$$\begin{aligned}
2.4(x-1.5)^2-4.1&=0\\
2.4(x-1.5)^2&=4.1\\
(x-1.5)^2&=\frac{4.1}{2.4}\\
x-1.5&=\pm\sqrt{\frac{4.1}{2.4}}\\
x&=1.5\pm\sqrt{\frac{4.1}{2.4}}\\
x&\approx 0.193,\ 2.807
\end{aligned}$$

Meeting Learners' Needs

If you think your students will find this example difficult because the a parameter is not equal to 1, try an easier example first such as $f(x)=x^2-7.2x+2.4$. The vertex is $(3.6,-10.56)$ and the vertex form is $f(x)=(x+3.6)^2-10.56$. The x-intercepts are approximately $(0.35,0)$ and $(6.85,0)$. Then have students solve the equation $f(x)=0$.

- Next, instruct students to work through another example, this time solving a quadratic equation without numerical values for *a, h,* and *k*. It may help students make sense of the computations by writing the examples next to each other. When students have finished working, invite a volunteer to share their work with the class.

$2.4(x-1.5)^2-4.1=0$	$a(x-h)^2+k=0$
$2.4(x-1.5)^2=4.1$	$a(x-h)^2=-k$
$(x-1.5)^2=\frac{4.1}{2.4}$	$(x-h)^2=\frac{-k}{a}$
$x-1.5=\pm\sqrt{\frac{4.1}{2.4}}$	$x-h=\pm\sqrt{\frac{-k}{a}}$
$x=1.5\pm\sqrt{\frac{4.1}{2.4}}$	$x=h\pm\sqrt{\frac{-k}{a}}$
$x\approx 0.193,\ 2.807$	

Meeting Learners' Needs

If students are having difficulty working the computation with the coefficients a, h, and k, you can have them first work through a few more specific examples of solving a quadratic equation in vertex form and write out the process in words for solving the equation.

- Explain to students that one reason for solving the equation using a, h, and k instead of numbers is that it is more generalizable. Once a formula is developed, the algebra has been done and all that is left is substituting the values and performing the calculations. It means that every vertex-form quadratic equation can be solved, and they are all solved the same way.

Instructional Rationale

If the value of the number under the square root is negative, then there are no real-number solutions to the equation. The solutions will have imaginary numbers, which are beyond the scope of Algebra 1. However, it is valuable for students to see that every vertex form of a quadratic equation can be solved using the same operations. This insight leads to the next part of the lesson: deriving the quadratic formula.

- The next lesson will delve into this more deeply, but it can be useful to point out to students now that the form of the solution for both a specific case and the general case is the same:

$$x = \begin{pmatrix} x\text{-coordinate} \\ \text{of vertex} \end{pmatrix} \pm \begin{pmatrix} \text{Same amount on} \\ \text{both sides of vertex} \end{pmatrix}$$

PART 2: DERIVING THE QUADRATIC FORMULA

In this part of the lesson, students derive the quadratic formula using the vertex form of a quadratic equation and the formulas for h and k, developed previously. This approach to deriving the quadratic formula is likely to be to be more accessible to students than the conventional approach of completing the square with the standard form of a quadratic equation.

- Begin by asking some guiding questions:
  - Think about the formula $x = h \pm \sqrt{\frac{-k}{a}}$. When is this formula most helpful—when working with standard form or vertex form?

    It is most useful when the quadratic equation is written in vertex form.
  - How could you use the formula $x = h \pm \sqrt{\frac{-k}{a}}$ to solve a quadratic equation that is written in standard form?

    You could transform the equation into vertex form using the formulas for h and k, and then solve the equation in vertex form. That would be a lot of work every single time.

Instructional Rationale

The upcoming portion of the lesson could be computationally challenging, especially for students who find it difficult to perform operations with variable quantities instead of numbers. However, by the end of the lesson, students should understand the utility of the quadratic formula as a method for solving all quadratic equations and that it is a logical result of solving a quadratic equation from vertex form.

- Explain to students that since there are formulas for h and k, to save time, we can do the algebra once by substituting the formulas from Lesson 3.6 into $x = h \pm \sqrt{\frac{-k}{a}}$. This gives us a new formula that will involve a, b, and c. Guide students through this process as shown in the table that follows.

Verbal Explanation	Symbolic Notation
State what we know.	$x = h \pm \sqrt{\frac{-k}{a}} \quad h = \frac{-b}{2a} \quad k = c - \frac{b^2}{4a}$
We can substitute for h and k.	$x = \frac{-b}{2a} \pm \sqrt{\frac{-\left(c - \frac{b^2}{4a}\right)}{a}}$
The most complicated part of the right side is the part under the square root. We can take it out, simplify it, and then replace it later.	$\frac{-\left(c - \frac{b^2}{4a}\right)}{a}$
Distribute the negative sign and rewrite the fraction using multiplication.	$\frac{1}{a}\left(\frac{b^2}{4a} - c\right)$
To add $\frac{b^2}{4a}$ and c, we need a common denominator. The easiest denominator is $4a$. To write a fraction in an equivalent form, multiply the numerator and denominator by the same value—in this case, $4a$.	$\frac{1}{a}\left(\frac{b^2}{4a} - \frac{4ac}{4a}\right)$
We can combine the numerators as much as possible.	$\frac{1}{a}\left(\frac{b^2 - 4ac}{4a}\right)$

Verbal Explanation	Symbolic Notation
We can multiply the two fractions in the square root.	$\dfrac{b^2-4ac}{4a^2}$
Replace $\dfrac{b^2-4ac}{4a^2}$ under the square root.	$x=\dfrac{-b}{2a}\pm\sqrt{\dfrac{b^2-4ac}{4a^2}}$
Simplify the denominator of the fraction inside the square root to be $2a$.	$x=\dfrac{-b}{2a}\pm\dfrac{\sqrt{b^2-4ac}}{2a}$

- Next, lead a whole-class discussion in which students state what they notice about the result. Encourage all responses, but try to lead students toward the following key conclusions:
  - The formula involves only a, b, and c. This means that every quadratic equation in standard form can be solved using this formula.
  - The x-coordinate of the vertex, h or $\dfrac{-b}{2a}$, is in the same position as in the earlier formula. This suggests that the solutions the quadratic formula yields will be graphically horizontally equidistant from the x-value of the vertex. (We will explore more about the symmetry of the parabola in later lessons.)
  - The quadratic formula can also be written in its more familiar form, $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.
- Now is a good time to have students test the quadratic formula with an equation from the beginning of the class, because they already know the solutions. Have students solve $2.4x^2-7.2x+1.3=0$. They may need some guidance identifying the parameters: $a=2.4$, $b=-7.2$, and $c=1.3$. Substituting these values into the formula yields this work:

$$x=\frac{7.2}{2(2.4)}\pm\frac{\sqrt{(-7.2)^2-4(2.4)(1.3)}}{2(2.4)}$$
$$x=1.5\pm\frac{\sqrt{39.36}}{4.8}$$
$$x\approx 0.193,\ 2.807$$

Meeting Learners' Needs

Some students will be inclined to memorize the quadratic formula and forget what it is a formula for. You can have students justify why the formula is of the form "$x=0$." Students might reason that they are determining the x-values that solve the equation $f(x)=0$, where f is a quadratic function.

Instructional Rationale

Students do not need to reduce the square roots to lowest terms at this point. What is more important right now is using the formula to solve a quadratic equation. The quadratic formula is particularly helpful when the coefficients for a quadratic equation are noninteger values, like the example in this lesson.

PART 3: SUMMARY AND PRACTICE

Students should practice using **Handout 3.7: Practice with the Quadratic Formula**. The handout includes several examples in which the coefficients for the quadratic equation are noninteger values.

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

Solve the following quadratic equations:

(a) $-\frac{2}{3}(x+0.2)^2+5.8=0$ $\quad x\approx -3.15,\ 2.75$

(b) $0.25x^2+0.65x-1.67=0$ $\quad x\approx -4.193,\ 1.593$

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 3.7: Practice with the Quadratic Formula

1. $\left\{\frac{1}{3},\ -\frac{13}{5}\right\}$
2. $\approx 7.6, 2.4\ \left\{5\pm\sqrt{7}\right\}$
3. $\approx 53.7, -3.7\ \left\{25\pm\sqrt{825}\right\}$
4. $\{1,13\}$
5. $\approx 0.3, 0.9\ \left\{\frac{14\pm\sqrt{722}}{48}\right\}$
6. $\approx 211.65, 28.35\ \left\{120\pm 20\sqrt{21}\right\}$
7. (a) no real roots

   (b) $x=-3$

   (c) $x=\frac{5\pm\sqrt{17}}{2}$

   (d) $x=\frac{-10\pm\sqrt{139.2}}{-9.8}$

8. (a) $x = \dfrac{-5 \pm \sqrt{26.6}}{-0.8} \approx -0.197, 12.697$

(b) no real roots

(c) $x = \dfrac{2 \pm \sqrt{10}}{2} \approx -0.581, 2.581$

(d) $x = -\dfrac{3}{2}$

(e) $x = \dfrac{5 \pm \sqrt{89}}{16} \approx -0.277, 0.902$

(f) no real roots

LESSON 3.8

The Symmetry of the Parabola

OVERVIEW

LESSON DESCRIPTION

Part 1: Connecting the Intercepts and the Vertex
Students explore the relationship between the x-intercepts of a parabola and the x-coordinate of the vertex.

Part 2: Connecting the Quadratic Formula and the Vertex
Students work through calculations to discover that the quadratic formula reveals the x-coordinate of the vertex and also how horizontally far away the x-intercepts are from the vertex.

Part 3: Summary and Practice
Students complete practice problems that extend what they can do with the knowledge that the vertex is halfway between the intercepts.

AREA OF FOCUS

- Connections Among Multiple Representations

SUGGESTED TIMING

~45 minutes

HANDOUT

Lesson

- 3.8: Practice with Parabolas

CONTENT FOCUS

In this lesson, students explore how the symmetry of the x-intercepts of a parabola is related to, and indicated by, the quadratic formula. The symmetry of the parabola is extremely useful because it often makes calculations with quadratic functions simpler. Symmetry also makes quadratic functions a useful tool for modeling physical phenomena that exhibit symmetry, such as the motion of an object thrown upward under the influence of gravity.

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings
Every quadratic equation, $ax^2 + bx + c = 0$, where a is not zero, has at most two real solutions. These solutions can be determined using the quadratic formula.

Learning Objectives	Essential Knowledge
3.2.3 Describe key features of the graph of a quadratic function in reference to an algebraic form of the quadratic function.	**3.2.3a** The graph of a quadratic function whose standard form is $f(x)=ax^2+bx+c$ has a vertex with the x-coordinate at $x=-\frac{b}{2a}$. The y-coordinate of the vertex can be calculated by evaluating the function rule using the x-coordinate of the vertex. The graph is symmetric about the vertical line $x=-\frac{b}{2a}$.
3.3.1 Describe the relationship between the algebraic and graphical representations of a quadratic equation.	**3.3.1b** If a quadratic equation $ax^2+bx+c=0$ has real solutions $x=r$ and $x=s$, then the parabola defined by $y=ax^2+bx+c$ has x-intercepts at $(r,0)$ and $(s,0)$.
3.4.3 Interpret the vertex and roots of a quadratic model in context.	**3.4.3a** If the values of the vertex are included in the contextual domain and range of the problem, the x-value of the vertex of the parabola represents the input value that corresponds to either the minimum or maximum output value, and the y-value of the vertex of the parabola represents the minimum or maximum output value. **3.4.3b** The x-value(s) of the root(s) of a parabola often represent the extreme values (of the input variable) in a contextual scenario.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

A parabola has x-intercepts at $(5,0)$ and $\left(-\frac{3}{2},0\right)$.

(a) What is the x-coordinate of the vertex of the parabola?

(b) Write a function whose graph could be the parabola.

(c) Explain why the function you wrote for part b is not the only function whose graph is a parabola with x-intercepts $(5,0)$ and $\left(-\frac{3}{2},0\right)$.

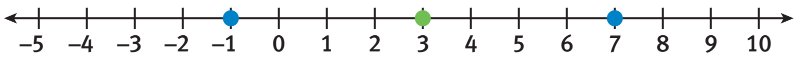
PART 1: CONNECTING THE INTERCEPTS AND THE VERTEX

In the first part of the lesson, students explore the relationship between the x-intercepts of a parabola and the x-coordinate of the vertex. The key observation for students to make is that the x-coordinate of the vertex is located halfway between the x-intercepts of the parabola. To help focus students' attention on this concept, the problem solution uses only the x-axis.

- To begin, have students determine the x-coordinate of the vertex of $f(x) = 2x^2 - 12x - 14$, which is $x = 3$. Then instruct students to factor the function and identify the x-intercepts.

  The factored form is $f(x) = 2(x+1)(x-7)$, and the x-intercepts are $(7, 0)$ and $(-1, 0)$.

- Next, instruct students to plot the x-coordinates of the vertex and of the x-intercepts (-1, 3, and 7) on a number line, as shown here.



  Ask students what they notice about the location of the x-coordinate of the vertex in relation to the x-coordinate of each x-intercept. Guide students to see that 3 is halfway between -1 and 7.

Instructional Rationale

The purpose of using a number line instead of a coordinate plane is to focus students' attention on the horizontal distance between the x-coordinates of the vertex and of the x-intercepts.

- Finally, ask students if they think the x-coordinate of the vertex will always fall halfway between the x-coordinates of the x-intercepts. Encourage students to engage in academic conversation about this topic by supporting their views and/or challenging those of their classmates.

Meeting Learners' Needs

If students are unconvinced about this relationship, you can have them work through several more examples. In Part 2 of this lesson, they will prove algebraically that the x-intercepts are horizontally equidistant from the x-coordinate of the vertex.

UNIT 3

PART 2: CONNECTING THE QUADRATIC FORMULA AND THE VERTEX

In this part of the lesson, students learn that the quadratic formula reveals the x-coordinate of the vertex and also see how horizontally distant the x-intercepts are from the vertex.

- To begin, have students solve the equation $2x^2 - 12x - 14 = 0$ using the quadratic formula. They already know the x-intercepts of the graph of $y = f(x)$ and can use these to check that they have worked the problem correctly. What is important in this investigation is for students to see that the quadratic formula can be interpreted to show both the x-coordinate of the vertex of the parabola and the horizontal distance from the axis of symmetry to the x-intercepts.
- As students work through the quadratic formula, have them stop before evaluating the final step, (3 ± 4):

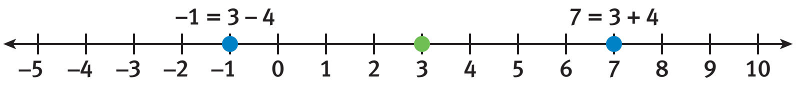
$$x = \frac{12}{4} \pm \frac{\sqrt{144 - 4 \bullet 2 \bullet -14}}{4} = 3 \pm \frac{\sqrt{256}}{4} = 3 \pm 4$$

  Invite students to closely observe and analyze their work at this step by asking:

  - What do you notice? How does this expression relate to the x-axis graph you made in Part 1?

  Allow students time to formulate and share their observations.

- Help students recognize that the 3 represents the x-coordinate of the vertex and that ±4 represents the distance on either side of the vertex to the x-intercepts. In particular, students should understand that their two answers of $x = 7$ and $x = -1$ actually came from $x = 3 - 4$ and $x = 3 + 4$. It can be helpful to write the calculations on the number line so that they make this connection:



- Lastly, show students that the quadratic formula can be written in a way that reveals both the x-coordinate of the vertex and the horizontal distance to the x-intercepts:

  In $x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$, the $\frac{-b}{2a}$ is the x-coordinate of the vertex, and $\pm\frac{\sqrt{b^2 - 4ac}}{2a}$ represents the horizontal distance to the x-intercepts from the vertex.

PART 3: SUMMARY AND PRACTICE

To conclude the lesson, have students complete practice problems that are meant to extend what they can do with the knowledge that the vertex is halfway between the intercepts. See **Handout 3.8: Practice with Parabolas**. Many of the practice problems on this handout provide two pieces of information about a parabola, such as the x-coordinate of the vertex and one of the x-intercepts, and ask students to figure out the third.

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

A parabola has x-intercepts at $(5, 0)$ and $\left(-\frac{3}{2}, 0\right)$.

(a) What is the x-coordinate of the vertex of the parabola?

The x-coordinate of the vertex is 1.75.

(b) Write a function that describes this parabola.

Answers can vary, but the function must take the form $f(x)=a(x-5)\left(x+\frac{3}{2}\right)$ or $f(x)=a(x-1.75)^2+k$.

(c) Explain why the function you wrote for part *b* is not the only function whose graph is a parabola with x-intercepts $(5, 0)$ and $\left(-\frac{3}{2}, 0\right)$.

The scale factor and/or the y-coordinate of the vertex can vary. The x-intercepts determine only the x-coordinate of the vertex of the parabola.

HANDOUT ANSWERS AND GUIDANCE

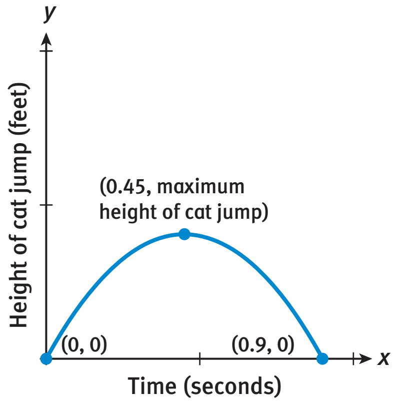
To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 3.8: Practice with Parabolas

1. The x-coordinate of the vertex is -4 (17 units away from either x-intercept).
2. The other x-intercept is $(-3, 0)$.
3. The other x-intercept is $(-23, 0)$.
4. A function in vertex form that matches the graph of question 3 is $f(x)=\frac{6}{196}(x+9)^2-6$.
5. The x-value of the vertex is $-\frac{7}{10}$ and the x-intercepts are approximately 1.578 units to the left and right of the vertex.

6. (a) The cat was moving upward for 0.45 seconds and then downward for 0.45 seconds.

   (b) The graph must look like the one shown, but the height could vary based on choices the student makes about how high the cat can jump.



7. It must be true that $g(13) = g(21)$ because $13 = 17 - 4$ and $21 = 17 + 4$. This means that 13 and 21 are each 4 horizontal units away from the vertex. Any two coordinates horizontally equidistant from the vertex of the parabola must have the same y-coordinate because the parabola is symmetric about the vertical line passing through the minimum or maximum value.

UNIT 3

LESSON 3.9

Interpreting the Discriminant

OVERVIEW

LESSON DESCRIPTION

Part 1: Parabolas Without x-intercepts
Students explore a parabola that does not intersect the x-axis.

Part 2: How Many Solutions?
Students explore the discriminant of a quadratic function and use it to determine how many real solutions a quadratic equation has.

Part 3: Determining How Many Intersections
Students use the discriminant to determine whether two parabolas have any intersection points.

CONTENT FOCUS

This lesson is an exploration of the discriminant of a quadratic function. Students identify connections between a parabola's intercepts and/or intersections and the existence or nonexistence of solutions to the algebraic equation, as indicated by the value of the discriminant.

The discriminant of a quadratic equation can be used to quickly tell whether the equation will have real-number solutions. Solving a quadratic equation that has a non-real solution is beyond the scope of this course and should be reserved for Algebra 2.

AREAS OF FOCUS

- Engagement in Mathematical Argumentation
- Connections Among Multiple Representations

SUGGESTED TIMING

~60 minutes

HANDOUT

Practice

- 3.9: Practice Using the Discriminant

MATERIALS

- graphing utility, one per student (optional)

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ Every quadratic equation, $ax^2+bx+c=0$, where a is not zero, has at most two real solutions. These solutions can be determined using the quadratic formula.	
Learning Objectives	**Essential Knowledge**
3.3.2 Solve quadratic equations by taking a square root.	**3.3.2c** There is no real number that will satisfy the equation $x^2=a$, when a is a negative real number.
3.3.5 Solve quadratic equations by using the quadratic formula.	**3.3.5b** The quadratic formula, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, can be used to solve any quadratic equation of the form $ax^2+bx+c=0$, but the solutions may not be real numbers.
3.3.6 Determine the number of real solutions to a quadratic equation.	**3.3.6b** Given a quadratic equation of the form $ax^2+bx+c=0$, the value of the discriminant of the quadratic equation $\left(D=b^2-4ac\right)$ can be used to determine whether the quadratic equation has two distinct real solutions $(D>0)$, one real solution $(D=0)$, or no real solutions $(D<0)$.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

> Use the discriminant to determine how many times the graphs of $f(x)=0.5x^2-1.4x+6.2$ and $g(x)=0.6x+4.2$ will intersect. Verify your calculation with a graphing utility.

PART 1: PARABOLAS WITHOUT *x*-INTERCEPTS

In the beginning part of this lesson, students explore a parabola that does not intersect the x-axis and identify the discriminant of the corresponding quadratic function.

- Begin by asking: What are the x-intercepts of the graph of the function $f(x)=x^2+1$? Provide students with some time to work on this problem. If appropriate, they can use a graphing utility. Students should conclude that the graph does not intersect the x-axis and therefore has no x-intercepts.

Guiding Student Thinking

Watch for students who mistake the y-intercept $(0, 1)$ for an x-intercept or incorrectly state that $x=1$. Students may be confused because the parabola does not intersect the x-axis. You can ask students to identify which axis, the vertical or horizontal one, is the x-axis.

- Once students have finished working the problem, ask them: Does the equation $x^2+1=0$ have any real-number solutions? Why or why not? Challenge students to justify why this equation does not have any real-number solutions; encourage them to engage in academic conversation as they support or refute each other's mathematical arguments. If students need help, here are some guiding questions to use:
    - What happens when you square a positive number? A negative number? Zero?

      The product will always be nonnegative. In fact, it will be positive anytime the original number is not zero.
    - What happens if you add 1 to a nonnegative number?

      The sum is positive. It will always be greater than or equal to 1.
    - If you square a number x and add 1 to it, what can you say about the resulting value?

      The number should be positive—that is, greater than or equal to 1.
    - What kinds of values will be the outputs of $f(x)=x^2+1$?

      The outputs will always be 1 or greater.
    - Will there be a value for x that will make $x^2+1=0$?

      No, since $x^2+1\geq 1$ and therefore is never equal to 0.

Meeting Learners' Needs

You may need to review or discuss with students what "nonnegative" means and why it is not synonymous with "positive."

- What does this mean about the x-intercepts of the function $f(x)=x^2+1$?

  There should be none, since there are no solutions to the equation $x^2+1=0$.

Guiding Student Thinking

Even though it could be daunting for students to make sense of the inequality symbols, encourage them to become familiar with using them. As they use them more, they will get comfortable with manipulating and reasoning with the symbols. Show students that because $x^2 \geq 0$, and adding a positive real number to both sides of an inequality does not change the inequality relationship, $x^2+1 \geq 0+1$ must also be a true statement.

- Once students are convinced that there are no x-intercepts for $f(x)=x^2+1$, ask them to solve the equation $x^2+1=0$ using the quadratic formula. Some sample work is shown here.

$$x=\frac{-0}{2}\pm\frac{\sqrt{0^2-4\bullet 1\bullet 1}}{2}=0\pm\frac{\sqrt{-4}}{2}$$

  Some students will recognize that $x=0$ is the x-coordinate of the vertex. This observation was the focus of the previous lesson.

Meeting Learners' Needs

Some students might be confused because there is no "bx" term here. If students are struggling, it could help to have them write the equation as $1x^2+0x+1=0$ to identify the values of a, b, and c.

- Direct students' attention to $\sqrt{-4}$ and guide them to see that it cannot be a real number. To help students toward this realization, you can ask questions such as the following:
  - What does it mean that $\sqrt{9}=3$? What does the square root of a number mean?

    The square root of 9 is 3, which means that 3^2 is equal to 9. In general if $\sqrt{a}=z$ then $z^2=a$.
  - If $\sqrt{-4}=z$, then what is another way of writing that, without using the square root symbol?

    Another way to write it is $z^2=-4$.
  - What is the square of 2? What is the square of -2?

    $(2)^2=4$ and $(-2)^2=4$. Neither one is equal to -4.
  - In other words, what real number can you square to give you -4?

    It is not possible to find a real number whose square is -4.

UNIT 3

- Finally, turn students' attention back to the quadratic formula calculations. Ask students which part of the calculation led to the "issue" of having a negative number underneath the square root symbol. Students should highlight $b^2 - 4ac$ as the problem area. Tell students that this expression is called the *discriminant*.

Instructional Rationale

It is not necessary to introduce students to complex numbers at this time. However, it is appropriate to let students know that in a later course they will learn another set of numbers that includes the real numbers and is large enough to include solutions to all quadratic, and other polynomial, equations. For now, in the case of a quadratic function whose discriminant is negative, have students state "There are no real-number solutions," rather than "There are no solutions."

PART 2: HOW MANY SOLUTIONS?

In this part of the lesson, students explore in more detail the three different cases of the discriminant of a quadratic function: when it is negative, when it is positive, and when it is zero.

- Have students solve several quadratic equations with the goal of identifying the three different cases of the discriminant. See **Handout 3.9: Practice Using the Discriminant**. As students are working, encourage them to engage in close observation and analysis of how different values of the discriminant lead to different kinds of solutions to the equations.
- Once students have completed the handout, lead a whole-class discussion in which students share what they noticed about the value of the discriminant. If students need more structure for the conversation, use these questions:
  - What happened when the discriminant was equal to zero? Did you get a real-number solution to the equation? If so, how many solutions?

    When the discriminant was equal to zero, there was one real-number solution.
  - What are the different kinds of values the discriminant could have?

    It can be positive, negative, or zero.

Classroom Ideas

This could be a good time for students to create a graphic organizer for the new vocabulary word *discriminant*. Have students define the word, explain the three different cases, and state what the three cases mean with regard to the graph of the parabola.

- How many solutions does the equation have in each case?

  When the discriminant is negative, there are no real-number solutions. When the discriminant is zero, there is one real-number solution. When the discriminant is positive, there are two real-number solutions.

- What would the graph look like in each of the three cases? (How many x-intercepts are there in each case?)

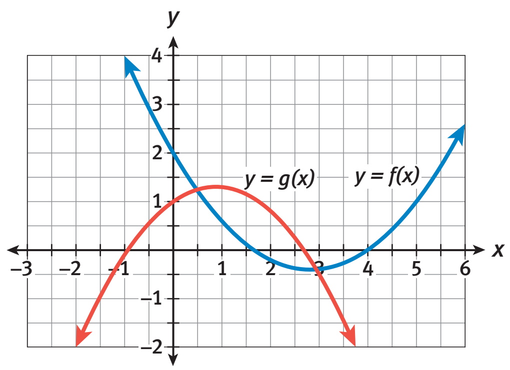
  When the discriminant is negative, there are no x-intercepts and the parabola is completely above or below the x-axis. When the discriminant is zero, there is one real-number solution and the vertex of the parabola is on the x-axis. When the discriminant is positive, there are two real-number solutions and the parabola will intersect the x-axis twice.

PART 3: DETERMINING HOW MANY INTERSECTIONS

In this portion of the lesson, students explore how the discriminant can be useful in verifying whether two parabolas have any intersection points. The principle is the same as solving one quadratic equation. There can be two intersection points, one intersection point, or no intersection points.

- Display the following functions and their graphs:

$$f(x) = 0.3x^2 - 1.7x + 2 \quad \text{and} \quad g(x) = -0.4x^2 + 0.7x + 1$$



  Ask students what they notice about the graphs. They may observe that the parabolas each have two x-intercepts, that they open in different directions, or that they intersect twice.

- Next, ask students whether it is possible to determine the intersection points algebraically, and, if so, how they could do it. Invite students to share their ideas and engage in academic conversation with their classmates to support or challenge each other's suggestions. If possible, encourage students to conclude that the best approach is to solve the equation $f(x)=g(x)$. Ask guiding questions to direct students to this conclusion if needed.

- Have students use the graph to estimate what the coordinates of the intersections are, so when they solve the equation they will know if they are close or if they have made a mistake. Then have students solve the equation $f(x)=g(x)$. They can work on it with a partner if they need more support. Some sample work is provided here:

$$0.3x^2-1.7x+2=-0.4x^2+0.7x+1$$

Simplify so that all nonzero terms are on the same side.

$$0.7x^2-2.4x+1=0$$

Using the quadratic formula, the solutions are:

$$x=\frac{2.4}{1.4}\pm\frac{\sqrt{(-2.4)^2-4\bullet 0.7\bullet 1}}{1.4}\approx 1.714\pm 1.229.$$

Meeting Learners' Needs

For students who struggle to begin, ask them what they would normally do to solve a quadratic equation. This may help them see that getting all terms to one side will be helpful.

Guiding Student Thinking

Some students will anticipate that the solutions to the quadratic equation are the x-intercepts of the functions. Remind students that the equation they originally set out to solve was $f(x)=g(x)$, so these x-values are the solutions to that equation.

- Once students have finished solving the equation, have them share the x-coordinates of the two intersection points (0.485 and 2.943). Ask students:
  - What will be the output of f and g when these x-values are used as the function inputs?

    The outputs will be the same. That is, $f(0.485)=g(0.485)$ and $f(2.943)=g(2.943)$.
  - Are there other inputs we could use where the corresponding output of f will equal the output of g? How do you know?

    These two input values (0.485 and 2.943) are the only ones that will give f and g the same output values. We know because the parabolas for f and g intersect twice, so there will only be two solutions to the equation $f(x)=g(x)$.

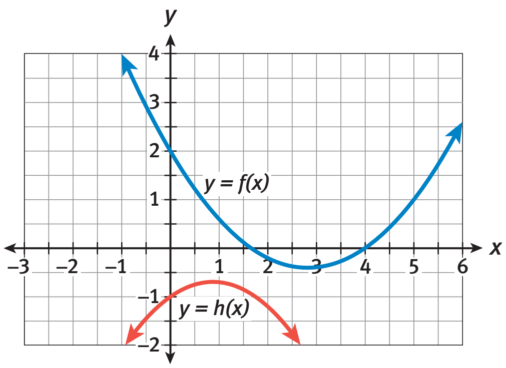
UNIT 3

- Connect this exercise back to the value of the discriminant. Ask students:
  - How can the value of the discriminant be used to determine the number of intersection points?

    We can conclude that the value of the discriminant is positive; this indicates that there are two intersection points.

- Next, show the class two more functions and their graphs.

$$f(x)=0.3x^2-1.7x+2 \quad \text{and} \quad h(x)=-0.4x^2+0.7x-1$$



In this situation, the two parabolas do not intersect. Have students conjecture what will be the value of the discriminant for this situation.

- Ask students what they notice about the graphs. They might suggest that only the graph of f intersects the x-axis, that the parabolas open in different directions, or that the graphs of f and h do not intersect. Ask students what equation they could solve to figure out the intersection, if there were an intersection point. Encourage students to make a prediction about the solution to $f(x)=h(x)$, then have them solve it. Some sample work is shown here:

$$0.3x^2-1.7x+2=-0.4x^2+0.7x-1$$

$$0.7x^2-2.4x+3=0$$

Using the quadratic formula like before:

$$x=\frac{2.4}{1.4}\pm\frac{\sqrt{(-2.4)^2-4\bullet 0.7\bullet 3}}{1.4}=\frac{2.4}{1.4}\pm\frac{\sqrt{-2.64}}{1.4}$$

- Finally, ask students what we can conclude from the calculations. Students should note that the discriminant is -2.64. They should also point out that to solve this equation, we would have to evaluate $\sqrt{-2.64}$, which is not a real number. Therefore, no real solutions to the equation exist, and there are no points where the two parabolas intersect.

UNIT 3

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

Use the discriminant to determine how many times the graphs of $f(x) = 0.5x^2 - 1.4x + 6.2$ and $g(x) = 0.6x + 4.2$ intersect. Verify your calculation with a graphing utility.

There is one point of intersection between the graphs of f and g.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 3.9: Practice Using the Discriminant

1. 13, 2
2. −23, 0
3. 0, 1
4. $-\frac{15}{4}$, 0
5. −20, 0
6. 48, 2
7. 44, 2
8. 0, 1
9. −8, 0
10. $\frac{1}{25}$, 2

LESSON 3.10

Pursuit Problems

OVERVIEW

LESSON DESCRIPTION

Part 1: A Different Kind of System of Equations
Students warm up with a system of equations involving a quadratic function and a linear function.

Part 2: The Cheetah and the Wildebeest
Students engage with a problem involving the pursuit of one animal by another.

Part 3: Extending the Problem
Students experiment with changing the parameters of the problem.

Part 4: Summary and Practice
Students have an opportunity to work on more pursuit problems.

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~60 minutes

HANDOUT

Practice

- 3.10: Pursuit Practice Problems

MATERIALS

- graphing utility

CONTENT FOCUS

The problem in this lesson is inspired by a physics and calculus problem and involves a cheetah pursuing a wildebeest along a straight-line path. However, the mathematics involved is very accessible for students at this level and provides a reason to solve a system of equations with a nonlinear function.

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings
▪ Quadratic functions have a linear rate of change. ▪ Quadratic functions can be used to model scenarios that involve a linear rate of change and symmetry around a unique minimum or maximum. ▪ Every quadratic equation, $ax^2+bx+c=0$, where a is not zero, has at most two real solutions. These solutions can be determined using the quadratic formula.

Learning Objectives	Essential Knowledge
3.3.5 Solve quadratic equations using the quadratic formula.	**3.3.5b** The quadratic formula, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, can be used to solve any quadratic equation of the form $ax^2+bx+c=0$, but the solutions may not be real numbers.
3.4.1 Model a contextual scenario with a quadratic function.	**3.4.1a** A contextual scenario where the output quantity increases and then decreases (or vice versa), such as accelerated motion, can be effectively modeled by a quadratic function.
3.4.2 Interpret solutions to quadratic equations derived from contextual scenarios.	**3.4.2a** A quadratic equation derived from a contextual scenario can be solved free of context, but the solution must be interpreted in context to be correctly understood. **3.4.2b** The solution to a quadratic equation derived from a context should involve the same units as the variables in the contextual scenario.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

For times $t>0$ in seconds, a superhero's height above the ground in meters is given by the function $S(t)=600-14.7t^2$. Directly below the superhero, a person is in free fall. The person's height above the ground is given by the function $P(t)=300-10t-4.9t^2$. At what time are the superhero and the person at the same height above the ground and how high above the ground are they?

PART 1: A DIFFERENT KIND OF SYSTEM OF EQUATIONS

To prepare students for the main problem in this lesson, the first part of the lesson presents an exercise in which one function is linear and the other is quadratic, but no contexts are used.

- First, display these two algebraic function rules for the whole class, and ask students what they notice.

  $f(x) = x^2 - 5x + 6 \qquad g(x) = 3x - 10$

  Expect students to recognize that one function is linear while the other is quadratic. Before they start to graph the functions or do any algebra, ask students if they think the graphs will intersect each other, and if the graphs do intersect how many intersection points will there be. Collect responses from the class, asking students to give a brief justification or rationale for their predictions.

- Then, have students graph the functions, and verify algebraically that there is only one intersection, by solving $f(x) = g(x)$. (The algebra of this equation connects back to Lesson 3.9, where students explored the discriminant.)

  The functions intersect once at $(4, 2)$. The discriminant of the quadratic equation has a value of zero, meaning there is only one solution.

- Before moving to the next part of the lesson, challenge students to change only one number in either f or g so that the graphs do not intersect, and then again so that the graphs intersect twice. (It will be helpful for students to think about how the parameters of the function affect the graph of the function. Remind students that nothing else in the scenario changes.) Answers will vary because students have lots of choices they could make. Look for one of the following responses.

  Changing the Parameters of *f*:

- a-values greater than 1 will yield graphs with no intersections, while a-values less than 1 will have two intersection points.
- b-values greater than -5 will produce no intersections, while b-values less than -5 will show two intersection points.
- c-values greater than 6 have no intersection for the graphs and c-values less than 6 have two intersection points.

Classroom Ideas

One good way to get students engaged in these investigations is by using a graphing utility with a "slider" capability. With a slider, students can quickly cycle through a multitude of values for a parameter in a function.

Changing the Parameters of g:

- Adjusting the y-intercept to be less than -10 will yield no intersections but values greater than -10 will produce two intersection points.
- Changing the value of the slope is particularly interesting. If the slope is -13, f and g will intersect only once. However, if the slope is greater than 3 or less than -13 then the graphs will intersect two times. When the slope is between -13 and 3, there will be no intersections.

PART 2: THE CHEETAH AND THE WILDEBEEST

In this part of the lesson, students connect the noncontextual exercise from the beginning to a scenario in which a cheetah is chasing a wildebeest. In the scenario, the cheetah moves at a constant speed, so its position can be modeled by a linear function. The wildebeest moves at an increasing speed, and its position can be modeled by a quadratic function. Figuring out whether the cheetah catches the wildebeest involves solving a system of equations.

- Instruct students to work in pairs or trios, and present them with the following scenario:

> A cheetah is hunting a wildebeest on the African plains. At time $t = 0$ seconds, the cheetah is 100 meters behind the wildebeest and is running at a constant speed of 30 meters per second. At $t = 0$, the wildebeest notices the cheetah and begins running faster and faster. In fact, the wildebeest's movement is modeled by $W(t) = 2.5t^2$, where $W(t)$ measures the distance from the wildebeest's starting point t seconds after the wildebeest noticed the cheetah. Will the cheetah catch the wildebeest?

- As students work, circulate around the room to monitor their progress. You can prompt any students who struggle to get started with the task with these questions:

  To determine the domain and range of the scenario:

  - What are the input and output quantities in this context?

    Distance traveled is a function of time.

  - What would make sense to use as the units of measurement for the axes?

    Time (in seconds) for the x-axis and distance from the wildebeest's starting point (in meters) for the y-axis.

  - Which quadrants would be used to solve this problem?

    The graphs will appear in the first and fourth quadrants, because nonpositive values for time are not relevant to the context. (However, because the cheetah starts 100 meters behind the wildebeest, the cheetah's starting point is –100 meters.)

To determine the expression that represents the cheetah's movement:

- What do you know about the cheetah's speed? What type of function could represent the cheetah's speed?

  The cheetah's speed is constant at 30 meters per second. Since speed is constant, the distance the cheetah travels can be represented by a linear function.

- What is the slope of this function?

  The slope is 30, because the cheetah is running at a constant speed.

- What is the y-intercept?

  Because the cheetah starts 100 meters away from where the wildebeest starts, the y-intercept is -100.

- What algebraic expression might be a good representative of the cheetah's distance in terms of elapsed time?

  $C(t) = 30t - 100$

To better understand the function that represents the wildebeest's movement:

- What algebraic function rule might be a good representation of the wildebeest's distance as a function of time?

  $W(t) = 2.5t^2$

- Is $(0, 0)$ a point on the graph of W? Does it make sense that $(0, 0)$ is a point on the graph that is represented by this function?

  Yes, because we are measuring the distance of the wildebeest as it runs away from the cheetah. At time $t = 0$, the wildebeest is at position 0.

- Could you make a table of values for this function? What is happening to the wildebeest's distance as time passes? What does that mean about the wildebeest's speed?

  The distance is increasing at an increasing rate. The speed is increasing.

Time (seconds)	Distance from Wildebeest's Starting Point (meters)
0	0
1	2.5
2	10.0
3	22.5
4	40.0

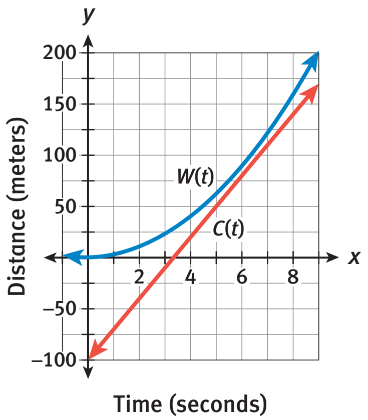
- What type of function does this represent? How do you know?

  It is quadratic, because the values are increasing at an increasing rate (and because the function's variable is squared); for equal changes in time, the second difference is constant.

- What is the shape of the graph that this function represents?

  This graph is a parabola that is opening upward. The minimum value of the parabola is $(0, 0)$.

- Encourage students to graph the two functions. Sketches of the two graphs are shown here.



- Once students have worked through the scenario and graphed the two functions, lead a whole-class discussion about the graphs using the following questions.
  - What key feature of the graph would let you know that the cheetah catches the wildebeest?

    The two functions would intersect.

  - If the cheetah had caught the wildebeest, what would have been true about the equation $C(t) = W(t)$?

    There would be a solution to the equation.

  - Does the cheetah catch the wildebeest in this situation?

    No, there is no point of intersection and the equation $C(t) = W(t)$ has no solutions.

  - How would you show this algebraically?

$$30t - 100 = 2.5t^2$$

$$2.5t^2 - 30t + 100 = 0$$

- How do you know this equation will have no solution? What is the discriminant?

  The discriminant is negative: $(-30)^2 - 4(2.5)(100) = -100$. So, there is no real-number solution.

PART 3: EXTENDING THE PROBLEM

In this part of the lesson students think about how the outcome would change if the cheetah started from a different location, if the cheetah ran more quickly, or if the wildebeest ran more slowly. Three different scenarios are presented below. The third scenario pertains to the wildebeest's speed, which is best understood through physics or calculus. But students can get a sense of how fast the wildebeest is moving by adjusting the parameter of $W(t)$.

Classroom Ideas

One good way to get students engaged in these investigations is by using a graphing utility with a "slider" capability. With a slider, students can quickly cycle through a multitude of values for a parameter in a function.

- Ask students to think about what parts of the problem and what parameters of the equation could be changed. Use the following questions to guide the discussion about the effects of changing values in the scenario:
  - The cheetah started 100 meters behind the wildebeest. If the cheetah had been closer to the wildebeest, could it have caught the wildebeest? What is the farthest away the cheetah could start and still catch the wildebeest?

    Any distance less than 90 meters will work. Encourage students to show this algebraically and by using the graph. Students might also solve an inequality to find this "tipping point."
  - The cheetah's speed was 30 meters per second. If the cheetah was still 100 meters behind the wildebeest, how fast would the cheetah need to be running to catch the wildebeest?

    Any speed faster than 31.62 meters per second would have allowed the cheetah to catch the wildebeest. Encourage students to show this algebraically and by using the graph.
  - The wildebeest's position is given by $W(t) = 2.5t^2$. The parameter 2.5 tells us how fast the wildebeest is moving. If the cheetah runs at 30 meters per second from 100 meters behind the wildebeest, what is the lowest value the wildebeest's parameter can be for the wildebeest to escape?

    The parameter must be greater than 2.25 or the cheetah will catch the wildebeest.

Guiding Student Thinking

Understanding the wildebeest's position function really requires some knowledge of physics or calculus. The "2.5" in $W(t) = 2.5t^2$ means that the wildebeest is accelerating at a constant 5 meters per second squared. It is not necessary for students to get the physics here. It is more important for them to adjust the parameters and determine graphically and algebraically whether the wildebeest gets caught.

PART 4: SUMMARY AND PRACTICE

Students should do a few more pursuit problems. See **Handout 3.10: Pursuit Practice Problems** for additional practice questions.

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

> For times $t > 0$ in seconds, a superhero's height above the ground in meters is given by the function $S(t) = 600 - 14.7t^2$. Directly below the superhero, a person is in free fall. The person's height above the ground is given by the function $P(t) = 300 - 10t - 4.9t^2$. At what time are the superhero and the person at the same height above the ground and how high above the ground are they?
>
> They are approximately 59 meters above the ground at 6.067 seconds.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 3.10: Pursuit Practice Problems

1. Bob will be behind Alice for approximately 5.464 seconds. He will catch up to Alice after they have swum approximately 7.464 meters, and he will beat her to the other side of the pool.
2. The graph shows that the police car does not catch the getaway car. The police car is as close as it gets, 12.5 meters and time $t = 5$ seconds.

UNIT 3

LESSON 3.11

Gravity and Free-Fall Investigations

OVERVIEW

LESSON DESCRIPTION

Part 1: Graphing the Height of a Thrown Object
Students graph the height of a ball thrown straight up in the air.

Part 2: Writing an Equation for the Height of a Thrown Object
Students use what they know about parabolas to write a function for the height of an object in free fall.

Part 3: Summary and Practice
Students have an opportunity to model more scenarios that involve gravity.

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~45 minutes

HANDOUT

Practice

- 3.11: Practice with Gravity and Free Fall

MATERIALS

- graphing utility

CONTENT FOCUS

In this lesson, students explore how quadratic functions can be used to model objects in free fall. They investigate a simple scenario of throwing a ball by using a quadratic function to model the height of the ball in terms of the time that the ball is in the air.

It might not be obvious to students, initially, that a mathematical model for gravity should be quadratic. This model is rooted in the work of Galileo. The quadratic model makes some simplifying assumptions: The most noticeable assumption is that air resistance is negligible. To simplify the situation, the ball in this investigation is thrown directly upward; the ball's motion is therefore an example of straight-line motion. The function h represents the height of the ball as a function of time. It is important that when students look at a parabolic graph of h, they don't mistake the graph for a picture of a parabolic trajectory over the ground. The parabolic graph of h versus t does not represent the actual path of an object.

UNIT 3

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings

- Quadratic functions have a linear rate of change.
- Quadratic functions can be used to model scenarios that involve a linear rate of change and symmetry around a unique minimum or maximum.
- Every quadratic equation, $ax^2+bx+c=0$, where a is not zero, has at most two real solutions. These solutions can be determined using the quadratic formula.

Learning Objectives	Essential Knowledge
3.2.4 Determine an algebraic rule for a quadratic function given a sufficient number of points from the graph.	**3.2.4c** The x-intercepts of a quadratic function are two convenient points that, with a third point, can be used to determine an algebraic rule for the quadratic function. **3.2.4d** An algebraic rule for a quadratic function can be determined from the vertex and one other point on the graph of the parabola.
3.4.1 Model a contextual scenario with a quadratic function.	**3.4.1a** A contextual scenario where the output quantity increases and then decreases (or vice versa), such as accelerated motion, can be effectively modeled by a quadratic function.
3.4.2 Interpret solutions to quadratic equations derived from contextual scenarios.	**3.4.2a** A quadratic equation derived from a contextual scenario can be solved free of context, but the solution must be interpreted in context to be correctly understood. **3.4.2b** The solution to a quadratic equation derived from a context should involve the same units as the variables in the contextual scenario.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

An astronaut on Mars is working to install a radio antenna on top of a 300-meter-high cliff. Suppose that the astronaut drops her wrench off the cliff. The height of the wrench above the ground, in meters, can be modeled by the function $h(t) = 300 - 1.86t^2$, where t is measured in seconds.

(a) What is the height of the wrench above the ground after 1 second?

(b) How long will it take the wrench to hit the ground?

PART 1: GRAPHING THE HEIGHT OF A THROWN OBJECT

The goal of this lesson is for students to develop a model for the height of an object in free fall, using a quadratic function. In this part of the lesson, students should build some intuition about when and why a quadratic function is a good tool to use. They draw graphs that reasonably approximate two thrown-object scenarios. They do not have to write algebraic expressions for the functions in this part; that will be done in Part 2 of the lesson.

- Present students with the following scenario, and have them work in pairs to draw a reasonable graph of the ball's height over time.

> Jenny throws a ball straight upward as high as she can. Her friend Marie times her throw and discovers that the ball was in the air for 3 seconds before it landed on the ground. Jenny is 5 feet tall, and this was the ball's height above the ground at the moment when she released it. Sketch a graph of the ball's height as a function of time.

As students work, circulate around the room to monitor their progress. If students struggle to get started with the task, you can prompt them to identify key information from the scenario with questions such as these:

- What was the height of the ball when it was released?

  The ball leaves Jenny's hand at 0 seconds, and at that time the ball is 5 feet from the ground: $h(0) = 5$.

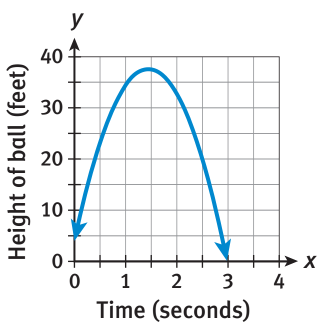
- For how long is the ball in the air?

  For 3 seconds; this means the contextual domain of the function is $0 \le t \le 3$.

- What kind of mathematical shape do we know that could represent an up-and-down motion?

  A parabola.

In response to the scenario, students should generate some kind of parabolic shape. The exact height will be almost impossible to determine at this point. An appropriate graph will look like this:



- Ask students, as a whole class, to estimate the maximum height they think the ball could reach.

  Students should make an appropriate guess about the maximum height of the ball. In this situation, the ball's maximum height will be approximately 40 feet.

Guiding Student Thinking

Students might wonder why a semicircle isn't a better model of the height of the ball with respect to time than a parabola. In this case, students' personal experiences can help them recognize that a parabola has the right shape. Students might say that when you throw a ball in the air, it reaches a maximum height and then comes back down. Near the peak of the throw, the ball seems to "slow down," but then seems to "speed up" on the way down. Explain that what actually happens, in terms of the height, is that the ball gains height at a decreasing rate on its way to the apex and then loses height at an increasing rate on its way to the ground. Quadratic functions are good models for scenarios that show these characteristics in rate of change.

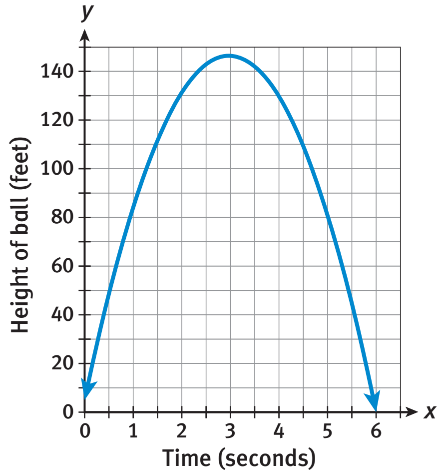
- After the class has come to a consensus on a graph of the ball's height, present them with the second scenario:

> Jenny and Marie decide to use a pitching machine to shoot a ball straight upward and time how long it takes for the ball to come back down. The pitching machine releases the ball at 5 feet in the air, the same height that the ball left Jenny's hand. This time, it takes 6 seconds for the ball to hit the ground. Sketch a graph of the ball's height as a function of time.

Just like in the first scenario, have students think about how high the ball could get.

The maximum height is approximately 150 feet, but students may not guess that high. Students should see that both graphs have the shape of a parabola.

An appropriate graph will look like this:



Guiding Student Thinking

In this scenario, a ball is shot into the air with a pitching machine; it takes twice as long as Jenny's throw to get back to the ground. Students may be tempted to say that because the pitching machine's ball is in the air twice as long as Jenny's thrown ball, its maximum height will be twice that of Jenny's ball. However, the physics of this situation is not linear, so the maximum height of the pitching machine's ball is actually more than three times the height of Jenny's ball. Because neither scenario is well-modeled by linear functions, reasoning with proportions will not be accurate.

PART 2: WRITING AN EQUATION FOR THE HEIGHT OF A THROWN OBJECT

In this part of the lesson, students learn how to determine the algebraic models for the ball problems they graphed in Part 1. Students should reference the scenarios from Part 1 to identify the initial height of the ball and how long it takes for the ball to reach the ground. Using this data and information about the effect of gravity on the ball (which you will probably have to tell them), they can write the function. This part of the lesson requires one formula from physics, but students should not be expected to memorize it. You can provide it to them whenever they need it. Expect students to determine the missing parameters of the function given information from the scenario or graph.

- First, ask the class to identify what features of the graphs from Part 1 of this lesson suggest that a quadratic function would be a good model for the ball's height.
  - The ball goes up and comes back down.
  - There is a maximum height.
  - The graph appears to have symmetry.

- Next, ask students to identify different forms that a quadratic function can take, and to think about what form might be useful for this situation.

  Expect students to list the standard form $\left(f(x)=ax^2+bx+c\right)$, the vertex form $\left(f(x)=a(x-h)^2+k\right)$, and the factored form $\left(f(x)=a(x-r)(x-s)\right)$.

  Guide students to understand the following about using each form.

  - If they knew the vertex and one other piece of information, like the initial height or how long it took for the ball to hit the ground or even how gravity affects the height of the ball, they could use vertex form. But since they can't be sure of the vertex, vertex form will not be the best one to use.
  - If the ball started on the ground and got back to the ground, they would know the two times when the ball had the height of zero, and they would be able to use factored form (also using the effect of gravity). Since the ball does not start on the ground for Jenny or for the pitching machine, factored form will not be very helpful.
  - Standard form may not appear to be the best choice, but the effect of gravity is the a parameter, the initial velocity of the ball is the b parameter, and the initial height of the ball is the c parameter. All of this will lead to the very useful formula:

    $$h(t)=(\text{effect of gravity})t^2+(\text{initial velocity})t+(\text{initial height})$$

    The effect of gravity is constant; on Earth, it is -16 ft/s^2, or -4.9 m/s^2. The constant would be different on another planet. (This is important for the formative assessment question in this lesson: The scenario involves an astronaut on Mars who drops a tool from a cliff. The a parameter in that scenario is -1.86 m/s^2.)

- Once they have determined what form to use, ask students what aspects of throwing the ball should affect the height.

  - How hard Jenny threw the ball or how fast the ball came out of the pitching machine. These are both examples of the initial velocity of the ball.
  - How tall Jenny is or how high off the ground the pitching machine was. These are examples of the initial height.
  - Gravity.

- Let students know that the three factors that affect the height of the ball will correspond to the three parameters in the standard form of a quadratic function. Display (or write) this function for the students:

  $$h(t)=(\text{effect of gravity})t^2+(\text{initial velocity})t+(\text{initial height})$$

Then, inform students that because the function is a little cumbersome to write, and because the effect of gravity is constant, the formula can be abbreviated to

$$h(t) = -16t^2 + v_0 t + h_0,$$

where v_0 represents the initial velocity of the ball and h_0 represents the initial height of the ball.

Guiding Student Thinking

Students may be confused about the meaning of the subscripts in the formula. Explain that mathematicians and scientists often use a subscript of zero to denote the initial value of a parameter.

- Have students identify what information they know from Jenny's throw, and then brainstorm how they might use this information to fill in some values of $h(t) = -16t^2 + v_0 t + h_0$.

  Jenny threw the ball from 5 feet in the air (the initial height), it took the ball 3 seconds to reach the ground (a piece of information), and gravity affects the ball (-16 ft/s^2). They should be able to get as far as $h(t) = -16t^2 + v_0 t + 5$. But to determine the initial velocity, they will need to use the fact that the ball took 3 seconds to get back to the ground.

- Next, ask students to identify the height (in feet) of the ball at 3 seconds and think about how this information could be used to determine the initial velocity.

  Because the ball is on the ground, the height is zero. They can write that information in function notation as $h(3) = 0$. Then, substituting 3 for t in $h(t)$, setting the expression equal to zero, and solving for v_0 will reveal the initial velocity, measured in feet per second (ft/sec):

$$\begin{aligned} -16(3)^2 + v_0(3) + 5 &= 0 \\ -144 + 3v_0 + 5 &= 0 \\ 3v_0 &= 139 \\ v_0 &= \frac{139}{3} \\ v_0 &\approx 46.33 \text{ ft/sec} \end{aligned}$$

- Have students use a graphing utility to graph the function for Jenny's throw and verify that points $(0, 5)$ and $(3, 0)$ lie on the parabola. Then, ask students to determine the maximum height of Jenny's throw.

- Now, have students repeat the process with the pitching machine situation. Some sample work is shown here:

$$\begin{aligned} -16(6)^2+v_0(6)+5&=0 \\ -576+6v_0+5&=0 \\ 6v_0&=571 \\ v_0&=\frac{571}{6} \\ v_0&\approx 95.17 \text{ ft/sec} \end{aligned}$$

- Finally, ask students how the formula $h(t)=-16t^2+v_0t+h_0$ might change if the scenario were different.
  - How would you deal with an object that is simply dropped instead of thrown? What if an object is thrown downward instead of upward? What if the object were thrown from ground level? What if the object were thrown upward from below ground level?

    The value of v_0 would be zero; the value of v_0 would be negative; the value of h_0 would be zero; the value of h_0 would be negative.
- Explain to students that in the last part of the lesson they will work on some practice problems relating to gravity and free fall.

PART 3: SUMMARY AND PRACTICE

Students write equations, graph functions, and answer questions to check their understanding of the key features of the graphs from Parts 1 and 2 of the lesson. See **Handout 3.11: Practice with Gravity and Free Fall** for additional practice questions.

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

An astronaut on Mars is working to install a radio antenna on top of a 300-meter-high cliff. Suppose that the astronaut drops her wrench off the cliff. The height of the wrench above the ground, in meters, can be modeled by the function $h(t) = 300 - 1.86t^2$, where t is measured in seconds.

(a) What is the height of the wrench above the ground after 1 second?

Approximately 298.14 meters.

(b) How long will it take the wrench to hit the ground?

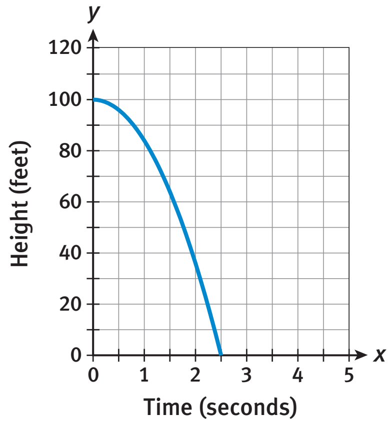
Approximately 12.7 seconds.

HANDOUT ANSWERS AND GUIDANCE

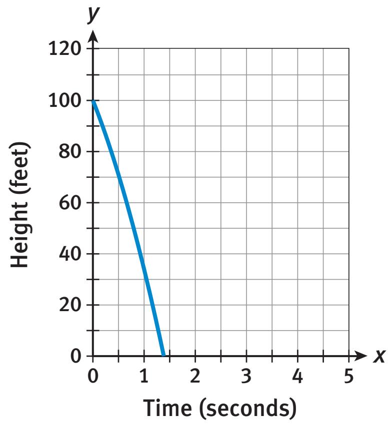
To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 3.11: Practice with Gravity and Free Fall

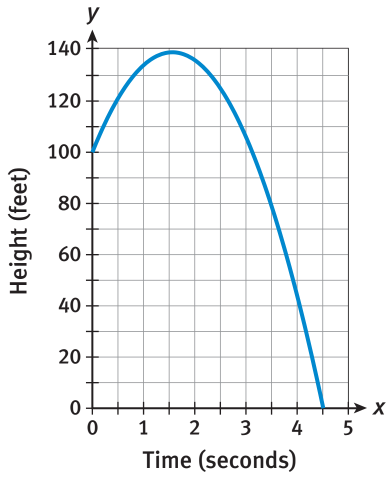
1. Function: $h(t) = -16t^2 + 100$



2. Function: $h(t) = -16t^2 - 50t + 100$



3. Function $h(t) = -16t^2 + 50t + 100$



4. (a) The ball hits the river at $t \approx 4.511$ seconds.

   (b) The ball is at its maximum height at $t \approx 1.563$ seconds.

   (c) The maximum height of the ball is ≈ 139.063 feet.

   (d) The ball will be at a height of 120 feet twice: Once at $t \approx 0.471$ seconds and again at $t \approx 2.654$ seconds.

   (e) The ball will never reach a height of 200 feet.

UNIT 3

LESSON 3.12

The Golden Ratio

OVERVIEW

LESSON DESCRIPTION

Part 1: Art Gallery
Students solve a problem about the proportions of a painting. Students pairs each work with different values.

Part 2: Collecting Data and Connecting Results
Students' solutions to Part 1 are collected and graphed, revealing that the ratio of the length to the width is the same number: the golden ratio.

Part 3: Summary and Practice
Students solve more problems related to the golden ratio.

AREA OF FOCUS

- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~60 minutes

HANDOUT

Practice

- 3.12: Practice with the Golden Ratio

MATERIALS

- graphing utility, one per student

CONTENT FOCUS

In this lesson, students practice solving quadratic equations in mathematically interesting contexts and use properties of the golden ratio to set up and solve an atypical quadratic equation. The properties of the golden ratio, also called the golden proportion, have been studied for thousands of years. It is often said that it is an aesthetically pleasing ratio and often occurs in art and architecture, both intentionally and unintentionally. The value of the golden ratio, which is approximately 1.618, occurs in several interesting calculations, shown here:

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\dots}}}} \qquad \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{\dots}}}}}} \qquad (0, 1, 1, 2, 3, 5, 8, 13, \dots)$$

In the first example, the expansion of the continued fraction can be calculated to be as close to the golden ratio as desired. The same is true for the second example, which shows a nested square root calculation. Finally, the third example shows that it is also the case that the ratio of consecutive terms in the Fibonnaci sequence approaches the golden ratio.

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ Quadratic functions can be used to model scenarios that involve a linear rate of change and symmetry around a unique minimum or maximum. ▪ Quadratic functions can be expressed as a product of linear factors.	
Learning Objectives	**Essential Knowledge**
3.3.5 Solve quadratic equations using the quadratic formula.	**3.3.5b** The quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, can be used to solve any quadratic equation of the form $ax^2 + bx + c = 0$, but the solutions may not be real numbers.
3.4.1 Model a contextual scenario with a quadratic function.	**3.4.1c** A contextual scenario that can be expressed as a product of two linear expressions, such as profit or area, can be effectively modeled by a quadratic function.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

What number is equal to 1 more than its reciprocal?

PART 1: ART GALLERY

In the first part of this lesson, students solve a problem about the proportions of a painting.

- To begin, share the following scenario with the class:

> Amira wants to submit one of her paintings to a local art gallery. The gallery has specific restrictions on the dimensions of paintings. The ratio of the width of the painting to its length must be the same as the ratio of the length of the painting to the total of its length and width. The art gallery claims that this proportion is the most pleasing to the human eye.

- Have students work in pairs, and assign each pair of students a different starting width. You can use a prompt like this one: "Suppose that Amira wants to produce a painting that is 2 feet wide. What would be the required length of the painting?"

Instructional Rationale

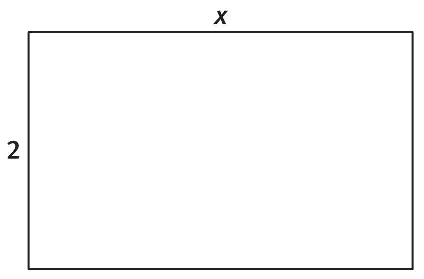
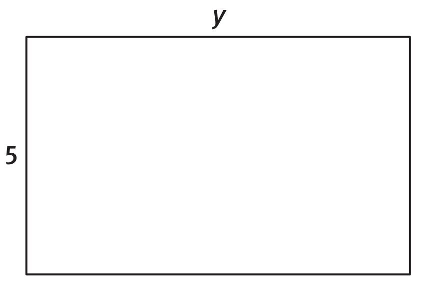
Rather than solving the problem using variables for the length and width of the painting, students will solve using specific lengths and widths. This will make the problem more tangible for students.

Because the ratio of the lengths to widths will be constant, you can assign every pair of students a different value. In the second part of the problem, students will share their lengths and widths and see that the ratio is a constant value.

- As students are working on the problem, circulate around the room to make sure they understand what the problem is asking and how to set it up. Some students may need help drawing the diagram, or setting up the equation.

Sample solutions are shown on the next page for two possible scenarios—paintings with widths 2 and 5.

UNIT 3

Verbal Explanation	Symbolic Notation	
	Painting 1	**Painting 2**
Model the known width and unknown length of the painting.		
Model the desired relationship of width to length. The ratio of width to length is equal to the ratio of length to width plus length.	$\frac{2}{x}=\frac{x}{x+2}$	$\frac{5}{y}=\frac{y}{y+5}$
Using algebra, begin to solve for the unknown length. Identify the result as a quadratic equation.	$2(x+2)=x^2$ $2x+4=x^2$	$5(y+5)=y^2$ $5y+25=y^2$
Solve the quadratic equation.	$x^2-2x-4=0$ $x=\frac{2\pm\sqrt{4-4\bullet 1\bullet -4}}{2}$ $x=\frac{2\pm\sqrt{20}}{2}\approx -1.236, 3.236$	$y^2+5y-25=0$ $y=\frac{5\pm\sqrt{25-4\bullet 1\bullet -25}}{2}$ $y=\frac{5\pm\sqrt{125}}{2}\approx -3.090, 8.090$
Identify the required dimensions of the painting.	2 feet × 3.236 feet	5 feet × 8.090 feet

Guiding Student Thinking

Students may simply leave the solution as, for example, $x = \frac{2 \pm \sqrt{20}}{2}$, but this is incomplete in the context of the problem. The value of $x = \frac{2 - \sqrt{20}}{2}$ is negative and has no meaning for the length of the painting. Ask students to get an approximate value for the length of the painting and explain what it means.

In general students should get the equation $\frac{w}{l} = \frac{l}{w+l}$ where w stands for the width of the painting and l stands for the length of the painting. Multiplying by the denominators will lead to the equation $w^2 + wl = l^2$. Because the width is given as a number, it will be most natural to rewrite the equation as $l^2 - wl - w^2 = 0$, to keep the unknown positive.

Solving for l shows that:

$$l = \frac{w \pm \sqrt{w^2 - 4 \bullet -w^2}}{2}$$

$$l = \frac{w \pm \sqrt{5w^2}}{2}$$

$$l = \frac{w \pm w\sqrt{5}}{2}$$

Because $w < w\sqrt{5}$ for positive values of w, there will always be one positive and one negative solution for the length. The positive value is the only valid one in this context. Another way to write the solution would be $l = w\left(\frac{1+\sqrt{5}}{2}\right)$. The approximate value of $\frac{1+\sqrt{5}}{2}$ is 1.618. Therefore, the length of Amira's painting will always be about 1.618 times as long as the width.

Classroom Ideas

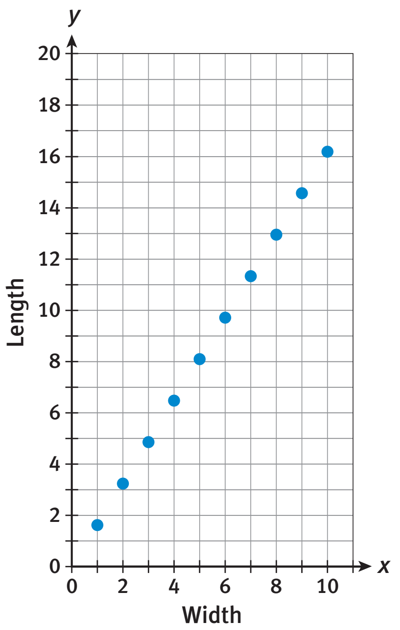
You can make this portion of the class more interactive by displaying the data points for everyone to see in the room and having students plot all the values with their own graphing utility. Students will get a lot of value out of seeing the linear shape reveal itself as more data are added to the graph.

PART 2: COLLECTING DATA AND CONNECTING RESULTS

In this part of the lesson, students' solutions to the Part 1 challenge are collected and graphed. The goal is to reveal that the ratio of length to width is the same number: the golden ratio.

- To begin, gather the various lengths and widths from the class. Use a graphing utility to plot the pairs (width, length) and display the graph for the class to see. Below are a table and graph of a sample data set. Because various widths could have been assigned, the data from your class may be different.

Width	Length
1	1.618034
2	3.236068
3	4.854102
4	6.472136
5	8.09017
6	9.708204
7	11.32624
8	12.94427
9	14.56231
10	16.18034



- Now, give students some time to think about the graph. Ask them what kind of shape the pairs of numbers appear to form. It could be obvious to some students that the points lie on a line, but others may be unconvinced. Ask students: How can we determine whether this shape actually is a line? What is the key characteristic of a linear function? Students should recall that a constant rate of change is the most important feature of a linear function.
- Next, have students calculate the rate of change between two points. Have different pairs of students use different data points so that the constant rate of change (≈ 1.618) will emerge. After the class has agreed that these data lie on a line, have students calculate an equation for the line. Because the relationship between the width and the length is a direct variation, the students should get the equation $y = 1.618x$.

- Ask students what they notice about the equation. Some students might say that the y-intercept is zero, or that the equation is a direct variation. Push students to make a connection to the context of the problem: This equation means that the length and width are in a ratio of 1.618. Another way to say this is that the length will always be 1.618 times the width, regardless of the numerical value of the width.

Guiding Student Thinking

It might be counterintuitive for students that the sides of the rectangle are in the same ratio. However, the criteria for the width and length of the painting were in terms of a ratio. Therefore it should make sense, in retrospect, that the ratio is the same number regardless of the rectangle's dimensions.

- Let students know that this value is called the "golden ratio," and it is usually denoted by the Greek letter φ. Its properties have fascinated people for many centuries. You can share some of the alternative forms listed in the content summary above, if you think students will find them interesting.

PART 3: SUMMARY AND PRACTICE

Students do not have to master or memorize anything specific about the golden ratio. This lesson presents an interesting problem that has a connection to solving quadratic equations. The formative assessment problem and **Handout 3.12: Practice with the Golden Ratio** highlight other accessible contexts where the golden ratio appears. This practice opportunity is also a chance to present students with other unconventionally written quadratic equations.

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

> What number is equal to 1 more than its reciprocal?
>
> The golden ratio is the solution to $g = 1 + \frac{1}{g}$.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 3.12: Practice with the Golden Ratio

1. The proportion for this problem would be:

$$\frac{\text{Younger child's inheritance}}{\text{Older child's inheritance}} = \frac{\text{Older child's inheritance}}{\text{Total inheritance}}.$$

This proportion is the same as the one in the lesson. It is convenient to replace the younger child's inheritance with \$20 million – x, where x represents the older child's inheritance. The older child should receive \$12,360,680 and the younger child should receive \$7,639,320.

2. Fibonacci sequence

(a) $\ldots, 13, 21, 34, \ldots$

(b) The 9th number is 21 and the 10th number is 34. The ratio is $\frac{34}{21} \approx 1.619$.

(c) The 49th number is 4,807,526,976 and the 50th number is 7,778,742,049. The ratio is $\frac{4{,}807{,}526{,}976}{7{,}778{,}742{,}049} \approx 1.618$.

(d) As the Fibonacci sequence continues, the ratio of the $(N+1)$th number to the Nth number will be better approximations of the golden ratio. That is $\frac{F_{N+1}}{F_N} \approx \varphi$.

3. $x = \frac{5 \pm \sqrt{41}}{2}$

4. $x=\frac{5}{2}, 4$. An efficient way to solve this problem is to recognize that $(6x-15)$ is $3(2x-5)$. Substituting A for $(2x-15)$ yields $A^2=3A$, which can be rewritten as $A(A-3)=0$.
5. $x=5$. An efficient way to solve this problem would be to rewrite it as $(x-5)^2=7(x-5)^2$, then notice that a graph of each side of the equation would be a parabola whose vertex is at $(5, 0)$, and identify the vertex as the only intersection point.
6. $x=7\pm\sqrt{59}$

UNIT 3

LESSON 3.13

Finding a Formula for Triangular Numbers

OVERVIEW

LESSON DESCRIPTION

Part 1: The Staircase Revisited
Students revisit the staircase pattern that began the unit.

Part 2: Writing a Formula for the Staircase
Students use what they know about quadratic functions to write a formula for the staircase pattern.

Part 3: Using the Formula
Students use the formula to answer questions about the sequence of triangular numbers.

Part 4: Summary and Practice
Students explore more quadratic sequences and write formulas for them.

CONTENT FOCUS

This lesson returns to the LEGO staircase problem that launched this unit on quadratic functions. At this point in the unit, students have developed many tools for writing a quadratic function when given three points of data. Earlier in the unit, students were able to write the equation of a quadratic function when given the zeros and the vertex. Here, students expand the criteria they can use to write a quadratic function. This lesson differs from students' first explorations of the LEGO staircase in that previously, students did not develop an explicit formula to determine the *n*th triangular number, only a recursive method for extending the quadratic sequence.

AREAS OF FOCUS

- Engagement in Mathematical Argumentation
- Connections Among Multiple Representations

SUGGESTED TIMING

~60 minutes

HANDOUT

Practice

- 3.13: Practice Writing Formulas for Quadratic Sequences

MATERIALS

- graphing utility, one per student

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings

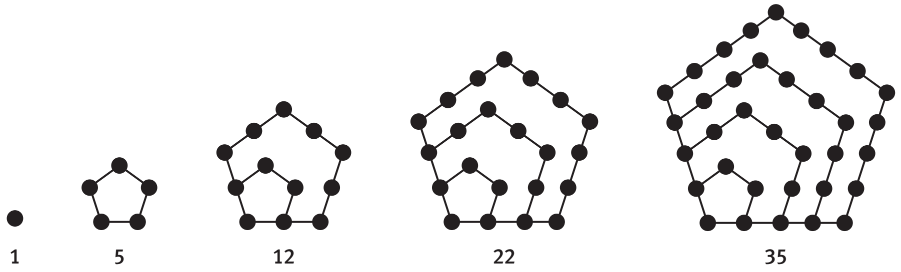
- Quadratic functions have a linear rate of change.
- Quadratic functions can be used to model scenarios that involve a linear rate of change and symmetry around a unique minimum or maximum.

Learning Objectives	Essential Knowledge
3.1.1 Determine whether a relationship is quadratic or nonquadratic based on a numerical sequence whose indices increase by a constant amount.	**3.1.1a** In a table of values that represents a quadratic relationship and that has constant step sizes, the differences in the values of the relationship, called the first differences, exhibit a linear pattern. The second differences of a quadratic sequence are constant. **3.1.1b** Successive terms in a quadratic sequence can be obtained by adding corresponding successive terms of an arithmetic sequence.
3.2.4 Determine an algebraic rule for a quadratic function given a sufficient number of points from the graph.	**3.2.4a** There is a unique parabola that includes any three distinct noncollinear points. A quadratic function whose graph is the parabola that contains these points can be determined using their coordinates.
3.3.5 Solve quadratic equations using the quadratic formula.	**3.3.5b** The quadratic formula, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, can be used to solve any quadratic equation of the form $ax^2+bx+c=0$, but the solutions may not be real numbers.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

Much like the triangular numbers, dots can be arranged in a pentagonal shape:



(a) Do the pentagonal numbers form a quadratic sequence? Explain your answer.

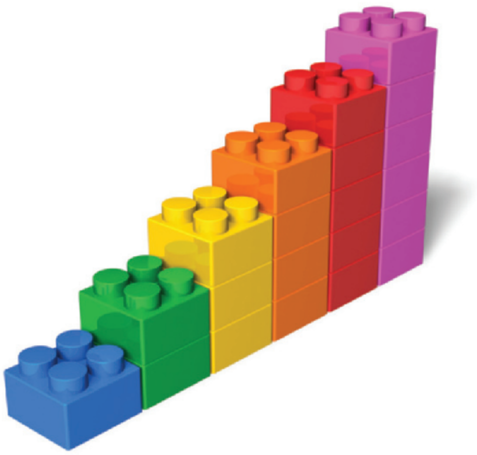
(b) What is the 15th pentagonal number? Show your work.

(c) Is 1,001 a pentagonal number? Why or why not?

PART 1: THE STAIRCASE REVISITED

In the first part of this lesson, students review the staircase pattern they explored at the beginning of the unit and its relevance to triangular numbers.

- Display the image of the staircase for students. Remind the class that they have seen this image before.



Meeting Learners' Needs

It has been some time since students saw the staircase, and they might need a few minutes to reorient themselves to it. It is possible that they will have new insights into the problem now that they have spent so much time thinking about quadratic functions.

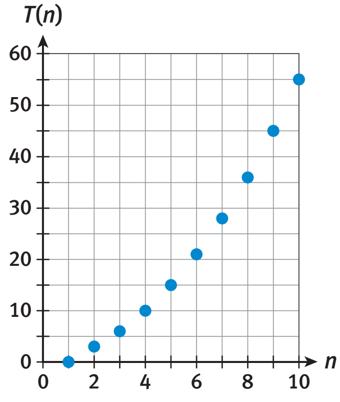
- Have students figure out how many total blocks they would need to build staircases from 1 block tall to 10 blocks tall. Guide students to organize the information in a table like the one shown here, in which n is the staircase height and $T(n)$ is the total number of blocks needed.

n	$T(n)$
1	1
2	3
3	6
4	10
5	15
6	21
7	28
8	36
9	45
10	55

- Ask students:
    - What patterns do you see? Does the table of values represent a linear relationship?

      You add the height of the last "stair" to the previous total blocks. For example, the total number of blocks needed for the 10th staircase is the amount of blocks needed in the 9th staircase plus 10. In general, the pattern is not linear because the rate of change is not constant.
    - Make an argument about whether the sequence of numbers is quadratic.

      Because the second differences are constant (the amount added for each successive staircase increases by 1 each time), this is a quadratic sequence.
- Have students use a graphing utility to plot the points from the table of values, with the staircase height displayed on the horizontal axis and the total blocks needed on the vertical axis. (Using the graphing utility at this step also sets up for later in the lesson.) Students should see the curved shape emerge, as in the graph shown here.



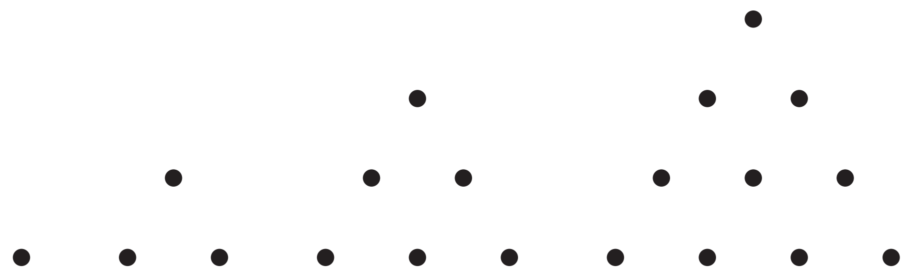
- Remind students that this sequence is called the "triangular numbers." Then ask students why it is called that.

  It is possible to arrange that many dots in an equilateral triangle shape, like bowling pins.

Meeting Learners' Needs

To support academic conversation, have students explain how they can use the bowling pin pattern to visualize that the 3rd triangular number is $1 + 2 + 3$. Students will likely see the pattern growing in different ways. Some students might see the pattern as a row added to the bottom with one more pin each time, while others might see the additional row added to the left or right side.

- Explain to students that another way to describe the triangular numbers is that the nth triangular number is the sum of the first n whole numbers. So, for example, the 3rd triangular number is $1+2+3=6$, as shown in the following pattern.



PART 2: WRITING A FORMULA FOR THE STAIRCASE

In this part of the lesson, students figure out how to use the data from Part 1 to write a quadratic function modeling the triangular numbers.

> **Meeting Learners' Needs**
> You may want to remind students of the gravity problems that they worked on in the previous lesson and identify important similarities and differences. Just like in the gravity problems, they need to determine the value of b. But in this case, they also need to find the value of a, which can be done using a system of equations.

- First, explain to students that we will use a quadratic function to model the triangular numbers. Elicit from students the different forms a quadratic function can have, and have them consider which form might be most useful in the current situation. Guide students to understand the following:
  - The vertex form will not be helpful in this situation because it is not clear from the data where the vertex of the parabola will be.
  - Likewise, the factored form will not be helpful because the x-intercepts are not clear from the data.
  - This means that, as in the free-fall problems in the previous lesson, the standard form of the quadratic will be the best to use. Therefore, the function T will have the form $T(n)=an^2+bn+c$.
- Have students consider what we can determine about a, b, and/or c based on the graph alone. The following are some potential questions you could use:
  - What does the parameter c correspond to in the graph of the function?

    It is the y-intercept. In other words, it is the value of $T(0)$.

UNIT 3

- Do we know the value of $T(0)$? In other words, what happens if you add up all the whole numbers up to 0?

  You will get 0. So, $T(0)=0$.

- What does that information above tell you about the value of c?

  This means that $c=0$.

- Should we expect a to be positive or negative? How can we tell?

  Since the graph opens upward, a must be positive.

Since $c=0$ and a must be positive, the equation is now:

$$T(n)=an^2+bn+0=an^2+bn$$

- Have students use different pairs of points to find the values of a and b. Assign students different pairs of points to work with.
- As students determine an algebraic function rule using their two points, collect both the function rules and the points used from the class. Display the functions where everyone can see them. Each pair of students should end up with the same values for a and b. This shows that any two pieces of data from this table of values will result in the same function rule. The table on the next page shows sample work using the points $(1,1)$ and $(2,3)$. The points $(1,1)$ and $(2,3)$ were chosen because they are the first two pieces of information from the table of values.

Meeting Learners' Needs

The algebra here should be accessible for students. It is a good opportunity to remind them about how to solve a system of equations, especially if they haven't solved one recently. The power of having an algebraic rule for the function T is that now they can use the formula to answer questions about triangular numbers.

Verbal Description	Symbolic Notation
Substitute the data points into the $T(n)=an^2+bn$ equation.	$T(1)=a\times1^2+b(1)=1$ $T(2)=a\times2^2+b\times2(2)=3$
Simplify the equations, which will show a system of equations with variables a and b.	$a+b=1$ $4a+2b=3$
Solve the system using any method. (Substitution is shown to the right.)	$b=1-a$ $4a+2(1-a)=3$ $4a+2-2a=3$ $2a+2=3$ $2a=1$ $a=\frac{1}{2}$
Use $a=\frac{1}{2}$ to find that $b=\frac{1}{2}$. Substitute the values of parameters a and b into the function to determine the function rule for the nth triangular number.	$T(n)=\frac{1}{2}n^2+\frac{1}{2}n$

PART 3: USING THE FORMULA

Now that students have determined the algebraic function rule, they will use it to answer questions about the triangular numbers, including whether 1,000 is a triangular number.

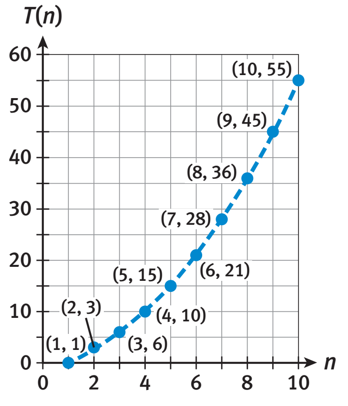
- Point out that the table of values from Part 1 says that the 8th triangular number is 36. Have students verify this by substituting 8 into $T(n)$:

$$T(8)=\frac{1}{2}(8^2)+\frac{1}{2}(8)=32+4=36.$$

- Ask students to determine the 15th triangular number. Students should use the algebraic function rule and not simply extend their original table of values. If you want to make sure they do not extend the table, ask for a triangular number that is much larger, like the 100th one.

$$T(15)=\frac{1}{2}(15^2)+\frac{1}{2}(15)=120.$$

- Next, have students verify graphically that the function they wrote matches the points. Using a graphing utility, students should graph the function T.



- Point out to students that they should use a dashed line to draw the curve. Ask:
  - Why is a dashed line the right way to present the graph?

    The only inputs that make sense in the context of the problem are the whole-number values of n. There is no such thing as the 3.5th triangular number, and drawing the parabola with a solid line implies that there is.

  It is important that students internalize that the function T is defined only for whole-number values of n. They will leverage this thinking in the next part of the lesson when trying to figure out if 1,000 is a triangular number. Here, students must first determine the value of n that results in $T(n)=1{,}000$. Then they must determine if their answer is reasonable given that T is only defined for whole-number inputs.

- Now, challenge students to find out if 1,000 is a triangular number. Students can work in pairs to determine the solution. Sample work is shown on the next page.

Meeting Learners' Needs

If students cannot figure out why a solid line is incorrect, ask whether or not there could be a 2.5th triangular number or a –1st triangular number. These questions should help students understand that the function only works for certain values of n.

Meeting Learners' Needs

Some students will benefit from first solving a problem that does result in a triangular number. In this case, you can have students determine if 378 is a triangular number. (It is the 27th triangular number.)

Verbal Explanation	Symbolic Notation
If 1,000 is a triangular number, there is some whole number n such that $T(n)=1{,}000$.	$\frac{1}{2}n^2+\frac{1}{2}n=1{,}000$
This gives us a quadratic equation to solve.	$\frac{1}{2}n^2+\frac{1}{2}n-1{,}000=0$
Then, use the quadratic formula to solve for n.	$n=-\frac{1}{2}\pm\frac{\sqrt{\left(\frac{1}{2}\right)^2-4\bullet\frac{1}{2}\bullet(-1{,}000)}}{1}=-\frac{1}{2}\pm\frac{\sqrt{2{,}000.25}}{1}$
Use a calculator to do the calculations.	$n=\sqrt{2{,}000.25}\approx 44.724$

Since the value for n is not a whole number, 1,000 is not a triangular number.

Guiding Student Thinking

Make sure students understand that just because a solution to the quadratic equation exists does not mean that 1,000 is a triangular number. The value of n must be a whole number in order for 1,000 to be triangular.

- Some students might be tempted to round the 44.724 up or down and say that 1,000 is the 44th or 45th triangular number. Challenge students to verify that neither the 44th or 45th triangular number is equal to 1,000:

$$T(44)=\frac{1}{2}\left(44^2\right)+\frac{1}{2}(44)=990\neq 1{,}000$$

$$T(45)=\frac{1}{2}\left(45^2\right)+\frac{1}{2}(45)=1035\neq 1{,}000$$

- Now students can write a function for other quadratic sequences such as the ones in the game *Snake* from Lesson 3.1, or in the case of pentagonal numbers shown in the formative assessment problem.

PART 4: SUMMARY AND PRACTICE

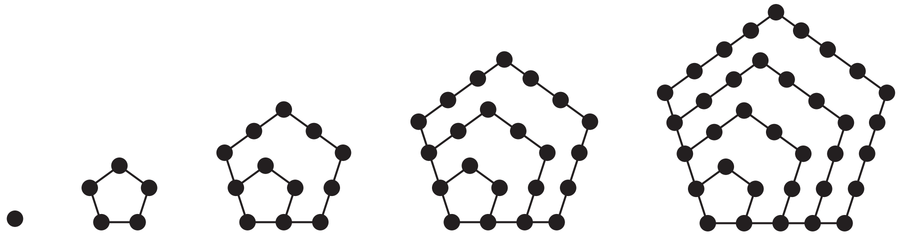
For practice, students should try to write an equation for another quadratic sequence and answer questions about the sequence. See **Handout 3.13: Practice Writing Formulas for Quadratic Sequences**.

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

Much like the triangular numbers, dots can be arranged in a pentagonal shape:



(a) Do the pentagonal numbers form a quadratic sequence? Explain your answer.

Yes, they do form a quadratic sequence because the second differences are constant.

(b) What is the 15th pentagonal number? Show your work.

The 15th pentagonal number is 330.

(c) Is 1,001 a pentagonal number? Why or why not?

Yes, 1,001 is the 26th pentagonal number.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 3.13: Practice Writing Formulas for Quadratic Sequences

1. $t(n) = 2n^2 - 5n - 3$
2. $t(n) = 4n^2 - 10n + 7$
3. $t(n) = 6n^2 - 12n + 6$
4. $t(n) = \frac{3}{2}n^2 - \frac{29}{2}n + 43$
5. $t(n) = \frac{1}{2}n^2 + \frac{1}{2}n$, 78 units

PRACTICE PERFORMANCE TASK

Weaving a Rug

OVERVIEW

DESCRIPTION

Students apply their understanding of modeling with quadratic functions to a novel context. They are provided an opportunity to evaluate their own performance and clarify their understanding. This practice performance task can be used after Key Concept 3.3 or 3.4.

CONTENT FOCUS

This practice performance task focuses on modeling a contextual scenario—weaving a rug—with a quadratic function, and answering problems related to the scenario. This task utilizes students' understanding that areas can often be modeled with a quadratic function, and is similar to the Unit 3 Performance Task.

AREAS OF FOCUS

- Engagement in Mathematical Argumentation
- Connections Among Multiple Representations

SUGGESTED TIMING

~45 minutes

HANDOUT

- Practice Performance Task: Weaving a Rug

MATERIALS

- graphing utility
- copies of scoring guidelines for student use (optional)

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ Quadratic functions can be used to model scenarios that involve a linear rate of change and symmetry around a unique minimum or maximum. ▪ Quadratic functions can be expressed as a product of linear factors. ▪ Every quadratic equation, $ax^2+bx+c=0$, where a is not zero, has at most two real solutions. These solutions can be determined using the quadratic formula.	
Learning Objectives	**Essential Knowledge**
3.2.2 Translate between algebraic forms of a quadratic function using purposeful algebraic manipulation.	**3.2.2a** Common algebraic forms of a quadratic function include standard form, $f(x)=ax^2+bx+c$; factored form, $f(x)=a(x-r)(x-s)$; and vertex form, $f(x)=a(x-h)^2+k$, where a is not zero.

3.3.5 Solve quadratic equations using the quadratic formula.	**3.3.5a** Any quadratic equation can be written in the form $mx^2 + nx + p = q$, which can be purposefully manipulated into the standard form of a quadratic equation, $ax^2 + bx + c = 0$. **3.3.5b** The quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, can be used to solve any quadratic equation of the form $ax^2 + bx + c = 0$, but the solutions may not be real numbers. **3.3.5c** The quadratic formula can be derived by completing the square on the standard form of a quadratic equation. **3.3.5d** The quadratic formula can be written as the sum of two terms, $x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$, which shows that the x-intercepts are each a horizontal distance of $\frac{\sqrt{b^2 - 4ac}}{2a}$ from the x-coordinate of the vertex.
3.4.1 Model a contextual scenario with a quadratic function.	**3.4.1c** A contextual scenario that can be expressed as a product of two linear expressions, such as profit or area, can be effectively modeled by a quadratic function.
3.4.3 Interpret the vertex and roots of a quadratic model in context.	**3.4.3a** If the values of the vertex are included in the contextual domain and range of the problem, the x-value of the vertex of the parabola represents the input value that corresponds to either the minimum or maximum output value, and the y-value of the vertex of the parabola represents the minimum or maximum output value.

SUPPORTING STUDENTS

BEFORE THE TASK

In this practice performance task, students are expected to use their understanding of quadratic functions as models of area problems. To be successful in the task, students need to know that the area of a rectangle is calculated by multiplying the length and width.

- Introduce students to the practice performance task. The student handout for the task is shown before the scoring guidelines for reference.

- To prepare students to engage in the task, you could begin by asking the following questions.
  - What kinds of scenarios can we use quadratic functions to model?

    Quadratic functions are good for modeling scenarios that have linear rates of change, for scenarios that involve area, for scenarios that involve gravity and freefall, and for scenarios that have a unique minimum or maximum and are symmetric about the minimum or maximum.
  - What do we know about the vertex of a parabola?

    The vertex of the parabola is the location of the unique minimum or maximum. We can find it by examining the graph of the parabola or by using the parameters of the algebraic rule for the quadratic function.
  - What are the ways we know to solve a quadratic equation?

    Some quadratic equations can be solved by factoring. All quadratic equations can be solved by completing the square or by using the quadratic formula. Not all quadratic equations have real number solutions.

DURING THE TASK

The goal of this task is for students to demonstrate their understanding about quadratic patterns in context. The type of problem in the task may be unfamiliar to students if they have not done many problems involving real-world scenarios.

- Students should work individually or in pairs to complete the task. The task is designed so students can complete it on their own. However, if they cannot, they may benefit from discussing the task with a partner.
- By this point in the course, students should have some experience engaging with these types of questions. Allow students time to engage with the entire task before discussing their solutions.

During the task, students might ask about which tools they can use to answer the questions. If it is your classroom norm that students can use calculators or other digital tools to complete assessments, then you should continue to allow students to use their familiar tools. If it is not in your classroom routine for students to use technology, then this would not be an appropriate time to introduce those tools. The implementation of this task should be adapted to your particular classroom and available materials.

AFTER THE TASK

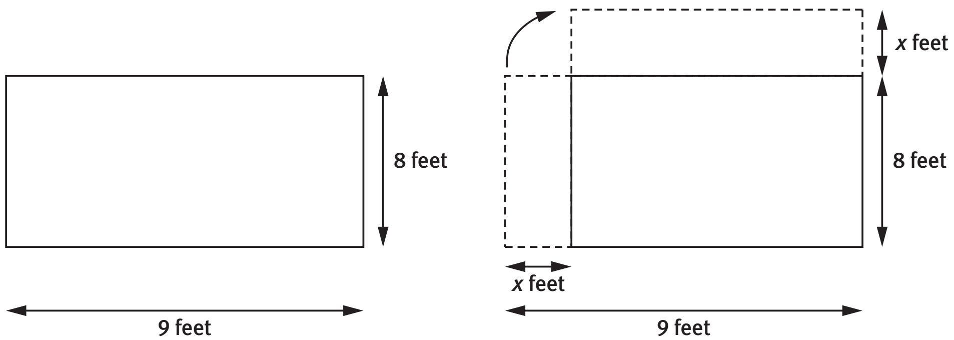
- After students have worked on the practice performance task for 20–30 minutes, you may want to have students assess their own work using the scoring guidelines. By now, students should be more comfortable with performance tasks and allowing them to score their own work helps them take ownership over identifying areas where they may need additional practice.

- You could have students first work together in groups of 2–3 to engage in some peer-to-peer review and revision to their answers. Then give each group a copy of the scoring guidelines, or clearly display them for the whole class to use to evaluate and revise their group's suggested answers.
- Students should understand that converting their score into a percent does not provide a good measure of how they performed on the task. You can use the suggested scoring conversion guide located after the scoring guidelines.
- To conclude, you may want to lead a whole-class discussion of the task by inviting students to ask questions, review their solutions, share their struggles, and examine any new insights they formed.

Weaving a Rug

A rug weaver wants to create a large rectangular rug. He has 34 linear feet of a material that he plans to use on the entire outside border of the rug. He wants to use all this material. So, the rug must have a perimeter of 34 feet.

The weaver already has a design for a rectangular rug that is 9 feet long and 8 feet wide. This design would produce a rug with the correct perimeter. However, he wants to be sure the rug will cover as much of the floor as possible, so he starts to experiment with changing the length and width of the rug design. To maintain the 34-foot border of the rug, he follows this rule: Whenever he decreases one dimension by x feet, he increases the other dimension by the same amount, x feet, as shown below.



Note: Figures are not drawn to scale.

(a) The area of the rug the weaver will make can be expressed as a function of x, the amount of material removed from the 9-foot side and added to the 8-foot side. Write an equation for this function and explain your work.

(b) Is it possible for the rug to have an area of 71.25 square feet? If it is possible, give approximate dimensions of the rectangle and explain your work. If it is not possible, explain why not.

(c) The weaver made a chart of possible areas and determined that the largest possible area is 72 square feet using the original 8- by 9-foot design with no changes. Determine if he is correct and explain why. If he is not correct, provide the dimensions of the rug with the largest possible area and explain your work.

UNIT 3

SCORING GUIDELINES

There are 9 possible points for this practice performance task.

Part (a)

Sample Solutions	Points Possible
Let A be a function defined by $A(x)=(9-x)(8+x)$ where $A(x)$ is the area of the rectangle, in square feet, whose length and width are $(9-x)$ feet and $(8+x)$ feet. The area of a rectangle is found by multiplying the length and the width of the rectangle. The length and width of the rectangle are calculated by subtracting an amount, x, from the original 9-foot side and adding that same amount to the original 8-foot side.	**3 points maximum** 1 point for a correct area equation 1 point for correct use of function notation 1 point for correct explanation

Targeted Feedback for Student Responses
If students have difficulty answering part (a), it could be because they are not sure how to incorporate the amount that is removed from the length and added to the width. In order to address this, students may benefit from additional numerical examples and making an organized table so they can see the structure of the quadratic function emerge.

TEACHER NOTES AND REFLECTIONS

Part (b)

Sample Solutions	Points Possible
Three possible approaches are given below. **Graphical approach:** Examine the graph of $y = A(x)$ and whether it intersects the graph of $y = 71.25$. The graph of $y = A(x)$ is a parabola, and $y = 71.25$ is a horizontal line. Because the line does intersect the parabola, there are x-values (cuts from the 9-foot side that are added to the 8-foot side) that will result in an area of 71.25 square feet. The dimensions are 7.5 ft × 9.5 ft. **Algebraic approach:** Set up the equation $A(x) = 71.25$ and attempt to solve. The discriminant will indicate that there are two rational solutions so it is possible to have an area of 71.25 square feet. $(9 - x)(8 + x) = 71.25$ $72 + x - x^2 = 71.25$ $-x^2 + x + 0.75 = 0$ $x = \frac{-1 \pm \sqrt{1^2 - 4(-1)(0.75)}}{2(-1)}$ $x = \frac{-1 \pm \sqrt{1+3}}{-2}$ $x = \frac{-1 \pm \sqrt{4}}{-2}$ $x = \frac{-1 \pm 2}{-2} = -0.5, 1.5$ Therefore, when $x = 1.5$ the dimensions are $9 - 1.5 = 7.5$ ft and $8 + 1.5 = 9.5$ ft.	**3 points maximum** **Graphical approach:** 1 point for showing or explaining that the parabola intersects line $y = 71.25$ at $x = 1.5$ 1 point for concluding that one dimension is 7.5 ft 1 point for concluding that the other dimension is 9.5 ft **Algebraic approach:** 1 point for showing that the solution to $(9 - x)(8 + x) = 71.25$ is $x = 1.5$ using the method shown or another correct algebraic method 1 point for concluding that one dimension is 7.5 ft 1 point for concluding that the other dimension is 9.5 ft

Continues on next page.

Numeric approach: Students might attempt to guess and check values or find the correct value using a calculator. Since the x-value of 1.5 is not too difficult to find, a calculator or a table of values would be a valid and appropriate technique for this problem.	**Numeric approach:** 1 point for showing that the solution to $(9-x)(8+x)=71.25$ is $x=1.5$ by providing a table or explaining their calculator use 1 point for concluding that one dimension is 7.5 ft 1 point for concluding that the other dimension is 9.5 ft *Scoring note:* If students have the wrong equation for $A(x)$, then they should be scored on their analysis of whether or not the function they wrote could be equal to 71.25.

Targeted Feedback for Student Responses

If students do not have sufficient practice with solving quadratic equations, they may try to use the parameters of $A(x)=72+x-x^2$ in the quadratic formula. If so, feedback should highlight that they would need to set $A(x)$ equal to 71.25 and then strategically manipulate the equation to the form $-x^2+x+0.75=0$ before solving.

TEACHER NOTES AND REFLECTIONS

Part (c)

Sample Solutions	Points Possible
The weaver is not correct. Students may use different approaches such as the two shown here. **Graphical approach:** The vertex of the parabola shows the maximum area of the rug and the changes in the side lengths that will result in it. The maximum area is 72.25 square feet, which occurs when the dimensions are 8.5 ft × 8.5 ft. **Algebraic/Numeric approach:** The maximum value of the function will occur at the vertex of the parabola, which is halfway between the x-intercepts. The x-intercepts are $x = 9$ and $x = -8$, which means the x-value of the vertex is at $x = 0.5$. Substitution shows that $A(0.5) = 72.25$, which means that the maximum area of the rug is 72.25 square feet when the dimensions of the rug are 8.5 ft × 8.5 ft.	**3 points maximum** 1 point for showing that the maximum area will occur when $x = 0.5$ ft 1 point for showing that the maximum area is 72.25 square feet 1 point for concluding that the dimensions for the maximum area will be 8.5 ft × 8.5 ft *Scoring note:* If students have the wrong equation for $A(x)$, then they should be scored on their analysis of whether or not the largest area as indicated by their function is 72 square feet.

Targeted Feedback for Student Responses
If students agree with the weaver, it may be because they have only considered integer values for the length and width of the rug. They may not recognize that the maximum (or minimum) value of a quadratic function occurs at the vertex of the associated parabola. To address this, try giving students some example problems where the vertex does not occur at an integer value.

TEACHER NOTES AND REFLECTIONS

Suggested point conversion, if assigning a grade to this problem:

Points Received	Appropriate Letter Grade (if Graded)	How Students Should Interpret Their Score
8 or 9 points	A	"I know all of this algebra really well."
6 to 7 points	B	"I know all of this algebra well, but I made a few mistakes."
4 to 5 points	C	"I know some of this algebra well, but not all of it."
2 to 3 points	D	"I only know a little bit of this algebra."
0 or 1 point	F	"I don't know much of this algebra at all."

Unit 3

Performance Task

PERFORMANCE TASK

The Path of a Football

OVERVIEW

DESCRIPTION

Students apply their understanding of quadratic functions to a novel context. This task requires students to analyze a physical scenario that is modeled by a parabola and its corresponding quadratic function to answer questions about the scenario. It is appropriate to use this performance task after Key Concept 3.3 or 3.4.

CONTENT FOCUS

In this performance task, students generate a function to model the parabolic path of a football, and construct a graph of the parabola to answer some questions about the scenario. This task expects students to use their understanding of the algebra and geometry of quadratic functions to construct mathematical arguments.

AREAS OF FOCUS

- Engagement in Mathematical Argumentation
- Greater Authenticity of Applications and Modeling
- Connections Among Multiple Representations

SUGGESTED TIMING

~45 minutes

HANDOUT

- Unit 3 Performance Task: The Path of a Football

MATERIALS

- graphing utility

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings
▪ Quadratic functions can be used to model scenarios that involve a linear rate of change and symmetry around a unique minimum or maximum. ▪ Quadratic functions can be expressed as a product of linear factors. ▪ Every quadratic equation, $ax^2+bx+c=0$, where a is not zero, has at most two real solutions. These solutions can be determined using the quadratic formula.

Learning Objectives	Essential Knowledge
3.4.1 Model a contextual scenario with a quadratic function.	**3.4.1b** A contextual scenario whose physical manifestation resembles a parabola, such as a satellite dish or solar collector, can be effectively modeled by a quadratic function.
3.4.2 Interpret solutions to quadratic equations derived from contextual scenarios.	**3.4.2a** A quadratic equation derived from a contextual scenario can be solved free of context, but the solution must be interpreted in context to be correctly understood. **3.4.2b** The solution to a quadratic equation derived from a context should involve the same units as the variables in the contextual scenario.
3.4.3 Interpret the vertex and roots of a quadratic model in context.	**3.4.3a** If the values of the vertex are included in the contextual domain and range of the problem, the x-value of the vertex of the parabola represents the input value that corresponds to either the minimum or maximum output value, and the y-value of the vertex of the parabola represents the minimum or maximum output value. **3.4.3b** The x-value(s) of the root(s) of a parabola often represent the extreme values (of the input variable) in a contextual scenario.

SCORING GUIDELINES

There are 9 possible points for this performance task.

Part (a)

Sample Solutions	Points Possible
The most efficient way to express the quadratic relationship is in vertex form. The parabola is concave down, so a must be negative, and the vertex (h, k) is $(20, 16)$, representing the height of the football halfway through its arc. We also know $(0, 6)$ represents the height of the ball when it has traveled 0 feet down the field (i.e., when it was thrown). So, $6 = a(0-20)^2 + 16$.	**3 points maximum** 1 point for identifying the correct vertex 1 point for getting the correct value for a and the correct equation 1 point for clear and correct explanations of the calculations
Solving for a shows that $a = -\frac{1}{40}$. The algebraic rule of the quadratic relationship is $y = -\frac{1}{40}(x-20)^2 + 16$. Students should use at least some of these reasons in their explanation: ▪ Parabola is concave down; therefore, a has a negative sign. ▪ The vertex is the point halfway between the quarterback and receiver and is the maximum of the curve. ▪ Rationale for selecting one of the two known points (not the vertex) by which to solve for y.	

Targeted Feedback for Student Responses
If students have difficulty with part (a), they may not understand how to translate the description of the situation into mathematical claims about associated pairs of values and the vertex. You can encourage students to begin by sketching a graph that models the situation, labeling the graph, and then using the pairs of values labeled on the graph to build the algebraic rule.

UNIT 3

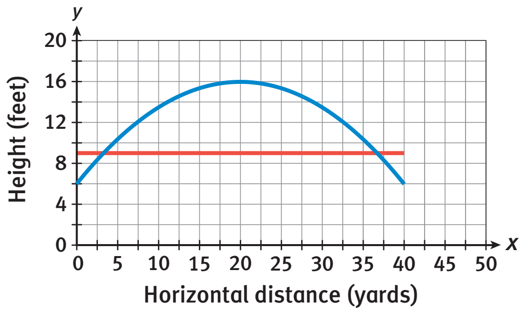
TEACHER NOTES AND REFLECTIONS

Part (b)

Sample Solutions	Points Possible
Algebraic solution: Substituting 35 for the value of the distance the defender is from the quarterback and solving for the height of the ball shows that the height of the ball is $10\frac{3}{8}$ or 10.375 feet. Therefore, the defender cannot prevent the catch. **Graphical solution:** Using a graphing utility, the student can show how the graphs of $x = 35$ and $y = -\frac{1}{40}(x-20)^2 + 16$ intersect at the point $\left(35, 10\frac{3}{8}\right)$. Therefore, the defender cannot prevent the catch.	**3 points maximum** 1 point for substituting 35 for x in their equation from part (a) or graphing their equation from part (a) and $x = 35$ 1 point for getting a correct solution for y either algebraically or graphically 1 point for explaining whether the defender can or cannot prevent the catch from their solution for y *Scoring note:* The graph of their equation for part (a) must yield a parabola that opens down and whose vertex is in the first quadrant in order to be eligible for these part (b) points.
Targeted Feedback for Student Responses	
Be on the lookout for students who might get a correct answer to part (b) by evaluating the function rule for $x = 5$. This does not accurately convey that the defender is 5 feet from the receiver as opposed to 5 feet from the quarterback.	

TEACHER NOTES AND REFLECTIONS

Part (c)

Sample Solutions	Points Possible
In order to determine the maximum distance, students should explain that they have to set up and solve the equation $9 = -\frac{1}{40}(x-20)^2 + 16$ for the value of x. Since x is the horizontal distance between the quarterback and the receiver, students should explain that subtracting the solution for x from 40 (the position of the receiver) will give the maximum distance between the defender and receiver. Students could also describe their reasoning graphically by stating that the maximum distance between the defender and the receiver could be found by determining the intersection of the parabola and the line $y = 9$. Then students should explain that the maximum distance between the defender and receiver is the difference between 40 and x.  (The solution was not required, but the maximum distance that the defensive player can be behind the receiver and prevent the catch is about 3.267 ft from either the receiver or 3.267 ft from the quarterback.)	**3 points maximum** 1 point for describing that the y-value must be equal to 9 (graphically or algebraically) 1 point for indicating that the equation should be solved for x, or that the x-value of the intersection point should be found 1 point for explaining that the maximum distance between the defender and the receiver is the difference between the solution for x and 40, the position of the receiver *Scoring note:* The graph of the equation for part (a) must yield a parabola that opens down and whose vertex is in the first quadrant in order to be eligible for these part (c) points.

Targeted Feedback for Student Responses

If students have difficulty with part (c), they may not understand that the question is asking them to determine which input corresponds to an output of 9 feet. Students then need to interpret that input in terms of the distance from the receiver, as opposed to the input value which is a distance from the quarterback.

TEACHER NOTES AND REFLECTIONS

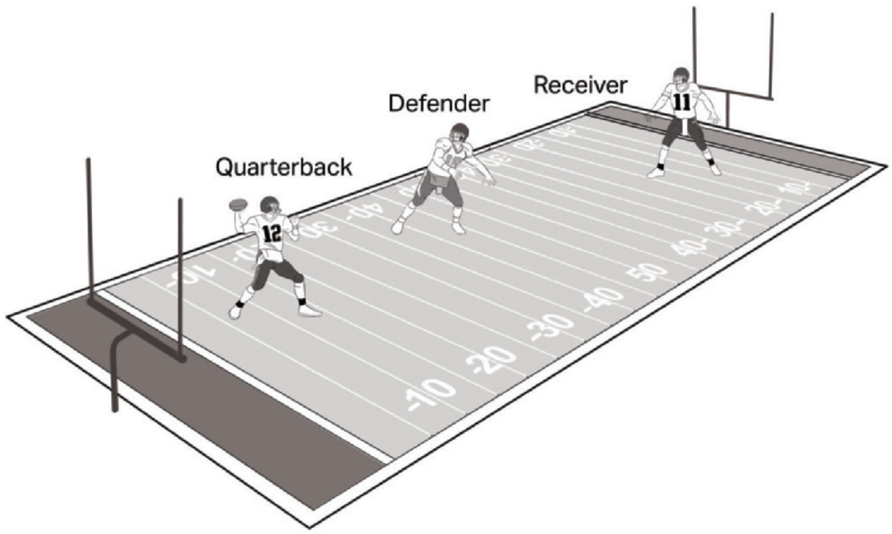
Suggested point conversion, if assigning a grade to this problem:

Points Received	Appropriate Letter Grade (if Graded)	How Students Should Interpret Their Score
8 or 9 points	A	"I know all of this algebra really well."
6 to 7 points	B	"I know all of this algebra well, but I made a few mistakes."
4 to 5 points	C	"I know some of this algebra well, but not all of it."
2 to 3 points	D	"I only know a little bit of this algebra."
0 or 1 point	F	"I don't know much of this algebra at all."

The Path of a Football

PERFORMANCE TASK

Quarterback Quick Release can throw a football in a perfect parabolic arc to a receiver 40 yards downfield.



A football field with players (not drawn to scale)

(a) Assume that Quick Release lets go of the ball 6 feet above the ground and the receiver catches it 6 feet above the ground. The ball reaches a maximum height of 16 feet above the ground halfway to the receiver. Write an algebraic rule that models the path of the football. Show all your work and explain your reasoning.

(b) If a defender guarding the receiver is 5 yards in front of the receiver (closer to the quarterback) and has a vertical stretch of 9 feet, will the defender be able to prevent the catch? Explain your answer.

(c) Describe how to determine the maximum distance the defensive player can be in front of the receiver and prevent the catch.

Unit 4

Unit 4

Exponent Properties and Exponential Functions

Overview

SUGGESTED TIMING: APPROXIMATELY 5 WEEKS

Students explore exponent rules as an extension of geometric sequences and the properties of multiplication and division for real numbers. Students should make sense of exponent rules and not simply memorize them without understanding how they arise. The unit culminates in students investigating how exponential functions can model physical phenomena that exhibit a constant multiplicative growth. Exponential functions are framed as multiplicative analogues of linear functions. Thus, a tight connection should be drawn between these two classes of functions and their shared properties.

ENDURING UNDERSTANDINGS

This unit focuses on the following enduring understandings:

- Properties of exponents are derived from the properties of multiplication and division.
- An exponential function has constant multiplicative growth or decay.
- Exponential functions can be used to model contextual scenarios that involve constant multiplicative growth or decay.
- Graphs and tables can be used to estimate the solution to an equation that involves exponential expressions.

KEY CONCEPTS

This unit focuses on the following key concepts:

- 4.1: Exponent Rules and Properties
- 4.2: Roots of Real Numbers
- 4.3: Sequences with Multiplicative Patterns
- 4.4: Exponential Growth and Decay

UNIT 4

UNIT RESOURCES

The tables below outline the resources provided by Pre-AP for this unit.

Lessons for Key Concept 4.1: Exponent Rules and Properties
There are no provided Pre-AP Lessons for this key concept. As with all key concepts, this key concept is addressed in a learning checkpoint.
All Key Concept 4.1 learning objectives and essential knowledge statements should be addressed with teacher-developed materials.

Practice Performance Task: Exponent Properties (~45 minutes)
This practice performance task assesses learning objectives and essential knowledge statements addressed up to this point in the unit.

Lessons for Key Concept 4.2: Roots of Real Numbers
There are no provided Pre-AP Lessons for this key concept. As with all key concepts, this key concept is addressed in a learning checkpoint.
All Key Concept 4.2 learning objectives and essential knowledge statements should be addressed with teacher-developed materials.

Learning Checkpoint 1: Key Concepts 4.1 and 4.2 (~45 minutes)
This learning checkpoint assesses learning objectives and essential knowledge statements from Key Concepts 4.1 and 4.2. For sample items and learning checkpoint details, visit Pre-AP Classroom.

Lessons for Key Concept 4.3: Sequences with Multiplicative Patterns

Lesson Title	Learning Objectives Addressed	Essential Knowledge Addressed	Suggested Timing	Areas of Focus
4.1: Counting Binary Strings	4.3.1	4.3.1a, 4.3.1b, 4.3.1c	~45 minutes	Engagement in Mathematical Argumentation

4.2: Multiplicative Patterns	4.3.1, 4.3.2	4.3.1a, 4.3.1b, 4.3.1c, 4.3.2b, 4.3.2c, 4.3.2d	~90 minutes	Engagement in Mathematical Argumentation
4.3: Finding Terms in a Geometric Sequence	4.3.1, 4.3.2	4.3.1a, 4.3.1b, 4.3.1c, 4.3.2b, 4.3.2c, 4.3.2d	~120 minutes	Engagement in Mathematical Argumentation

The following Key Concept 4.4 essential knowledge statement is not addressed in Pre-AP lessons. Address this in teacher-developed materials.

- Essential Knowledge Statement: 4.3.2a

Lessons for Key Concept 4.4: Exponential Growth and Decay

Lesson Title	Learning Objectives Addressed	Essential Knowledge Addressed	Suggested Timing	Areas of Focus
4.4: Graphing Exponential Functions	4.3.3, 4.4.1, 4.4.2, 4.4.3	4.3.3a, 4.3.3b, 4.3.3c, 4.4.1a, 4.4.1b, 4.4.2a, 4.4.2b, 4.4.2c, 4.4.2d, 4.4.3a, 4.4.3b, 4.4.3c, 4.4.3d	~135 minutes	Connections Among Multiple Representations
4.5: Modeling with Exponential Functions	4.4.3, 4.4.4, 4.4.5	4.4.3a, 4.4.3b, 4.4.3c, 4.4.3d, 4.4.4a, 4.4.4c, 4.4.5a, 4.4.5b, 4.4.5c	~60 minutes	Greater Authenticity of Applications and Modeling, Engagement in Mathematical Argumentation

The following Key Concept 4.4 essential knowledge statements are not addressed in Pre-AP lessons. Address these in teacher-developed materials.

- Essential Knowledge Statements: 4.4.1c, 4.4.1d, 4.4.4b

Learning Checkpoint 2: Key Concept 4.4 (~45 minutes)
This learning checkpoint assesses learning objectives and essential knowledge statements from Key Concept 4.4. For sample items and learning checkpoint details, visit Pre-AP Classroom.

Performance Task for Unit 4 (~45 minutes)
This performance task assesses learning objectives and essential knowledge statements from the entire unit.

UNIT 4

PRACTICE PERFORMANCE TASK

Exponent Properties

OVERVIEW

DESCRIPTION

In this practice performance task, students apply their understanding of exponent properties to calculations with exponent expressions. This task is designed to engage students in crafting arguments about expressions involving exponents.

CONTENT FOCUS

This practice performance task provides students an opportunity to synthesize their knowledge of exponent properties. In the task, students strategically use the rules of exponents to perform calculations; the task is designed to be completed without a calculator. The problem also has students make conjectures about the value of an expression involving exponents, which serves as a preview of exponential functions.

AREA OF FOCUS

- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~45 minutes

HANDOUT

- Practice Performance Task: Exponent Properties

MATERIALS

- graphing utility
- calculator
- graph paper
- copies of scoring guidelines for student use (optional)

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ Properties of exponents are derived from the properties of multiplication and division.	
Learning Objectives	**Essential Knowledge**
4.1.1 Use exponent rules to express products and quotients of exponential expressions in equivalent forms.	**4.1.1a** Exponential expressions involving multiplication can be rewritten by invoking the rule $n^a \bullet n^b = n^{a+b}$, where $n > 0$. **4.1.1c** Exponential expressions involving powers of powers can be rewritten by invoking the rule $\left(n^a\right)^b = n^{ab}$, where $n > 0$.

4.1.2 Use exponent rules to express numerical and variable expressions that involve negative exponents using positive exponents, and vice versa.	**4.1.2a** Any non-zero real number raised to the zero power is equal to 1. That is, $n^0 = 1$, where $n \neq 0$. **4.1.2c** A negative exponent of -1 can be used to represent a reciprocal. That is, $n^{-k} = \frac{1}{n^{-k}}$, where $n \neq 0$. **4.1.2d** The properties of negative integer exponents, and those of an exponent of zero, are extensions of the properties of positive integer exponents.

SUPPORTING STUDENTS

BEFORE THE TASK

In this performance task, students are expected to use their understanding of properties of exponents to simplify expressions and to make conjectures about expressions involving exponents. This task can be used as a review of exponent properties and also as a preview of exponential functions.

- Introduce students to the practice performance task. The student handout for the task is included before the scoring guidelines for reference.
- To prepare students to engage in the task, you could begin by asking the following questions.
  - Suppose you have an expression where you are multiplying two monomials that have the same base. What could you do to simplify any exponents in the expression?

    If you multiply two monomials that have the same base, you can add the exponents to simplify the expression.
  - Suppose you have an expression where you are dividing two monomials that have the same base. What could you do to simplify any exponents in the expression?

    If you divide two monomials that have the same base, you can subtract the exponents to simplify the expression.
  - Suppose you have an expression where you are raising a monomial to a power. What could you do to simplify any exponents in the expression?

    If you raise a monomial to a power, you can multiply the exponents to simplify the expression.

- Suppose you are taking the square root or the cube root of an expression. What exponent could you use to express the square root or the cube root?

  You can use the exponent $\frac{1}{2}$ to express the square root and the exponent $\frac{1}{3}$ to express the cube root.

- Suppose you have an expression in the denominator of a fraction. What exponent could you use to express that?

  You can use the exponent -1 for the entire expression to express a denominator.

DURING THE TASK

The goal of this task is for students to demonstrate their understanding of exponent properties. The parts of the problem that ask them to make conjectures about the behavior of an exponent expression based on values of the exponent may be unfamiliar and challenging for students. These parts of the problem are intended to have students start thinking about exponential functions.

- You can have students work individually or in pairs to complete the task. There may be ample work and enough potential discussion areas for up to, but not more than, two students in each group. In larger groups, some students may not have an opportunity to legitimately engage in the task.
- You can divide the task into four parts and have students complete one part at a time. Students could check their solutions with you or the scoring guidelines before moving on to the next part. During the check, spend a few moments discussing the solution with each student. Focus on what changes, if any, students could make to their solution to craft a more complete response the next time they engage in a performance task.

During the task, students may ask about which tools they can use to answer the questions. If it your classroom norm that students can use calculators or other digital tools to complete assessments, then you should continue to allow students to use their familiar tools. If it is not in your classroom routine for students to use technology, then this would not be an appropriate time to introduce those tools. The implementation of this task should be adapted to your particular classroom and available materials.

AFTER THE TASK

- After students have worked on the practice performance task for 20–30 minutes, you may want to have them assess their own work using the scoring guidelines. Even if students have not completed the task, it could be advantageous to regroup as a class and discuss the task. Students will get better at answering these performance task questions as they do more of them and as they understand how the scoring is designed.

UNIT 4

- Whether you decide to have students score their own solutions, have students score their classmates' solutions, or score the solutions yourself, the results of the practice performance task should be used to inform instruction.
- Students should understand that converting their score into a percentage does not provide a good measure of how they performed on the task. You can use the suggested scoring conversion guide located after the scoring guidelines.
- To conclude, you may want to lead a whole-class discussion about the task by inviting students to ask questions, review their solutions, share their struggles, and examine any new insights they formed.

Exponent Properties

(a) Consider the equation below:

$$9^2 \bullet 9^3 = 59{,}049$$

Using only this information and the laws of exponents, explain how you could compute 9^6.

(b) Consider another equation:

$$(81)^3 = 531{,}441$$

Rewrite this equation using powers of 9, and explain your reasoning.

(c) Consider the expression 9^x. Describe the behavior of 9^x for the following values of x:

$$x < 0$$
$$x = 0$$
$$0 < x < 1$$
$$x = 1$$
$$x > 1$$

Support your conclusion with a numerical example for each of the specified values of x.

(d) Based on your analysis from part *c*, sketch a graph of what $y = 9^x$ could look like.

SCORING GUIDELINES

There are 12 possible points for this practice performance task.

Part (a)

Sample Solutions	Points Possible
Because $9^2 \bullet 9^3 = 59{,}409$ and $9^2 \bullet 9^3 = 9^{2+3} = 9^5$, you can multiply 9^5 by 9 to get $9^5 \bullet 9 = 9^{5+1} = 9^6$.	**3 points maximum** 1 point for recognizing $9^2 \bullet 9^3 = 9^5$ 1 point for showing $9^5 \bullet 9 = 9^6$ 1 point for a clear explanation
Targeted Feedback for Student Responses	
If students make a mistake in part (a), they may be unsure how to re-express products where the two factors are exponential expressions. Consider having students look for patterns by re-expressing $9^2 \bullet 9^3$ as $(9 \bullet 9) \bullet (9 \bullet 9 \bullet 9)$.	

TEACHER NOTES AND REFLECTIONS

Part (b)

Sample Solutions	Points Possible
$81 = 9^2$, so 81^3 can be rewritten as $(9^2)^3$, which simplifies to 9^6.	**3 points maximum** 1 point for recognizing that $81 = 9^2$ 1 point for showing $(9^2)^3 = 9^6$ 1 point for a clear explanation
Targeted Feedback for Student Responses	
If students make mistakes in part (b), it could mean they do not understand that $(a^b)^c = a^{(bc)}$ means that an expression of the form $a^{(bc)}$ can be re-expressed in the form $(a^b)^c$. This reversible way of thinking about equality can be challenging for many students. Alternatively, students might confuse multiplication and exponentiation in this problem, thus determining $(81)^3 = (9^2)^3 = 9^{(2^3)} = 9^8$.	

UNIT 4

TEACHER NOTES AND REFLECTIONS

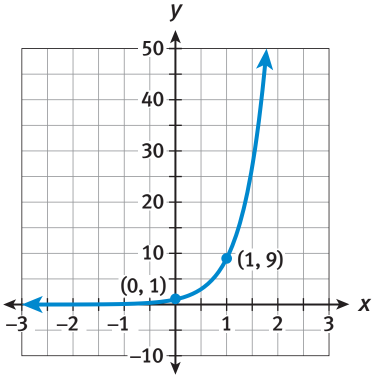
Part (c)

Sample Solutions	Points Possible
When $x < 0$, $0 < 9^x < 1$. (When the value of x is less than 0, the value of 9^x is between 0 and 1.) When $x = 0$, $9^x = 1$. (When the value of x is 0, the value of 9^x is 1.) When $0 < x < 1$, $1 < 9^x < 9$. (When the value of x is between 0 and 1, the value of 9^x is between 1 and 9.) When $x = 1$, $9^x = 9$. (When the value of x is 1, the value of 9^x is 9.) When $x > 1$, $9^x > 9$. (When the value of x is greater than 1, the value of 9^x is greater than 9.)	**3 points maximum** 1 point for describing the behavior when x is less than or equal to 0 1 point for correctly describing the behavior when x is between 0 and 1 1 point for correctly describing the behavior when x is greater than or equal to 1

Targeted Feedback for Student Responses
If students make mistakes in part (c), they may be having difficulty thinking about an exponential expression having a non-whole-number exponent. You could have students practice evaluating exponent expressions with various bases and non-whole-number exponents and have them look for patterns in the values of the expressions.

TEACHER NOTES AND REFLECTIONS

Part (d)

Sample Solutions	Points Possible
Sketch should resemble the following graph: 	**3 points maximum** 1 point for identifying both coordinates: $(0, 1)$ and $(1, 9)$ 1 point for showing the curve between 0 and 1 for $x < 0$ 1 point for showing the curve increasing for $x > 0$ *Scoring note:* If students sketch the graph with line segments instead of curves, but the placement of the segments is generally correct, they should receive full points.

Targeted Feedback for Student Responses
If students make mistakes in part (d), they may be having difficulty thinking about the value of 9^x as the value of x varies, and instead may be thinking about plotting a few points instead of a continuous curve.

TEACHER NOTES AND REFLECTIONS

Suggested point conversion, if assigning a grade to this problem:

Points Received	Appropriate Letter Grade (if Graded)	How Students Should Interpret Their Score
11 or 12 points	A	"I know all of this algebra really well."
8 to 10 points	B	"I know all of this algebra well, but I made a few mistakes."
5 to 7 points	C	"I know some of this algebra well, but not all of it."
2 to 4 points	D	"I only know a little bit of this algebra."
0 or 1 point	F	"I don't know much of this algebra at all."

UNIT 4

LESSON 4.1

Counting Binary Strings

OVERVIEW

LESSON DESCRIPTION

Part 1: Exploring Binary Strings
Students investigate how many binary strings there are of a given length.

Part 2: Creating a Formula for Binary Strings
Students use the patterns they observe in the number of binary strings of a given length to write a formula for the number of binary strings of any length.

Part 3: Exploring Ternary Strings
Students use what they know about binary strings to investigate sequences composed of three symbols.

AREA OF FOCUS

- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~45 minutes

HANDOUT

Lesson

- 4.1: Video Game Cheat Codes

CONTENT FOCUS

In this first lesson, students are introduced to geometric sequences and exponential growth by exploring a real-world method of storing data called *binary strings*. Unlike a character string, which usually stores digital data as text, binary strings use a sequence of 0s and 1s to code for digital data such as pictures or video. Therefore, a binary string is an example of an ordered list where there are only two options for each spot in the list. The list can be any length. Students work on developing strategies to determine how many unique binary strings there are for a given length.

Even though students spend much of the lesson counting how many unique strings there are of a given length, the goal of the lesson is for students to understand that the quantity of binary strings of a given length is related to the length of the string. They will come to see that the number of binary strings doubles every time the length of the string increases by one. Students eventually develop both a recursive rule and an explicit rule to describe this relationship, and they extend their work by exploring ternary strings—ordered lists where there are only three options for each spot in the list.

No vocabulary related to geometric sequences is used in this lesson, but the concepts learned here prepare students to understand these terms when they are introduced in Lessons 4.2 and 4.3. In later lessons, students also compare and contrast geometric sequences with arithmetic sequences, which they explored in Unit 1.

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ An exponential function has constant multiplicative growth or decay.	
Learning Objectives	**Essential Knowledge**
4.3.1 Determine whether a relationship is exponential or nonexponential based on a numerical sequence whose indices increase by a constant amount.	**4.3.1a** A geometric sequence is an exponential relationship whose domain consists of consecutive integers. **4.3.1b** The ratios of successive terms of a geometric sequence are equivalent. **4.3.1c** A geometric sequence can be determined from the common ratio and any term in the sequence.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

Ali the Adventurer fell into a trap and poison gas is filling the room! To escape from the room, Ali must find the correct arrangement of four levers. Each lever can be positioned up or down.

(a) What are all the different ways Ali could position the levers?

(b) Ali estimates that she has only about a half hour of breathable air left. It takes Ali one minute to rearrange all four levers. If she has to try all the possible combinations before escaping, will she be able to get out of the room in time? Explain your answer.

PART 1: EXPLORING BINARY STRINGS

In this part of the lesson, students explore binary strings, which are ordered lists of symbols where there are only two choices for the symbol. This is a contextual example of doubling, because every time the sequence gets one character longer, twice as many strings can be formed. The patterns that students identify from working through this example will be used in Parts 2 and 3 of this lesson to develop two rules for determining the number of binary strings of a given length.

- To begin the lesson, share the following information with students:

> Computers transmit and store information using a combination of digits with only two choices per spot. A popular way to do this is using a combination of 0s and 1s, but information can be coded in other ways, too. For example, some hard disk drives use magnetic positives and negatives. DVDs use laser light that either is or is not reflected back. All of these systems are binary, since there are only two options per space. A sequence of symbols that uses only 0s and 1s is called a *binary string*.

Guiding Student Thinking

Students might wonder if there is a special significance to 0 and 1. You can explain to them that a computer might be able to use 0 and 1 more easily than other options, but binary strings can be made with any choice of two symbols, such as A and B or ☺ and ☹.

- Ask students some questions to get them thinking about how to make a binary string:
  - What are the binary strings that have a length of one? How many of them exist?

    There are two of them: 0 and 1.
  - What are the binary strings that have a length of two? How many of them exist?

    There are four binary strings of length two: 00, 01, 10, and 11.
  - What are the binary strings that have a length of three? How many of them exist?

    There are eight strings: 000, 001, 010, 011, 100, 101, 110, and 111.

Classroom Ideas

Drawing a blank space like "⎵" for each symbol in the string can help students easily see the length of the string.

- After students have found all eight binary strings that have a length of three, it is a good time to have them think about how they can be sure that they did not miss any strings. Have students organize their list of strings if they haven't done so already. One possible way to organize the list is by the first symbol, as shown in this table:

Starts with "0"	Starts with "1"
000	100
001	101
010	110
011	111

- Once students have checked that they did not miss any strings, ask them what they notice or wonder about the strings. Allow them time to make some close observations, and record all student observations and questions for the whole class to see.

  Students may notice that the binary strings of length two are contained in the strings of length three and that they are repeated with a different first symbol. Students may wonder if that pattern can be used to find all the strings of length four.

- Now, you can ask students:
  - How many binary strings are there that have a length of four? List all of them.

    There are 16 strings of length four: 0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, and 1111.

  - How do you know you found all the strings? Organize your list like we did for the length-three strings to make sure we've found them all.

    The list of 16 strings could be split into two groups of eight strings, each with a different initial symbol.

Starts with "0"	Starts with "1"
0000	1000
0001	1001
0010	1010
0011	1011
0100	1100
0101	1101
0110	1110
0111	1111

- What do you notice about your organized list?

  There are eight strings that start with "0" and eight strings that start with "1." The strings that start with "0" can be organized into two sets of four, where the second symbol is either a "0" or a "1." The same thing is true for the strings that start with "1."

  Students may come up with other ways to organize the lists and other things that they notice. It is important to validate these as long as they are correct. For example, students may draw a tree diagram to make sure they have all possible sequences of 0s and 1s.

- Once the class is convinced that they have all the length-four strings, ask students to summarize how many binary strings there are of lengths one, two, three, and four.

  There are 2, 4, 8, and 16 strings, respectively.

- After students have summarized how many strings there are of the different lengths, ask them what they notice. If students don't respond, you could prompt them:
  - It seems like the number of strings is starting to grow rather quickly—do you see a pattern?

    Students should suggest that the number of possible strings seems to be doubling.
- Next, have students predict how many binary strings of length five there would be and share their reasoning. Ideally, they should do this without writing out all of the strings. The patterns from the earlier investigation should lead them to a method for determining the strings. Ask students:
  - How many length-five strings are there? How did you figure out your answer?

    There are 32 strings with a length of five. I know because there are 16 length-four strings, and you can put a "0" or a "1" in the front of each string. So, there will be 16 strings that start with a "0" and 16 strings that start with a "1."

Meeting Learners' Needs
If some students need more support getting started, you can suggest that they continue to write out all of the strings as they did in the previous examples. It will take more time, but the process might help students see that they are doubling the number of strings every time the length of the string increases by one symbol.

- Ask students some clarifying questions before moving on to the next part of the lesson:
  - How many binary strings are there of length six? How do you know?

    There are $32 \cdot 2 = 64$, because the number of strings doubles when the length of the sequence is increased by one (each string of length five can now have a 0 or a 1 in front). The strings of length six can be classified into two groups: There are 32 strings that look like 0[string of length five] and 32 strings that look like 1[string of length five].
  - It turns out that there are 1,024 binary strings of length 10. How many strings are there of length 11? How do you know?

    There are $1{,}024 \cdot 2 = 2{,}048$, because every string now of length 10 is extended and can have a 0 or a 1 in front of it.
  - In general, if we know how many binary strings there are of a certain length, how could we figure out the number of binary strings when the length is increased by 1?

    You multiply the known number of binary strings by 2.

PART 2: CREATING A FORMULA FOR BINARY STRINGS

In this part of the lesson, students work on developing a formula to determine the number of binary strings for any length. From Part 1, students know how to determine the number of binary strings with a length of $n+1$ if they know the number of strings with a length of n. Now the focus is on developing a formula for determining the number of strings from just the length. Students also make connections with their prior knowledge of exponents and continue to build a foundation for understanding geometric sequences.

- To begin this part of the lesson, ask students:
  - Thinking about what you did earlier, would you be able to figure out how many binary strings of length eight there are without knowing the number of strings of length seven? Why or why not?

    It is not possible, because we always double the number of strings of length n to get the number of strings of length $n+1$.
- Now, ask students to complete a table of values for the length of the strings, n, and the total number of binary strings, b.

n	*b*
1	2
2	4
3	8
4	16
5	32
6	64
7	128

- Have students create a third column for the table where they will write down the calculations they used to determine the total number of binary strings of a length. It is important to write out the multiplication in terms of the factor 2, as shown in the following table, so students can see that an exponent will help simplify the entire process.

n	*b*	Calculations
1	2	$= 2$
2	4	$= 2 \cdot 2$
3	8	$= 2 \cdot 4 = 2 \cdot 2 \cdot 2$
4	16	$= 2 \cdot 8 = 2 \cdot 2 \cdot 2 \cdot 2$
5	32	$= 2 \cdot 16 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
6	64	$= 2 \cdot 32 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
7	128	$= 2 \cdot 64 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

- Ask students:
  - How do we get from one row of the table to the next row? What pattern do you notice?

    You multiply by 2 to get to the next row. The number of 2s in the calculation column increases by 1.
  - Is it possible to express each of the multiplications in the calculation column in a different way?

    Yes, you can represent repeated multiplication with an exponent.

- Now have students again extend the table by one column so they can express the calculations using an exponent.

n	*b*	Calculations	Exponent form
1	2	$=2$	2^1
2	4	$=2 \bullet 2$	2^2
3	8	$=2 \bullet 4 = 2 \bullet 2 \bullet 2$	2^3
4	16	$=2 \bullet 8 = 2 \bullet 2 \bullet 2 \bullet 2$	2^4
5	32	$=2 \bullet 16 = 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2$	2^5
6	64	$=2 \bullet 32 = 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2$	2^6
7	128	$=2 \bullet 64 = 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2$	2^7

- Ask students:
  - Is it possible to use exponents to determine how many binary strings there are with a length of 12?

    Yes, it is. The number of binary strings with a length of 12 will be 2^{12}.
  - Are there any advantages to using the exponent formula instead of multiplying each row by 2?

    It is easier to go directly from the length of the string to how many strings there are using the exponent formula. Using the multiplication method, it could take a long time to get to the answer for a long string.

Guiding Student Thinking

Students may wonder if there is any connection between “multiplying by 2” and the base of the exponent form 2^n. It is not a coincidence that the same number appears in both rules. You can let students know that this is an important insight to have and that they will explore this connection more deeply later in the unit (in Lessons 4.2 and 4.3).

- At this point, it is appropriate to reintroduce sequence notation and have students write a formal rule for this sequence. Tell students that they can use the expression b_n to represent the total number of binary strings with a length of n. Ask students some questions to ensure that they remember how to use sequence notation:
  - If we wrote $b_3 = 8$, what does that mean in the context of binary strings?

    There are 8 binary strings with a length of three.
  - How would we use sequence notation to write, "There are 1,024 binary strings with a length of 10"?

    We would write $b_{10} = 1{,}024$.
  - What is the general rule (or formula) for determining the number of binary strings with a length of n written in sequence notation?

    A rule for this sequence is $b_n = 2^n$.

Instructional Rationale

This lesson should prompt students to realize they used a similar process and similar notation to extend arithmetic sequences in Unit 1 and quadratic sequences in Unit 2. If they do notice a connection, validate their thinking, but hold off explaining the general rules of geometric sequences. In this lesson, students have an opportunity to work through a specific sequence. In Lesson 4.2, they will make a much more explicit connection to arithmetic sequences.

PART 3: EXPLORING TERNARY STRINGS

In this part of the lesson, students have an opportunity to extend their understanding of strings. Here, students explore a contextual scenario where there are three symbols for each spot in the string instead of two. Students engage in the same reasoning process that they used with binary strings. First, they determine the number of strings with a length of $n+1$ based on the number of strings with a length of n. Then they work on developing a sequence notation rule for this scenario that allows them to predict the number of strings for sequences of any given length.

- Have students work independently or in pairs on the questions involving ternary form (See **Handout 4.1: Video Game Cheat Codes**). Circulate around the room and provide help and guidance as students work.

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

Ali the Adventurer fell into a trap, and poison gas is filling the room! To escape from the room, Ali has to find the correct arrangement of four levers. Each lever can be positioned up or down.

(a) What are all the different ways Ali could position the levers?

There are 16 possible lever arrangements, as shown in the table.

Lever 1	Lever 2	Lever 3	Lever 4
Up	Up	Up	Up
Up	Up	Up	Down
Up	Up	Down	Up
Up	Up	Down	Down
Up	Down	Up	Up
Up	Down	Up	Down
Up	Down	Down	Up
Up	Down	Down	Down
Down	Up	Up	Up
Down	Up	Up	Down
Down	Up	Down	Up
Down	Up	Down	Down
Down	Down	Up	Up
Down	Down	Up	Down
Down	Down	Down	Up
Down	Down	Down	Down

(b) Ali estimates that she has only about a half hour of breathable air left. It takes Ali one minute to rearrange all four levers. If she has to try all the possible combinations before escaping, will she be able to get out of the room in time? Explain your answer.

Since there are 16 lever arrangements and it takes 1 minute to try each arrangement, it would take 16 minutes to try them all. So, yes, Ali would still be able to escape before her air runs out.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 4.1: Video Game Cheat Codes

1. There are 9 possible sequences: AA, AB, AC, BA, BB, BC, CA, CB, and CC.
2. There are 27 possible sequences. Each of the 9 two-press sequences has another button-press that has three choices, so $3 \bullet 9 = 27$.
3. There are 81 possible sequences. Each of the 27 three-press sequences has another button-press that has three choices, so $3 \bullet 27 = 81$.
4. To get the number of sequences with $n+1$ button-presses if you know the number of cheat codes that have n button-presses, you should multiply by 3.
5. A formula for the number of sequences with n button-presses is $S_n = 3^n$.
   I made a table of values and expressed the repeated multiplication with exponents:

n	$S_n = 3^n$
1	$3 = 3^1$ $= 3^1$
2	$9 = 3 \bullet 3$ $= 3^2$
3	$27 = 3 \bullet 3 \bullet 3$ $= 3^3$
4	$81 = 3 \bullet 3 \bullet 3 \bullet 3$ $= 3^4$

6. Using the formula from question 5, I determined that the number of cheat codes with 10 button-presses would be $S_{10} = 3^{10} = 59{,}049$.

UNIT 4

LESSON 4.2

Multiplicative Patterns

OVERVIEW

LESSON DESCRIPTION

Part 1: Comparing Arithmetic and Geometric Sequences
Students explore and compare sequences that increase or decrease in different ways.

Part 2: Modeling Pizza Night with Sequence Formulas
Students explore a thought experiment involving different rates or ratios of change.

Part 3: Investigating Fractal Patterns
Students extend fractal image sequences and write a sequence formula for a particular feature of the fractal.

AREA OF FOCUS

- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~90 minutes

HANDOUTS

Lesson

- 4.2.A: Fractal Investigations

Practice

- 4.2.B: Practice with Geometric Sequences

CONTENT FOCUS

Arithmetic and geometric sequences have many similar features, such as a repetitive recursive process to get consecutive terms in the sequence. Using what they learned about geometric sequences in Lesson 4.1, students write an explicit formula for the geometric sequences in this lesson. They also explore some fractal images that exhibit multiplicative growth in a particular facet of the image. Through this lesson and Lesson 4.3, students deepen their understanding of both arithmetic and geometric sequences by examining how their forms and processes are alike.

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings
▪ An exponential function has constant multiplicative growth or decay.

<table>
<tr><th>Learning Objectives</th><th>Essential Knowledge</th></tr>
<tr><td>4.3.1 Determine whether a relationship is exponential or nonexponential based on a numerical sequence whose indices increase by a constant amount.</td><td>4.3.1a A geometric sequence is an exponential relationship whose domain consists of consecutive integers.
4.3.1b The ratios of successive terms of a geometric sequence are equivalent.
4.3.1c A geometric sequence can be determined from the common ratio and any term in the sequence.</td></tr>
<tr><td>4.3.2 Convert a given representation of a geometric sequence to another representation of the geometric sequence.</td><td>4.3.2b Successive terms in a geometric sequence are obtained by multiplying the previous term by the common ratio. To find the value of the term that occurs n terms after a specified term, multiply the specified term by the common ratio n times.
4.3.2c A geometric sequence can be algebraically expressed with the formula $a_n = a_k \bullet c^{(n-k)}$, where a_n is the nth term, a_k is the kth term, and c is the common ratio between successive terms.
4.3.2d A verbal representation of a geometric sequence describes a discrete domain and a constant multiplicative growth or decay.</td></tr>
</table>

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

One of these lists of numbers is part of a geometric sequence, one is part of an arithmetic sequence, and one is not part of either kind of sequence. Determine which one is which, and justify your choice. Write a sequence formula for the arithmetic and geometric sequences.

(a) 2, –1, –4, –7, –10, …

(b) 2, –1, –6, –13, –22, …

(c) 2, –6, 18, –54, 162, …

PART 1: COMPARING ARITHMETIC AND GEOMETRIC SEQUENCES

In this part of the lesson, students compare arithmetic sequences, which they explored in Lessons 1.4 and 1.5, with geometric sequences. The goal is for students to gain some insight into multiplicative growth patterns, which will help them make sense of exponent properties.

- To get students thinking about different kinds of rates of change, display the following sequences. Ask students to note any patterns they see and to make a conjecture about the next three terms in the sequence if there seems to be a predictable pattern.

  Sequence 1: 167, 172, 177, 182, ____, ____, ____

  187, 192, 197, …

  Sequence 2: 2, 10, 50, 250, ____, ____, ____

  1,250, 6,250, 31,250, …

- After students have had some time to think, invite them to share their conjectures and reasoning. Then ask students to compare the two sequences. How are they similar? How are they different?

  Both sequences are increasing. In sequence 1, the terms increase slowly and steadily because the next term is 5 added to the previous term. The terms in sequence 2 get much bigger much faster than sequence 1, because in sequence 2 the next term is the previous term multiplied by 5. Because sequence 1 has a constant rate of change, it is an arithmetic sequence. Sequence 2 is not an arithmetic sequence.

- Now, display two different sequences for the class to see. These sequences are decreasing; they will help students think about how addition and multiplication can be used to make numbers smaller. Again, ask students to compare the two sequences, as well as to note any patterns they see and make a conjecture about the next three terms in the sequence if there seems to be a predictable pattern.

  Sequence 3: 72, 24, 8, ____, ____, ____

  $\frac{8}{3}, \frac{8}{9}, \frac{8}{27}, \ldots$

  Sequence 4: 1.9, 0.5, –0.9, ____, ____, ____

  –2.3, –3.7, –5.1, …

- After students have had some time to think about these sequences, again invite them to share their conjectures and reasoning. Ask students to compare these sequences, as they did with the previous two sets. Again, how are they similar? How are they different?

  Both sequences are decreasing. The terms in sequence 3 seem to get smaller but remain positive, while the terms in sequence 4 become negative and get more negative. The terms in sequence 3 can be found by multiplying the previous term by $\frac{1}{3}$, and the terms in sequence 4 can be found by adding -1.4. Because sequence 4 has a constant rate of change, it is an arithmetic sequence. Sequence 3 is not an arithmetic sequence.

Instructional Rationale

In most advanced mathematics, it is preferable to refer to division as multiplying by the inverse and to refer to subtraction as adding the opposite. So in sequence 3, it is better to say the terms change by multiplying by $\frac{1}{3}$ instead of dividing by 3. In sequence 4, it is better to say that the terms change by adding -1.4 rather than by subtracting 1.4. Even though the thinking behind the process is "divide" or "subtract," having only two distinct operations (multiply or add) makes it easier to generalize. By the end of the lesson, students should understand that all arithmetic sequences have a common difference that you add to get consecutive terms, and all geometric sequences have a common ratio that you multiply to get consecutive terms.

PART 2: MODELING PIZZA NIGHT WITH SEQUENCE FORMULAS

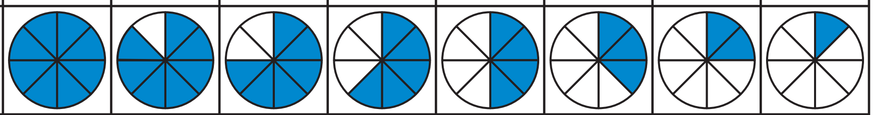
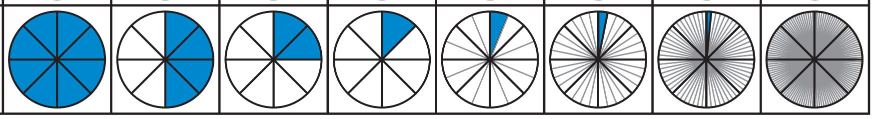
In this part of the lesson, students engage in a thought experiment that highlights both arithmetic and geometric sequences. This naturally introduces the need to define geometric sequences.

- To begin, display the following scenario for the class:

> Two friends, Freddy and Harry, each have their own 8-slice pizza, but they eat their pizza differently. "Slow and Steady" Freddy eats one slice of pizza at a time. "Half Some More" Harry takes half a pizza at first, then every time he goes for more pizza, he takes half as much as he had the previous time.

- Ask students what they notice and what they wonder about the scenario. Students may point out similarities and differences in the ways the friends eat pizza. Or, they may wonder who finishes the pizza first, or whether the pizzas are the same size. Record all observations and questions so all students can see them.
- Once students have had some time to comment on the scenario, ask them the following questions to get them thinking about how to represent the pizza party using number sequences.
    - How many slices are in Freddy's pizza the 2nd time he reaches for a slice? What about the 4th time? How many slices are in Harry's pizza the 2nd and 4th times he reaches for a slice? Explain how you found the answers. You may also want to draw a picture or create a table to help you answer the question.

      There would be 7 slices as Freddy is reaching the 2nd time and 5 slices as he is reaching the 4th time. There would be 4 slices as Harry is reaching the 2nd time and 1 slice as he is reaching the 4th time.
- Invite students to share responses to the question. You may want to work as a class to create a diagram like the one shown here.

Number of Reaches for Pizza	1	2	3	4	5	6	7	8
Freddy's pizza *before* he takes a slice								
Harry's pizza *before* he takes a slice								

- Lead students through a series of questions that will help them compare Freddy's and Harry's pizza-eating styles. This series of questions culminates in students learning the definition of a geometric sequence and writing formulas for both Freddy's and Harry's sequences.
    - Does Freddy finish his entire pizza? Does Harry? How do you know?

      Freddy will finish the pizza because there will be no slices left by the 9th time he reaches for pizza. Harry doesn't finish the pizza because he only takes half of what remains, so there is always a little bit left.

- If you haven't already, write a sequence that models the number of slices available in Freddy's pizza. The first value in the sequence should represent the full pizza. What kind of sequence is it? How do you know?

  The sequence is 8, 7, 6, 5, 4, 3, 2, 1, 0. This is an arithmetic sequence, because there is a common difference of –1; that is, Freddy eats one slice at a time.

- If you haven't already, write a sequence that models the number of slices available in Harry's pizza. Again, the first value in the sequence should represent the full pizza. What kind of sequence is it? How do you know?

  The sequence is 8, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ... This is not an arithmetic sequence, because there is not a common difference. Harry eats less and less pizza each time.

- Now, ask students if it is possible to write a sequence formula that models the number of slices in Freddy's pizza. Students may struggle a bit and many students will try to use a formula such as $F_n = 8 - 1n$, where F_n represents the number of slices in the pizza (as he reaches for a slice) and n represents the number of times he reaches. This is not a correct formula in this case because the first time Freddy reaches for pizza, there should be 8 total slices. Substituting $n = 1$ into the formula will give $F_1 = 8 - 1 = 7$.
- To help students understand how to write the sequence formula, have them complete a table of values like the one on the following page.

Number of Times Freddy Reaches for Pizza (n)	Slices Available (F_n)	Calculations
1	8	$F_1 = 8$
2	7	$F_2 = 8 - 1$
3	6	$F_3 = 7 - 1$ $= 8 - 1 - 1$ $= 8 - 1(2)$
4	5	$F_4 = 6 - 1$ $= 8 - 1 - 1 - 1$ $= 8 - 1(3)$
5	4	$F_5 = 5 - 1$ $= 8 - 1 - 1 - 1 - 1$ $= 8 - 1(4)$
6	3	$F_6 = 4 - 1$ $= 8 - 1 - 1 - 1 - 1 - 1$ $= 8 - 1(5)$
7	2	$F_7 = 3 - 1$ $= 8 - 1 - 1 - 1 - 1 - 1 - 1$ $= 8 - 1(6)$
8	1	$F_8 = 2 - 1$ $= 8 - 1 - 1 - 1 - 1 - 1 - 1 - 1$ $= 8 - 1(7)$

- Once students have completed a table of values, ask them what patterns they see. Make sure to direct their attention to the number of times the common difference (-1) gets added to the 8 slices of pizza. Students should observe that the number of times the common difference is added is one less than the number of times Freddy reaches for pizza.

- Now students are ready to write a formula for the sequence. To help lead them through this process, ask the following questions:
  - In the table of values we made, what calculation is the same in each row?

    The thing that is the same in each calculation is $8-1(\text{some number})$.
  - How do you figure out how many times to add -1? What is the value of the number in parentheses?

    The number is always one less than the number of times he reaches for pizza.
  - If we let n be the number of times that Freddy reaches for pizza and F_n be the number of slices in the pizza before he takes a slice, what is the sequence formula?

    The sequence formula is $F_n = 8-1(n-1)$.

Instructional Rationale

Writing a formula for an arithmetic sequence should be familiar for students because they did it in Unit 1. However, it is useful to work through the process again with students, because they will use it, with some modifications, to write a formula for a geometric sequence.

- Once students understand how to write the formula for an arithmetic sequence, you can turn their attention to Harry's pizza. Ask students this question:
  - How do you figure out how many slices are available for Harry to take every time he reaches for a slice?

    You divide the previous fraction or number of slices by 2. Another way to say this is that you multiply the previous amount by $\frac{1}{2}$.
- Have students make a table of values for the slices that Harry has in his pizza before he takes half. The table should look very similar to Freddy's table. Make sure that students include a column for calculations, even if they struggle to make sense of how to write the repeated multiplication.

Number of Times Harry Reaches for Pizza (n)	Slices Available (H_n)	Calculations
1	8	$H_1 = 8$
2	4	$H_2 = 8 \bullet \frac{1}{2}$
3	2	$H_3 = 4 \bullet \frac{1}{2}$ $= 8 \bullet \frac{1}{2} \bullet \frac{1}{2}$ $= 8 \bullet \left(\frac{1}{2}\right)^2$
4	1	$H_4 = 2 \bullet \frac{1}{2}$ $= 8 \bullet \frac{1}{2} \bullet \frac{1}{2} \bullet \frac{1}{2}$ $= 8 \bullet \left(\frac{1}{2}\right)^3$
5	$\frac{1}{2}$	$H_5 = 1 \bullet \frac{1}{2}$ $= 8 \bullet \frac{1}{2} \bullet \frac{1}{2} \bullet \frac{1}{2} \bullet \frac{1}{2}$ $= 8 \bullet \left(\frac{1}{2}\right)^4$
6	$\frac{1}{4}$	$H_6 = \frac{1}{2} \bullet \frac{1}{2}$ $= 8 \bullet \frac{1}{2} \bullet \frac{1}{2} \bullet \frac{1}{2} \bullet \frac{1}{2} \bullet \frac{1}{2}$ $= 8 \bullet \left(\frac{1}{2}\right)^5$

Number of Times Harry Reaches for Pizza (n)	Slices Available (H_n)	Calculations
7	$\frac{1}{8}$	$H_7 = \frac{1}{4} \bullet \frac{1}{2}$ $= 8 \bullet \frac{1}{2} \bullet \frac{1}{2} \bullet \frac{1}{2} \bullet \frac{1}{2} \bullet \frac{1}{2} \bullet \frac{1}{2}$ $= 8 \bullet \left(\frac{1}{2}\right)^6$
8	$\frac{1}{16}$	$H_8 = \frac{1}{8} \bullet \frac{1}{2}$ $= 8 \bullet \frac{1}{2} \bullet \frac{1}{2} \bullet \frac{1}{2} \bullet \frac{1}{2} \bullet \frac{1}{2} \bullet \frac{1}{2} \bullet \frac{1}{2}$ $= 8 \bullet \left(\frac{1}{2}\right)^7$

- Ask students what patterns they see in the tables they have made. Make sure to direct their attention to the number of times that the 8 slices of pizza are multiplied by $\frac{1}{2}$. Students should observe that the number of times the 8 slices of pizza are multiplied by $\frac{1}{2}$ is 1 less than the number of times Harry reaches for pizza.
- Now ask students the following questions to lead them through writing a formula for the sequence:
  - In the table of values we made, what calculation is the same in each row?

    The thing that is the same in each calculation is $8 \bullet \left(\frac{1}{2}\right)^{\text{some number}}$.
  - How do you figure out how many times to multiply by $\frac{1}{2}$? What is the value of the exponent?

    The exponent is always one less than the number of times he reaches for pizza.
  - If we let n be the number of times that Harry reaches for pizza and H_n be the number of slices in the pizza before he takes a slice, what is the sequence formula?

    The sequence formula is $H_n = 8 \bullet \left(\frac{1}{2}\right)^{n-1}$.

- To summarize what you've done so far, and to give students some academic vocabulary, define *geometric sequences* for students:

  A geometric sequence is a sequence of numbers where each term is found by multiplying the previous term by a fixed value. The fixed value is called the common ratio.

Guiding Student Thinking

Students might be confused about why a fixed multiplication value is called a *ratio*, which usually implies division. You can point out that in an arithmetic sequence, the common difference (the fixed addition value) is found by subtracting two consecutive terms. Similarly, in a geometric sequence, the common ratio can be found by dividing two consecutive terms.

PART 3: FRACTAL PATTERNS

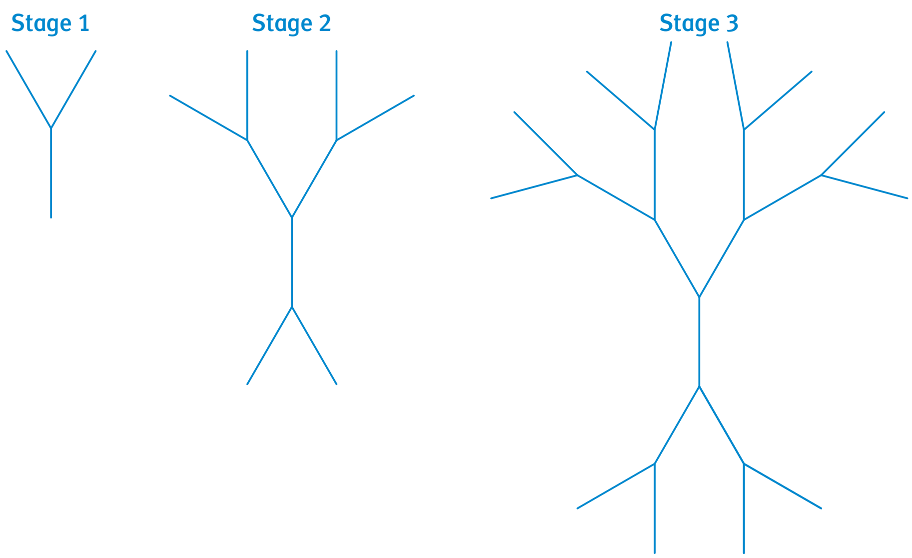
In this part of the lesson, students investigate three visual presentations of geometric sequences in the form of fractal figures. Many fractals can be constructed using a recursive process, where repeating a rule at each successive stage of the fractal sequence results in an increasingly complex figure.

- Have students work in pairs on **Handout 4.2.A: Fractal Investigations**. If you think that students will need help engaging with the fractals, you could choose to do the first investigation (shown on the next page) as a class. Answers for this handout are available in the Assess and Reflect section of the lesson.

Fractal Investigations

In the three investigations below, you will explore sequences of geometric objects known as *fractals*. A fractal has the quality of being "self-similar," which means that a small part of it has the same features as the entire figure. Fractals are usually formed by repeating a process over and over again.

1. **Tree Fractal Investigation:** Look at the first three stages of the tree fractal shown here. For this investigation, closely observe the number of branches the tree has in each stage.



(a) How is each stage formed?

Handout 4.2.A

(b) Draw the next stage in the pattern.

(c) Make a table of values that relates the number of new branches at each stage to the stage number. Does the number of new branches at each stage form a geometric sequence? How do you know?

(d) Use your table of values to write a sequence formula for the number of new branches in any stage.

Handout 4.2.A

- While students are working, circulate around the room and provide help and suggestions as needed.
- After a student pair has completed the three investigations on the handout, have them join with another pair to form a group of four. In the group of four, students should compare solutions, discuss any differences, and come to a consensus.
- Once the groups of students have come to a consensus, summarize the results as a class. Here are some questions to help guide the conversation:
    - What did the three investigations have in common?

      They all involved multiplicative patterns, or geometric sequences.
    - What was different about the different fractals?

      The common ratios were different in each fractal. Also, most of the investigations involved increasing values, but in the Koch snowflake, each segment was $\frac{1}{3}$ as long as the segments in the previous stage.
- If you want to give students an opportunity to explore geometric sequences more deeply, you can provide them with **Handout 4.2.B: Practice with Geometric Sequences**. However, students will continue their investigation of geometric sequences in Lesson 4.3, so the practice handout could be held until after that lesson.

UNIT 4

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

One of these lists of numbers is part of a geometric sequence, one is part of an arithmetic sequence, and one is not part of either kind of sequence. Determine which one is which, and justify your choice. Write a sequence formula for the arithmetic and geometric sequences.

(a) 2, –1, –4, –7, –10, …

Arithmetic sequence. There is a common difference of –3. To get to the next term, you add –3 (or subtract 3). The formula is $a_n = 2 + (-3)(n-1)$.

(b) 2, –1, –6, –13, –22, …

Neither. There is no common difference or common ratio.

(c) 2, –6, 18, –54, 162, …

Geometric sequence. There is a common ratio of –3. To get to the next term, you multiply by –3. The formula is $a_n = 2 \bullet (-3)^{(n-1)}$.

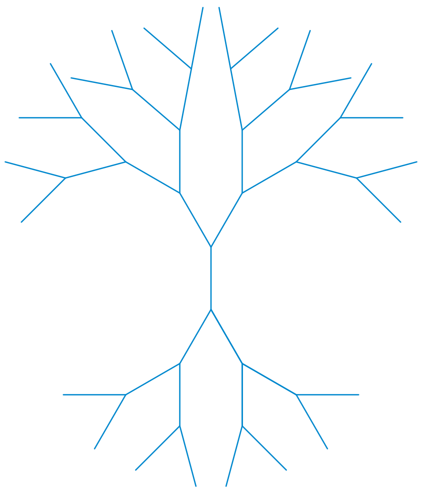
HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 4.2.A: Fractal Investigations

1. (a) At each stage, every branch grows two more branches.

   (b) The fourth stage will look like this:



(c)

Stage Number	New Branches
1	3
2	6
3	12
4	24

Each stage has twice as many branches as the previous stage. Each branch "grows" two new branches. This is a geometric sequence because the common ratio is 2.

(d) A formula for the number of new branches at any stage would be $B_n = 3 \bullet 2^{n-1}$, where n is the stage number and B_n is the new branches grown in that stage.

2. (a) You form the next stage by inscribing a white, equilateral triangle inside each black triangle.

(b)

Stage Number	Black Triangles
1	1
2	3
3	9
4	27

Multiply by 3 to get the number of black triangles in the next stage. This is a geometric sequence and the common ratio is 3.

(c) There would be 81 black triangles in the 5th stage. I multiplied the number of black triangles in the 4th stage by 3.

(d) A formula for the number of black triangles in any stage would be $S_n = 1 \bullet 3^{n-1}$, where n is the stage number and S_n is the number of black triangles in that stage.

3. (a) To get the next stage, remove the middle third of each line segment and create an inverted V-shape in its place. All new line segments have the same length.

   (b) There are 1, 4, 16, and 64 line segments in the first four stages, respectively. It is a geometric sequence, because you can multiply by 4 to get the number of line segments in the next stage.

   (c) There should be 1,024 line segments in stage 6. I multiplied 64 by 42 to get from the 4th stage to the 6th stage.

   (d) A formula for the number of line segments in any stage would be $K_n = 1 \bullet 4^{n-1}$, where n is the stage number and K_n is the number of line segments in that stage.

4. (a) A table of values for the length of an individual line segment is:

Stage Number	Segment Length
1	1
2	$\frac{1}{3}$
3	$\frac{1}{9}$
4	$\frac{1}{27}$

   (b) This does form a geometric sequence, because you can get the length of the line segment in the next stage by multiplying by $\frac{1}{3}$.

   (c) A formula for the length of an individual line segment in any stage would be $L_n = 1 \bullet \left(\frac{1}{3}\right)^{n-1}$, where n is the stage number and L_n is the length of an individual line segment in that stage.

Handout 4.2.B: Practice with Geometric Sequences

1. (a) Next three terms: 135, 405, 1,215; common ratio: $r = 3$

   (b) Next three terms: 18, 6, 2; common ratio: $r = \frac{1}{3}$

   (c) Next three terms: 40.5, 60.75, 91.125; common ratio: $r = 1.5$

   (d) Next three terms: 78.72, −31.488, 12.5952; common ratio: $r = -\frac{2}{5}$

2. (a) The sequence formed by the total amount of money you raise is an arithmetic sequence. The common difference is 3, because you raise \$3 for each lap around the track.

   (b) The sequence formed by the total number of cells is geometric. The common ratio is 2, because the cells double every hour.

   (c) The sequence formed is geometric. The common ratio is $\frac{2}{3}$. (Note for teachers: Students may need help understanding that losing one-third of a quantity is the same as two-thirds remaining.)

   (d) The sequence formed by the total number of steps is arithmetic. The common difference is 12 because there are 12 steps between each floor.

UNIT 4

LESSON 4.3

Finding Terms in a Geometric Sequence

OVERVIEW

LESSON DESCRIPTION

Part 1: Investigating the Koch Snowflake
Students explore a fractal pattern and learn how to write a geometric sequence formula using any known term.

Part 2: Modeling with Geometric Sequences
Students see how to use a geometric sequence to model knocking down larger and larger dominoes.

Part 3: Summary and Practice
Students have an opportunity to practice with arithmetic and geometric sequences.

AREA OF FOCUS

- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~120 minutes

HANDOUT

Practice

- 4.3: Practice with Arithmetic and Geometric Sequences

MATERIALS

- LCD projector, electronic whiteboard, or other technology for displaying an online video to students
- access to the internet to show “Domino Chain Reaction (short version),” available at **https://youtu.be/5JCm5FY-dEY**

CONTENT FOCUS

In Lessons 4.1 and 4.2, students learned how to write a formula for a geometric sequence given the common ratio and the first term in the sequence. The goal of this lesson is for students to develop a general geometric sequence formula that uses the common ratio and any known term. This lesson mirrors the conceptual development of a general sequence formula for arithmetic sequences (Lesson 1.5). Students have an opportunity to connect geometric sequences with their prior knowledge of arithmetic sequences. By the end of this lesson set, students should understand that they can find any term in an arithmetic or geometric sequence if they know one term in the sequence and either the common difference or the common ratio, respectively. Students also learn that a geometric sequence can be expressed as $s_n = s_k \bullet r^{n-k}$, where s_n represents the nth term in the sequence and s_k represents any known term in the sequence. The focus for students should be on applying their knowledge of geometric sequences in flexible ways rather than on memorizing formulas associated with geometric sequences.

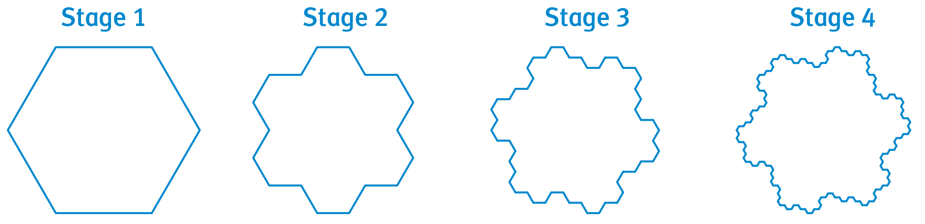
COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ An exponential function has constant multiplicative growth or decay.	
Learning Objectives	**Essential Knowledge**
4.3.1 Determine whether a relationship is exponential or nonexponential based on a numerical sequence whose indices increase by a constant amount.	**4.3.1a** A geometric sequence is an exponential relationship whose domain consists of consecutive integers. **4.3.1b** The ratios of successive terms of a geometric sequence are equivalent. **4.3.1c** A geometric sequence can be determined from the common ratio and any term in the sequence.
4.3.2 Convert a given representation of a geometric sequence to another representation of the geometric sequence.	**4.3.2b** Successive terms in a geometric sequence are obtained by multiplying the previous term by the common ratio. To find the value of the term that occurs n terms after a specified term, multiply the specified term by the common ratio n times. **4.3.2c** A geometric sequence can be algebraically expressed with the formula $a_n = a_k \bullet c^{(n-k)}$, where a_n is the nth term, a_k is the kth term, and c is the common ratio between successive terms. **4.3.2d** A verbal representation of a geometric sequence describes a discrete domain and a constant multiplicative growth or decay.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

The fractal known as Gosper Island is a variation on the Koch Snowflake. Instead of an equilateral triangle in stage 1, the Gosper Island fractal begins with a regular hexagon. The first four stages of Gosper Island are shown below:



(a) Does the number of line segments at each stage of the Gosper Island fractal form a geometric sequence? Why or why not?

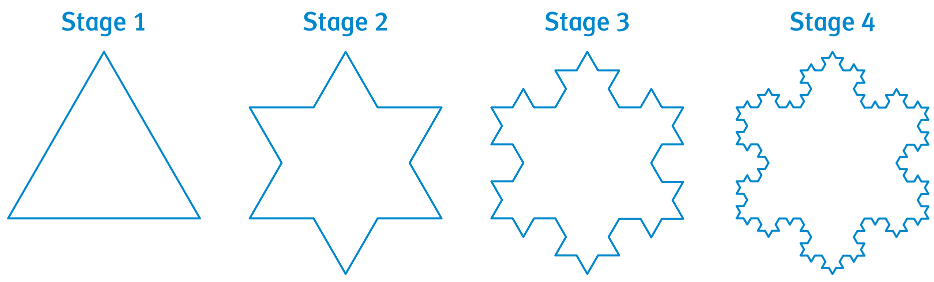
(b) How many line segments would you expect in the 5th and 6th stages of the fractal? Explain your reasoning.

(c) Write an expression in sequence notation that you could use to determine the number of line segments in the 10th stage of the fractal.

PART 1: INVESTIGATING THE KOCH SNOWFLAKE

In this part of the lesson, students continue their exploration of the Koch curve fractal from Lesson 4.2. When the Koch curve pattern is applied to an equilateral triangle, the result is called the "Koch snowflake." Using this fractal, students can develop a formula for a geometric sequence using the value of any stage as the known value. This process should feel familiar to students because it is modeled on the arithmetic sequence process developed in Unit 1.

- To begin, display the following image of the Koch snowflake for the whole class to see. It is a slightly modified version of the curve used in the fractal investigations in Lesson 4.2. Ask students to closely observe the image and share their observations with the class.



- Once students have had time to share their observations, have them look at the number of line segments in each stage. Ask them some questions to get them thinking about the fractal:
  - How many line segments are there at each of the four stages? Does the sequence of total line segments at each stage form a geometric sequence? How do you know?

    There are 3, 12, 48, and 192 line segments in the first four stages. This does form a geometric sequence, because consecutive terms can be found by multiplying by 4, the common ratio of this geometric sequence.
  - Determine the number of line segments in the 5th and 6th stages.

    The 5th stage has 768 line segments, and the 6th stage has 3,072 line segments.
  - Since we know that this is a sequence, we can give the sequence a name and write the number of line segments at each stage using sequence notation. How would we write, "The number of line segments in stage 4 is 192"?

    We can call the sequence s_n. Then, "The number of line segments in stage 4 is 192" can be expressed as $s_4 = 192$.
- Have students construct a table of values organized by stage number that contains an expression for the term, the number of line segments, and a column to describe the calculations used to get each term. An example is shown below.
- Circulate around the room while students work and encourage them to look for patterns in the calculations. You may want to suggest that students use an expanded calculation to show how many times they are multiplying by 4 to get the number of line segments, if they are not already doing so.
- After students have had sufficient time to work on their tables, ask them to add a new column called "Exponent Form" and express the calculations using exponents. The following table is an example of a student response.

Meeting Learners' Needs

If students struggle to count the line segments, direct them to the Koch curve from the previous lesson (see **Handout 4.2.A: Fractal Investigations**). Ask them to make some observations about the relationship between the Koch snowflake and Koch curve. Students will recognize that the snowflake is three curves put together to make a triangle-like shape.

Stage Number	Term	Line Segments	Calculation	Exponent Form
1	s_1	3	3	$3 \cdot 4^0$
2	s_2	12	$3 \cdot 4$	$3 \cdot 4^1$
3	s_3	48	$3 \cdot 4 \cdot 4$	$3 \cdot 4^2$
4	s_4	192	$3 \cdot 4 \cdot 4 \cdot 4$	$3 \cdot 4^3$
5	s_5	768	$3 \cdot 4 \cdot 4 \cdot 4 \cdot 4$	$3 \cdot 4^4$
6	s_6	3,072	$3 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$	$3 \cdot 4^5$

- Now that students have a table of values with calculations and exponents, they should be ready to write a formula for the sequence. If students need support, you can use these guiding prompts:
  - What patterns do you see in the table? In the exponent form, what is the same in each row and what is different?

    Each row is 4 times the previous row. In the exponent form column, each row has "$3 \cdot 4$," but the exponent is different.
  - What connection can we make between the stage number and the exponent for 4?

    The exponent for 4 is always one less than the stage number.
  - What could be a formula for the sequence s_n?

    A formula for the sequence is $s_n = 3(4)^{n-1}$, where n is the stage number of the fractal.

This next portion of the lesson should help students begin to understand that they can write a formula for a geometric sequence using any known term and the common ratio. They also have opportunities to make connections with arithmetic sequences, which can be written using any known term and the common difference.

- Ask students some questions to get them thinking about the relationship between the term number (the stage of the Koch snowflake) and the number of times they multiply by 4. This series of questions will help students understand that using the first term to write the formula for a geometric sequence is a convenient choice, not a necessary one.
  - Suppose we start at the 3rd term and we multiply by four 2 times. What term would we get to? How do you know?

    If we multiply the 3rd term by four 2 times, we would be at the 5th term. We can see from the table of values that the 5th row is two rows away from the 3rd row, and multiplying by 4 "moves" you one row down.
  - If we start at the 2nd term and multiply by four 17 times, what term would we get? How can we figure it out?

    If we multiply the 2nd term by the common ratio 17 times, we should get the 19th term. We can figure it out because $2+17=19$.
  - Suppose we start with the 4th term and we want to get to the 10th term. How many times should we multiply by 4? How can we be sure?

    You would have to multiply the 4th term by four 6 times to get to the 10th term. We know because the 10th row would be 6 rows under the 4th row. We know it must be 6 because $10-4=6$.
  - If we wanted to get from the 5th term to the 78th term without having to calculate all the terms in between, how many times do you think we should multiply by 4? How did you figure it out?

    You should multiply the 5th term by four 73 times. We can subtract the term numbers, $78-5=73$, to determine the number of multiplications needed.
  - In general, if we know a specific term and we want a different term, how many times should we multiply the term we know by the common ratio to get the term we want? (What should be the exponent of the common ratio?)

    We should figure out the difference between the term number we know and the term number we want, and then multiply the term we know by the common ratio that many times. The exponent of the common ratio is the difference between the term numbers.

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Guiding Student Thinking

If students do not understand why subtracting the term numbers tells you how many multiplications there are between two terms, it could help to show the multiplications underneath the terms. The example here shows the multiplications needed to get from the 3rd term to the 10th term in a geometric sequence.

$$s_3 \xrightarrow{\times r} s_4 \xrightarrow{\times r} s_5 \xrightarrow{\times r} s_6 \xrightarrow{\times r} s_7 \xrightarrow{\times r} s_8 \xrightarrow{\times r} s_9 \xrightarrow{\times r} s_{10} = s_3 \cdot r^7$$

Students must count how many "$\times r$" there are. Because this is only a counting problem, a faster way to do it is to subtract the term numbers.

- You can summarize the discussion with your students by writing out these verbal explanations with the accompanying symbolic notation. The following table is similar to the summary table for arithmetic sequences used in Lesson 1.5.

Verbal Explanation	Symbolic Notation
1. Figure out how far apart the term you want (s_w) is from the term you know (s_k) by subtracting the term numbers.	$w-k$
2. Multiply the common ratio (r) by itself $w-k$ number of times. We can express this by using an exponent of $w-k$ for the common ratio. The quantity $w-k$ is how many ratios there are between the term you want (s_w) and the term you know (s_k).	r^{w-k}
3. Multiply that exponent expression and the term you know (s_k) to get the term you want (s_w).	$s_w = s_k(r)^{w-k}$

Instructional Rationale

During discussion in this part of the lesson, it is more important for students to understand the verbal description than the formula. Students should understand that the formula is an abbreviation of the process; if they understand the reasoning behind the formula, they can recreate the formula when they need it.

PART 2: MODELING WITH GEOMETRIC SEQUENCES

In this part of the lesson, students use their knowledge of geometric sequences to answer the question: How many dominos would it take to knock over a domino that is as tall as the Empire State Building?

- Begin by playing a short (approximately 1 minute) YouTube video, "Domino Chain Reaction (short version)," by Stephen Morris of the University of Toronto. It can be accessed at **https://youtu.be/5JCm5FY-dEY**. After the video is over, ask students to summarize some of the important information in it. You might need to show the video a second time. Record the important information for the whole class to see.

  You can expect students to respond with facts, such as, "The first domino is 5 mm tall," or "The 13th domino is 1 meter tall and weighs more than 100 pounds." Some other important statements are that each domino is approximately 1.5 times larger than the previous one and that the 29th domino would be as tall as the Empire State Building.

- Take a moment to explain that while "largeness" could refer to height, width, length, area, volume, or even mass, we will focus only on height. So when the speaker in the video says that each domino is 1.5 times as large as the previous one, we will assume each domino is 1.5 times as tall as the previous one.
- Ask students some questions to get them thinking about using a geometric sequence to model the domino heights:
  - The speaker says that each domino is 1.5 times as large as the previous one. We know that the first domino is 5 mm tall. How could we determine the height of the second domino?

    We should multiply 5 mm by 1.5 to get 7.5 mm.

  - How would we calculate the height of the third domino?

    We should multiply 7.5 by 1.5 to get 11.25 mm. Or, you could multiply 5 mm by 1.5 two times.

  - If we calculate the heights of dominoes by repeatedly multiplying, how can we model the domino heights?

    We can use a geometric sequence.

  - What is the common ratio of the geometric sequence? How do you know?

    The common ratio is 1.5, because that is what we should multiply by to get the next domino's height.

- Now have students use what they know about the first three terms in the domino height sequence and the common ratio 1.5 to determine the height of the 13th domino. Encourage different students to use different known terms or to verify

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their calculations by trying a different known term. In each case, the term they want to calculate is the 13th term.

Students should name the sequence and write one of these equations, using the correct notation, to calculate the height of the 13th domino: $h_{13} = 5(1.5)^{12}$, $h_{13} = 7.5(1.5)^{11}$, or $h_{13} = 11.25(1.5)^{10}$. In each case, the 13th domino is approximately 648.7 mm tall.

- At this point, remind students about the claims in the video. (It might be beneficial to watch the video again.) Point out that early in the video the speaker states that the 13th domino is "more than a meter tall." Ask students these questions:
  - Is the speaker correct? Based on the information we have, is the 13th domino 1 meter tall?

    According to our work, the 13th domino is a little more than half a meter tall.
  - What could account for why our numbers seem to contradict what the speaker says?

    Students will make varied suggestions. One possible reason is our choice to focus on height only rather than other aspects of size. Or, the speaker's estimate of "about" 1.5 times as large could be a good bit smaller than the actual ratio.
  - What is the last thing the speaker said about his chain of dominos?

    The 29th domino would be as tall as the Empire State Building.
- Before having students do any calculations, have them predict whether or not the 29th domino will be as tall as the Empire State Building. Record all predictions, and have students give a reason or rationale for their choice. Then ask these questions:
  - Using what we know, what will be the height of the 29th domino in this chain?

    The 29th domino is approximately 426,113 mm tall, or 426.113 meters tall.
  - Is the 29th domino as tall as the Empire State Building? (The Empire State Building is 381 meters without the antenna and 443 meters with the antenna.)

    The 29th domino is about as tall as the Empire State Building. The speaker was correct.

PART 3: SUMMARY AND PRACTICE

In this part of the lesson, students summarize what they know about geometric sequences and make connections with their knowledge of arithmetic sequences. They also have an opportunity to practice distinguishing between contextual scenarios that model arithmetic and geometric sequences.

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- To summarize key ideas about geometric sequences, you could have students develop a Frayer organizer; they will identify some important characteristics, develop a definition, and provide some examples and nonexamples. (See **Handout 1.2.C: Vocabulary Graphic Organizer** for a blank organizer.)
- Students can then work individually or collaboratively on the practice problems. The problem set integrates geometric and arithmetic sequences, so students can observe their similarities and differences. See **Handout 4.3: Practice with Arithmetic and Geometric Sequences** for the problem set.
- After students have completed the problems, have them list as many similarities between arithmetic and geometric sequences as they can; then you can fill in any gaps as needed. You may want to lead a class discussion to develop a table like the one shown here.

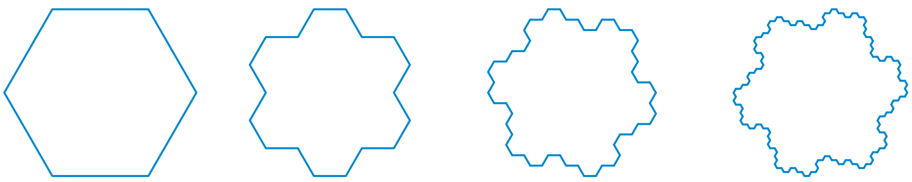
	Arithmetic Sequences	**Geometric Sequences**
How do the terms change?	▪ Common difference ▪ Calculate the common difference by subtracting two consecutive terms: $d = a_{n+1} - a_n$	▪ Common ratio ▪ Calculate the common ratio by dividing two consecutive terms: $r = \frac{a_{n+1}}{a_n}$
How do you get from one term to the next term?	Add the common difference: $a_{n+1} = a_n + d$	Multiply by the common ratio: $a_{n+1} = a_n \bullet r$
What is an explicit formula for the sequence using the first term?	The nth term in the sequence is the first term in the sequence plus $(n-1)$ differences: $a_n = a_1 + d(n-1)$	The nth term in the sequence is the first term in the sequence times $(n-1)$ ratios: $a_n = a_1 \bullet r^{(n-1)}$
What is an explicit formula for the sequence using the kth term?	The nth term in the sequence is the kth term in the sequence plus $(n-k)$ differences: $a_n = a_k + d(n-k)$	The nth term in the sequence is the kth term in the sequence times $(n-k)$ ratios: $a_n = a_k \bullet r^{(n-k)}$

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

The fractal known as Gosper Island is a variation on the Koch snowflake. Instead of an equilateral triangle in stage 1, the Gosper Island fractal begins with a regular hexagon. The first four stages of Gosper Island are shown below:



(a) Does the number of line segments at each stage of the Gosper Island fractal form a geometric sequence? Why or why not?

The number of line segments does form a geometric sequence. I know because there is a common ratio of 3. So to get the number of line segments in the next stage, I multiply the current stage by 3.

(b) How many line segments would you expect in the 5th and 6th stages of the fractal? Explain your reasoning.

I expect 486 line segments in the 5th stage and 1,458 line segments in the 6th stage. I extended the sequence of line segments from stage 4 by multiplying by 3.

(c) Write an expression in sequence notation that you could use to determine the number of line segments in the 10th stage of the fractal.

Suggested responses could include $s_{10} = 6 \bullet 3^9$ or $s_{10} = 164 \bullet 3^6$. Any correct solution has the form $s_{10} = s_k \bullet 3^{10-k}$.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 4.3: Practice with Arithmetic and Geometric Sequences

1. The 1st term in the sequence is 10. I know because the common ratio is 10. This means the 4th term would be 1,000.

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2. The common difference is 16. I know there are three differences between the 2nd and 5th terms, because 5 – 2 = 3. So, I subtracted 6 from 54 and divided by 3 to get 16. The 3rd term would be 22.
3. (a) The common ratio of the sequence is $\frac{1}{3}$. A formula for the sequence is $a_n = 4 \bullet \left(\frac{1}{3}\right)^{n-1}$.
   (b) The 15th term in the sequence can be determined this way:
   $a_{15} = 4 \bullet \left(\frac{1}{3}\right)^{14} = \frac{4}{3^{14}} = \frac{4}{4{,}782{,}969}$
4. (a) The common difference is 14.4. A formula for the sequence is $a_n = 1.8 + 14.4(n-1)$.
   (b) The 20th term in the sequence can be determined this way:
   $a_{20} = 1.8 + 14.4(19) = 1.8 + 273.6 = 275.4$
5. (a) The total number of squares at stages 1, 2, and 3 are 1, 5 (4 smaller squares and 1 larger square), and 9 (4 small squares, 4 medium squares, 1 large square).
   (b) The sequence of squares is an arithmetic sequence, because there are 4 new squares formed at each stage.
   (c) A formula to determine the total number of squares at the *n*th stage is $s_n = 1 + 4(n-1)$.
   (d) There are 37 squares in the 10th stage, because $s_{10} = 1 + 4(9) = 37$.
6. (a) The areas of the squares in stages 1, 2, 3, 4, and 5 are $\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \frac{1}{256}, \frac{1}{1{,}024}$.
   (b) The sequence of areas is a geometric sequence. There is a common ratio of $\frac{1}{4}$.
   (c) A formula for the area of a square in the *n*th stage is $A_n = \frac{1}{4}\left(\frac{1}{4}\right)^{n-1}$ or $A_n = \left(\frac{1}{4}\right)^n$.
   (d) The area of the square in the 10th stage is $A_{10} = \left(\frac{1}{4}\right)^{10} = \frac{1}{1{,}048{,}576}$.
7. (a) The height of the ball after the first time it hits the floor is $15(0.85) = 12.75$ ft.
   (b) The height of the ball after the first time it hits the floor is $15(0.85)^3 \approx 9.21$ ft.
   (c) One possible formula for the height of the ball after the *n*th time it hits the floor is $h_n = 15(0.85)^n$.
8. (a) The amount of oil in the reserves is an arithmetic sequence because the amount decreases by the same amount each year. The common difference is –8.
   (b) There will be 68 million barrels of oil in the reserves in year 5 because 76 + −8 = 68.
   (c) There will be 28 million barrels of oil in the reserves in year 10 because 68 + −8(5) = 28.

UNIT 4

LESSON 4.4

Graphing Exponential Functions

OVERVIEW

LESSON DESCRIPTION

Part 1: Graphing an Exponential Function
Students explore a contextual scenario with a continuous domain as an introduction to graphs of exponential functions.

Part 2: Matching Sequences, Functions, Descriptions, and Graphs
Students engage in a card-matching activity that will help them make connections between multiple presentations of a function.

Part 3: Exploring the Parameters of $f(x) = a \cdot c^x$
Students explore and summarize the features of exponential functions.

CONTENT FOCUS

This lesson introduces students to graphs of exponential functions through a scenario about caffeine consumption and helps students transition from using sequence notation to function notation. Because geometric sequences are exponential functions whose domain is whole numbers, students have been working with exponential functions throughout this unit. The skills they acquired in writing formulas for geometric sequences are the same ones needed to write algebraic expressions for exponential functions. The focus of this lesson is on defining exponential functions and making connections between the graph and the parameters of the algebraic form of the function.

AREA OF FOCUS

- Connections Among Multiple Representations

SUGGESTED TIMING

~135 minutes

HANDOUT

Lesson

- 4.4: Matching Sequences, Functions, Descriptions, and Graphs

MATERIALS

- graphing utility

UNIT 4

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ Exponential functions can be used to model contextual scenarios that involve constant multiplicative growth or decay. ▪ Graphs and tables can be used to estimate the solution to an equation that involves exponential expressions.	
Learning Objectives	**Essential Knowledge**
4.3.3 Determine whether a relationship is exponential by analyzing a graphical, numerical, algebraic, or verbal representation.	**4.3.3a** A graph of an exponential function is a curve that exhibits asymptotic behavior to the left or right. **4.3.3b** The numerical representation of an exponential function will have ordered pairs where, if the inputs differ by a constant amount, then the ratios of corresponding outputs are equivalent. **4.3.3c** An algebraic representation of an exponential function often takes the form $f(x) = a \bullet c, c > 0$, where c is the constant growth or decay factor.
4.4.1 Calculate a growth or decay factor of an exponential relationship.	**4.4.1a** In an exponential relationship, the output values grow by equal factors over equal intervals. **4.4.1b** Given two input–output pairs in an exponential relationship, $(a, f(a))$ and $(b, f(b))$, where b is n units more than a, the n-unit growth (or decay) factor is the quotient of the corresponding outputs, $\frac{f(b)}{f(a)}$.
4.4.2 Create graphical or numerical representations of an exponential function using the common growth or decay factor.	**4.4.2a** Given any point on the graph of an exponential function, the common growth or decay factor can be used to generate all points on the graph of the curve that contains the point.

	4.4.2b Given any input-output pair from an exponential function, the common growth or decay factor can be used to generate all other pairs of values that satisfy the relationship. **4.4.2c** If the relationship represented in a table of values has a common ratio in outputs over equal differences of inputs, then the points on the associated graph will lie on a curve that is asymptotic to the left or the right. **4.4.2d** If the output values change by a factor of c when the input values differ by 1, then the output values change by a factor of c^k when the input values differ by k, where k is a real number.
4.4.3 Convert a given representation of an exponential function to another representation of the exponential function.	**4.4.3a** The graphical representation of an exponential function displays ordered pairs that satisfy the relationship. The exact coordinates of the ordered pairs may or may not be evident from the graph of the function. **4.4.3b** A numerical representation of an exponential function consists of only a subset of the ordered pairs that satisfy the relationship, but can be used to compute the common ratio of growth or decay and determine all other aspects of the exponential function. **4.4.3c** An algebraic representation of an exponential function contains the complete information about the function because any output value can be determined from any given input value. **4.4.3d** A verbal representation of an exponential function describes the common ratio of growth or decay and/or known input–output pairs from the function.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

We know that newer computers are faster than older computers. One reason for this is that the number of transistors per computer microchip tends to increase over time as technology progresses. Gordon Moore, a cofounder of Intel, observed that the number of transistors seemed to double every 2 years. This observation has come to be called "Moore's law." In 1971, Intel produced a computer that had approximately 2,000 transistors.

(a) Make a table of values and sketch a graph for the number of transistors per microchip that Moore's law predicted for 6 years after 1971.

(b) Write an algebraic function for Moore's law.

(c) If Moore's law has held since 1971, how many transistors per microchip would you expect in a computer produced in 2015?

PART 1: GRAPHING AN EXPONENTIAL FUNCTION

In this part of the lesson, students explore a contextual scenario that exhibits multiplicative decay. The problem, about the biological half-life of caffeine, shares some common features with geometric sequence problems (Lessons 4.2 and 4.3); however, unlike geometric sequences, the function in the caffeine problem has a continuous domain. Students first construct a table of values for this scenario and then plot the points. The resulting shape graphically distinguishes it from linear and quadratic functions. Students use the graph and table of values together to develop an algebraic exponential function and then formally define exponential functions.

Classroom Ideas

You may wish to use this contextual scenario to prompt students to evaluate their own caffeine intake. Some students who consume caffeine may not realize that it remains in the body for several hours.

- To begin, display this scenario for the whole class. Give students time to read the scenario and form some questions about it. Then ask students what they notice or wonder about the scenario; record all student observations and questions for the whole class to see.

> Mr. Java drinks a 20-ounce coffee every morning. The biological half-life of caffeine is 5 hours, which means the amount of caffeine in the body is reduced by 50 percent every 5 hours. Mr. Java's coffee contains 330 milligrams (mg) of caffeine.

Students may notice the size of the drink, the half-life of caffeine, and the quantity of caffeine in the coffee. Students may wonder how long it takes him to drink the coffee, how much time it takes for the caffeine to be eliminated from his system, or if these are real values for coffee.

- If students do not ask about how long it takes for the caffeine to be eliminated from Mr. Java's body, prompt them to think about it. Let students know that for the sake of making the problem a little easier, we will assume that Mr. Java drinks the entire coffee instantly. Ask students some questions to get them thinking about the problem:
  - Suppose we wanted to know how much caffeine was remaining in Mr. Java's system 8 hours after he drank the coffee. How could we figure it out?

    Students may suggest making a table of values, making a graph, or writing a formula for the scenario.

    If students do not suggest creating a table of values, guide them in that direction.

- If we were going to make a table of values for the amount of caffeine in Mr. Java's system, what information should we include? What do we know about the scenario?

  We should include the time in hours and the milligrams of caffeine. We know that there is 330 mg of caffeine at time 0 and half of 330 mg 5 hours later.

- Have students work in pairs to construct a table of values for the scenario. Circulate around the room while they work. Students may struggle to determine the amount of caffeine after 1 hour; explain that it is not necessary to calculate that amount at this point. Instead, suggest that they make a table of values using increments of 5 hours, since the amount of caffeine will be easy to figure out.

Hours after Drinking Coffee	Milligrams of Caffeine
0	330
5	165
10	82.5
15	41.25
20	20.625
25	10.313

- After student pairs have completed their tables, like the one shown above, construct a table for the whole class to see. Once students have agreed on the table of values, ask:
  - Does the sequence of milligrams of caffeine in Mr. Java's body form a geometric sequence? How do you know?

    It does look like a geometric sequence, because the amount of caffeine at each 5-hour increment is half the previous amount.

  - How is this sequence the same as, and different from, the other geometric sequences we've explored?

    Usually the increments in a geometric sequence are by 1. These increments are by 5.

Guiding Student Thinking

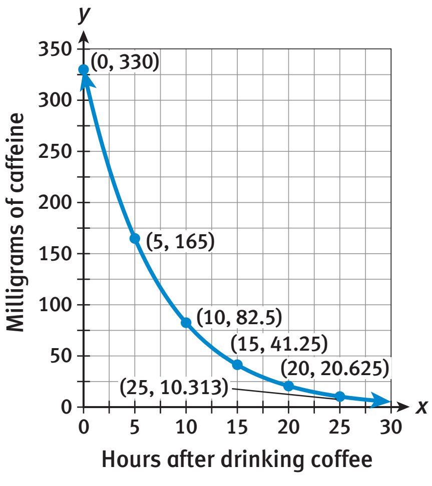
Even though the increments of the time variable are not 1, students should recognize that there is a multiplicative decay occurring. If students ask about writing an equation for the scenario, let them know that they will do that later in the lesson.

- Now have students work in pairs again to plot the points from the table of values on a coordinate grid. After students have completed their graphs, ask them what they notice or wonder about the graph.

  Students will probably point out that the y-values start fairly large and decrease quickly; graphically, the vertical distance between the points gets smaller while the horizontal distance stays the same; the graph is not linear because there is not a constant rate of change.

- If students conjecture that the graph might be a parabola, have them test their conjecture by calculating the second differences of the sequence. Since the second differences are not constant, they will be able to conclude that the sequence is not quadratic. This is a good opportunity to introduce students to exponential functions, a new class of functions that can model the phenomenon of caffeine decrease in Mr. Java's system.
- If students do not ask about the kind of function, lead them through a discussion to help them reach the conclusion that the sequence is neither linear nor quadratic. First have students sketch a curve that connects the points that they plotted, then ask:
  - What do you notice about the curve? What do you wonder about the curve?

    Students should have a curve that looks something like the graph shown. Students may notice that there does not seem to be a vertex, as with parabolas, or that the curve seems to get very close to the x-axis. They may wonder if the curve ever intersects the x-axis.



- Can you estimate how many milligrams of caffeine are in Mr. Java's body after 1 hour? What do you think it will be?

  Mr. Java should have around 285 mg of caffeine 1 hour after drinking the coffee.

- Suppose that Mr. Java has about 50 mg of caffeine in his system. Can you estimate how many hours have passed since he drank the coffee?

  Approximately 13.5 hours have passed since he drank the coffee.

- Once students have determined that the graph is neither linear nor quadratic, explain that there is another class of functions that can be used to model the phenomenon of caffeine decrease in Mr. Java's system. Let students know that this kind of function is called an *exponential function*. You can provide this initial definition:

  Exponential functions grow (or decay) by equal factors over equal intervals.

Guiding Student Thinking

If students find the definition of exponential functions difficult to understand, you can compare it to this definition of linear functions: *Linear functions grow by equal differences over equal intervals*. Exponential functions have multiplicative growth while linear functions have constant growth. One thing that distinguishes these two classes of functions is the kind of growth they exhibit.

- At this point, students are ready to write a formula for the caffeine decay scenario. Ask students some guiding questions to get them to recall or uncover the important details of the problem:

  - Did the amount of caffeine in Mr. Java's system look like a geometric sequence? How did we know?

    The amount of caffeine did look like a geometric sequence because there was a common ratio between the terms, but the term numbers had increments of 5 instead of 1.

    What was the common ratio? How do we know? What is the length of the interval between the terms?

    The common ratio is $\frac{1}{2}$, because the amount of caffeine is reduced by half every 5 hours. So, the interval between the terms is 5.

- Using what we know about geometric sequences, how would we start to write a formula for the sequence?

  We need an amount of caffeine. We can use the fact that at time 0, Mr. Java has 330 mg in his system. We will also need the common ratio. Our formula should look something like $c_n = 330\left(\frac{1}{2}\right)^n$.

Instructional Rationale

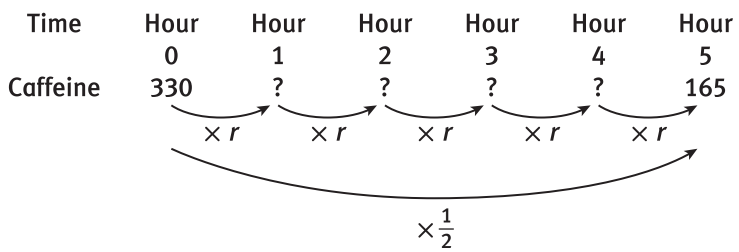
Students can use any pair of hours and milligrams of caffeine that they have in the table. Since they were used to using the first term in the sequence to write formulas in Lessons 4.2 and 4.3, they may be more comfortable writing a formula such as $c_n = 165\left(\frac{1}{2}\right)^{n-5}$. This would also be a good starting formula. If students seem confused by the formula $c_n = 330\left(\frac{1}{2}\right)^n$, you can show them that it follows the usual format because it could be written as $c_n = 330\left(\frac{1}{2}\right)^{n-0}$, since 330 is the 0th term.

- The sequence formula is not quite correct yet. Give students an opportunity to point this out by having them investigate whether the formula gives the correct values by substituting 5 or 10 into the formula and observing the outputs.
- After students have identified a problem with the formula, ask them the following question:
  - What is missing from the formula? What important information did we not use yet?

    We did not use the information that the interval for the ratio was 5, not 1.
- Let students know that a formula that correctly expresses the relationship between the amount of caffeine and the time will need to incorporate the “5.” Give students some time to experiment with different ways to incorporate it. For students who are hesitant to begin, you can suggest that they try to use the 5 in the exponent somehow. When students make a conjecture about how to incorporate the 5, make sure they test their conjecture by substituting an hour from the table of values into their new formula and comparing the output to the known output.
- As students close in on the correct formula of $c_n = 330\left(\frac{1}{2}\right)^{\frac{n}{5}}$, suggest that they graph their function, using graphing technology, and compare it to their original plotted points.

UNIT 4

- Students are likely to wonder why dividing the exponent by 5 produces the correct result. Show them a diagram like the one below to help them visualize what's happening. We know that the ratio of $\frac{1}{2}$ is multiplied once from hour 0 to hour 5. If this were a normal geometric sequence, there would be a ratio, r, between consecutive terms. That ratio, r, has to be multiplied by itself five times to equal one multiplication of $\frac{1}{2}$, or $(r)^5 = \left(\frac{1}{2}\right)^1$.



- You can then show students that one way to get the value of r is by dividing the exponent by 5:

$$(r)^5 = \left(\frac{1}{2}\right)^1$$

$$(r)^{\frac{5}{5}} = \left(\frac{1}{2}\right)^{\frac{1}{5}}$$

$$(r)^1 = \left(\frac{1}{2}\right)^{\frac{1}{5}}$$

These manipulations show that one multiplication by r is the same as one multiplication by $\left(\frac{1}{2}\right)^{\frac{1}{5}}$.

Guiding Student Thinking

You may also want to show students that dividing hours 0, 5, 10, 15, 20, and 25 by 5 will give the values 0, 1, 2, 3, 4, and 5. Since the hours are the exponent in the formula, dividing the exponent by 5 will give the usual inputs 0, 1, 2, 3, 4, and 5.

- Explain to students that because the caffeine scenario is an exponential function, not just a geometric sequence, we can write the formula using function notation: $c(t)=330\left(\frac{1}{2}\right)^{\frac{t}{5}}$, where t is the time in hours since the coffee was consumed, and $c(t)$ is the amount of caffeine remaining in Mr. Java's system.
- At this point you can expand the definition of exponential functions that you gave students earlier by incorporating the algebraic form of an exponential function:

  Exponential functions grow (or decay) by equal factors over equal intervals. Exponential functions have the form $f(x)=a \bullet c^x$*, where* c *is the common ratio.*

PART 2: MATCHING SEQUENCES, FUNCTIONS, DESCRIPTIONS, AND GRAPHS

In this part of the lesson, students deepen their understanding of the various ways a function can be presented by matching geometric sequences, exponential functions, graphs, and verbal descriptions that model the same example.

- You can assign students to pairs or have them complete this part of the lesson individually. See **Handout 4.4: Matching Sequences, Functions, Descriptions, and Graphs**. Circulate around the room as students work, providing support as necessary. Answers are provided in the Assess and Reflect section of the lesson.
- As students get close to finishing, prompt them to analyze their matches by asking some probing questions:
  - Which sequences are increasing? Which sequences are decreasing?

    The increasing sequences are 3, 4, and 5. The decreasing sequences are 1, 2, and 6.
  - What is the common ratio of each sequence? How are the ratios of increasing and decreasing sequences different?

    The ratios are S1, $r=0.5$; S2, $r=0.25$; S3, $r=1.2$; S4, $r=1.015$; S5, $r=1.05$; S6, $r=0.85$. Increasing sequences have a ratio that is greater than 1, while decreasing sequences have a ratio that is between 0 and 1.
  - Which graphs are increasing and which are decreasing?

    The increasing graphs are 2, 3, and 5. The decreasing graphs are 1, 4, and 6.

- What is the common ratio of each graph? How are the increasing and decreasing graphs different?

  The ratios are G1, $r = 0.25$; G2, $r = 1.015$; G3, $r = 1.05$; G4, $r = 0.85$; G5, $r = 1.2$; G6, $r = 0.5$. Increasing graphs have a ratio that is greater than 1. Decreasing graphs have a ratio between 0 and 1.

- What are the y-intercepts of the graphs? Do the y-intercepts of the graphs appear in the algebraic form of the function?

  The y-intercept for graphs 2, 3, 4, and 6 is $(0, 50)$. For graph 1 it is $(0, 60)$. For graph 5 it is $(0, 25)$. The y-intercepts are in the formula as the parameter a in the form $f(x) = a \bullet c^x$.

Instructional Rationale

It is not necessary for students to correctly answer all the questions. What is most important here is that students start to look for connections between the graphs and the parameters of the algebraic form of the function. They will explore these connections more completely in the final part of the lesson.

- Summarize for students that there are two different types of exponential functions: growth and decay. When the ratio is greater than 1, the exponential function exhibits growth. When the ratio is less than 1 but greater than 0, it exhibits decay.

Guiding Student Thinking

Students may be tempted to think about the graphs of exponential growth scenarios as rising from left to right and the graphs of exponential decay scenarios as falling from left to right. It is important for students to observe that over equal horizontal intervals, the graphs of exponential growth show an increase in vertical distance between points and the graphs of exponential decay show a decrease in vertical distance between points. This is an important distinction to make when the a parameter of $f(x) = a \bullet c^x$ is negative. This will be explored in more detail in Part 3.

PART 3: EXPLORING THE PARAMETERS OF $f(x) = a \bullet c^x$

In this part of the lesson, students work on identifying examples of exponential growth and decay by analyzing the algebraic presentation of the function and apply their understanding of the definition of exponential functions, properties of the graphs, and parameters of the algebraic presentation of the function. This part of the lesson synthesizes what students have explored in the lesson so far.

- This is a good time to summarize what students know so far about exponential functions. You can ask some questions to elicit the important information:
  - If an exponential function is described as "growth," what do you know about the graph? What do you know about the ratio?

    The vertical distance between two points will increase as the horizontal distances stay constant. The ratio will be greater than 1.
  - If an exponential function is described as "decay," what do you know about the graph? What do you know about the ratio?

    The vertical distance between two points will decrease as the horizontal distances stay constant. The ratio will be greater than 0 but less than 1.
  - In an exponential function $f(x)=a \bullet c^x$, which parameter is the ratio?

    The ratio is c.
  - Based on the previous part of this lesson, what do we expect the parameter a to be?

    The parameter a should be the y-intercept of the graph.
- Display the three functions below for all students to see:

$f(x)=\frac{1}{2}(3)^x \qquad g(x)=2\left(\frac{4}{3}\right)^x \qquad h(x)=-1(2)^x$

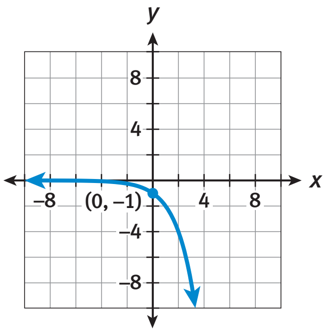
- Lead a whole-class discussion using the questions below. Before students turn to a graphing utility, ask them to analyze the functions using what they know from the matching activity. (It is okay if students are incorrect right now because they will have an opportunity to graph the functions and check their answers.) Record all answers for the class to see, especially if there is disagreement or uncertainty among the students.
  - Are these functions examples of growth or decay? How do you know?

    All three functions are examples of growth, because the ratios are larger than 1.
  - Which of these functions would you expect to grow the fastest? Explain your reasoning?

    Function f should grow the fastest, because its ratio is the greatest.
  - Where do you expect the graph of each function to cross the y-axis?

    Function f will have a y-intercept of $\left(0,\frac{1}{2}\right)$. Function g will have a y-intercept of $(0,2)$. Function h will have a y-intercept of $(0,-1)$.

- Once students have answered the questions, have them use a graphing utility to graph the three functions to check their answers. Have students correct their initial answers if they were incorrect.
- Students will likely observe that the graph for function h is dissimilar to the graphs in the matching activity. The graph of h, shown here, is decreasing but still exhibits typical growth behavior: Vertical distances between points are increasing over equal horizontal intervals. Point out to students that here the a parameter is negative.



- Ask students what else they notice about the graphs of the functions. You can use these questions to focus their attention on the end behavior of the graphs:
  - As the x-values become more negative, what happens to the y-values?

    The y-values get closer to 0.
  - As the x-values become more positive, what happens to the y-values?

    The y-values get farther away from the x-axis. (They grow without bound in the positive or negative direction.)
- Before summarizing these observations for students, have them analyze three more functions. Then lead them through a whole-class discussion by asking the following questions. Once again, have students analyze the functions before graphing them.

  $f(x) = 5(0.9)^x$ $\quad$ $g(x) = 1\left(\frac{1}{4}\right)^x$ $\quad$ $h(x) = -2\left(\frac{2}{3}\right)^x$
  - Are these functions examples of growth or decay? How do you know?

    All three functions are examples of decay, because the ratios are greater than 0 but less than 1.

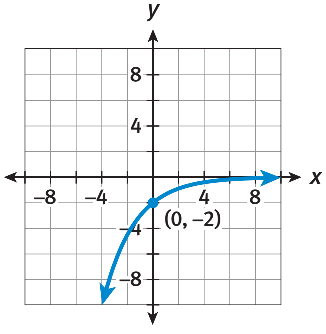
- Which of these functions would you expect to decay the fastest? Why do you think that?

  Function g should decay the fastest, because its ratio is the smallest.

- Where do you expect the graph of each function to cross the y-axis?

  Function f will have a y-intercept of $(0, 5)$. Function g will have a y-intercept of $(0, 1)$. Function h will have a y-intercept of $(0, -2)$.

- With a graphing utility, now have students graph the three functions to check their answers. As before, have students correct their initial answers if they were incorrect.
- Students will likely observe that the graph for function h is dissimilar to the graphs in the matching activity. The graph of h, shown here, is increasing but still exhibits the typical decay behavior: Vertical distances between points decrease over equal horizontal intervals.



- Ask students what else they notice about the graphs of the functions. You can use these questions to focus their attention on the end behavior of the graphs:
  - As the x-values become more negative, what happens to the y-values?

    The y-values get farther away from the x-axis. (They grow without bound in the positive or negative direction.)

  - As the x-values become more positive, what happens to the y-values?

    The y-values get closer to 0.

- At this point, you can have students create a Frayer vocabulary organizer for exponential functions. (See **Handout 1.2.C: Vocabulary Graphic Organizer** for a blank organizer.) Have students summarize the important features of exponential growth and decay functions and of their graphs. The table shown includes both similarities and differences between the functions.

	Exponential Growth	Exponential Decay
Algebraic function	$f(x)=a \bullet c^x$	$f(x)=a \bullet c^x$
Value for c	Greater than 1	Greater than 0, less than 1
y-intercept	$(0, a)$	$(0, a)$
Graph as x gets more positive	Increased distance from x-axis (grows without bound)	Gets closer to 0
Graph as x gets more negative	Gets closer to 0	Increased distance from x-axis (grows without bound)
Description of change	Vertical intervals increase over equal horizontal intervals	Vertical intervals decrease over equal horizontal intervals

UNIT 4

ASSESS AND REFLECT ON THE LESSON

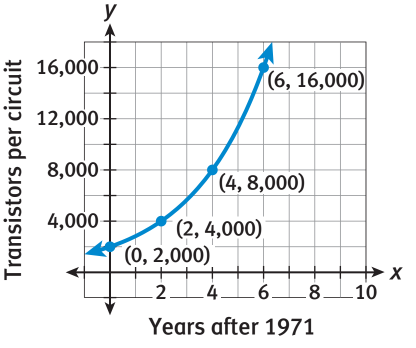
FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

We know that newer computers are faster than older computers. One reason for this is that the number of transistors per computer microchip tends to increase over time, as technology progresses. Gordon Moore, a cofounder of Intel, observed that the number of transistors seemed to double every 2 years. This observation has come to be called "Moore's law." In 1971, Intel produced a computer that had approximately 2,000 transistors.

(a) Make a table of values and sketch a graph for the number of transistors per microchip that Moore's law predicted for 6 years after 1971.

Years After 1971	Transistors per Microchip
0	2,000
2	4,000
4	8,000
6	16,000



(b) Write an algebraic function for Moore's law.

$M(t) = 2{,}000 \bullet 2^{\frac{t}{2}}$, where t represents the years since 1971 and $M(t)$ represents the number of transistors per circuit.

(c) If Moore's law has held since 1971, how many transistors per microchip would you expect in a computer produced in 2015?

If Moore's law has held, then I would expect there to be $M(44) = 2{,}000 \bullet 2^{\frac{44}{2}} = 8{,}388{,}608{,}000$ transistors. The input is 44, because $2015 - 1971 = 44$.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 4.4: Matching Sequences, Functions, Descriptions, and Graphs

Sequence 1, Function 2, Description 3, Graph 6

Sequence 2, Function 3, Description 6, Graph 1

Sequence 3, Function 1, Description 4, Graph 5

Sequence 4, Function 6, Description 5, Graph 2

Sequence 5, Function 4, Description 1, Graph 3

Sequence 6, Function 5, Description 2, Graph 4

UNIT 4

LESSON 4.5

Modeling with Exponential Functions

OVERVIEW

LESSON DESCRIPTION

Part 1: Modeling Depreciation
Students develop an exponential function to model the depreciating value of money over time and use it to make predictions.

Part 2: Modeling Copying Genes
Students explore a method that geneticists commonly use to generate many copies of a gene for their laboratory experiments.

Part 3: Summary and Practice
Students have an opportunity to practice using exponential functions in modeling scenarios. They also summarize the important features of exponential functions.

CONTENT FOCUS

A primary focus of this lesson is to use exponential functions to model realistic scenarios. While exponential functions are the focus of Unit 4, it may be helpful to notice that exponential functions share many common features with linear functions. Once it is known whether the function is linear or exponential, knowing two more pieces of information makes it possible to write an equation: the slope or ratio and a point on the line/curve, or two points on the line/curve. A linear function grows by equal differences over equal intervals and an exponential function grows by equal factors over equal intervals. One algebraic form of both functions includes parameters for the y-intercept of the graph and the slope or ratio of the graph: $f(x)=mx+b$ and $f(x)=a\bullet c^x$.

In this lesson, students continue to develop flexibility in moving between the various presentations of a function: table of values, algebraic equation, graph, and verbal description. Each of these presentations is useful for investigating different aspects of a scenario; students decide which is the best tool to use in each context.

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~60 minutes

HANDOUTS

Lesson

- 4.5.A: Copying Genes

Practice

- 4.5.B: Modeling with Exponential Functions

MATERIALS

- graphing utility

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings

- An exponential function has constant multiplicative growth or decay.
- Exponential functions can be used to model contextual scenarios that involve constant multiplicative growth or decay.
- Graphs and tables can be used to estimate the solution to an equation that involves exponential expressions.

Learning Objectives	Essential Knowledge
4.4.3 Convert a given representation of an exponential function to another representation of the exponential function.	**4.4.3a** The graphical representation of an exponential function displays ordered pairs that satisfy the relationship. The exact coordinates of the ordered pairs may or may not be evident from the graph of the function. **4.4.3b** A numerical representation of an exponential function consists of only a subset of the ordered pairs that satisfy the relationship, but can be used to compute the common ratio of growth or decay and determine all other aspects of the exponential function. **4.4.3c** An algebraic representation of an exponential function contains the complete information about the function because any output value can be determined from any given input value. **4.4.3d** A verbal representation of an exponential function describes the common ratio of growth or decay and/or known input–output pairs from the function.
4.4.4 Approximate input and output values of an exponential function using representations of the exponential function.	**4.4.4a** An algebraic rule for an exponential function can be used to determine an exact output for a specified input and can be used to approximate an input for a specified output.

	4.4.4c A graph of an exponential function can be used to approximate an output for a specified input and to approximate an input for a specified output.
4.4.5 Model a contextual scenario with an exponential function.	**4.4.5a** A contextual scenario that exhibits constant multiplicative growth or decay in its outputs over equal differences in the corresponding inputs can be modeled by an exponential function. **4.4.5b** Estimated inputs and outputs for exponential functions derived from a contextual scenario can be determined free of context, but the values must be interpreted in context to be correctly understood. **4.4.5c** The estimated inputs and outputs for an exponential function derived from a context should involve the same units as the variables in the contextual scenario.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

The population of African elephants has been declining over the past century due to both expansion of human settlements and changing climate conditions. The best estimates suggest that the number of African elephants has been reduced by 60 percent over the past 100 years. In 2015, there were about 400,000 African elephants in the world.

(a) Write a function that you could use to estimate the population of African elephants in terms of hundreds of years from 2015.

(b) Based on the information in the problem, approximately how many African elephants were there in 1915? Explain how you calculated your answer.

(c) If the decrease in the African elephant population continues without any changes, in approximately how many hundreds of years would you expect to see a population of less than 1,000 elephants? Explain how you calculated your answer.

PART 1: MODELING DEPRECIATION

In this part of the lesson, students apply their knowledge of exponential functions to explore how the buying power of money depreciates over time due to inflation. Students should be able to recognize this as an exponential decay scenario because the value of money decreases by a percentage each year. The process of modeling this scenario is a familiar one for students: constructing a table of values to verify that it is an exponential function, writing an algebraic form of the function, and finally using the function to answer questions about the scenario.

- To begin, present this scenario to students:

> Have you noticed that prices seem to increase over time? This is called *inflation*. In fact, a little bit of inflation is considered good for an economy. The Federal Reserve tries to keep inflation constant, because a slowly increasing price level keeps businesses profitable. A slightly different way to describe inflation is that the buying power of your money decreases a little bit every year. In recent times, money has been worth about 98 percent of what it was worth the previous year.

- Have students investigate what would happen to the spending power of $1,000 over subsequent years by asking them these guiding questions:
    - If every year money keeps 98 percent of its worth, how much will your $1,000 be worth in 1 year?

      In 1 year, $1,000 will be worth $1{,}000 \bullet 0.98 = \$980$.
    - What about the year after that? How much will your $980 be worth?

      In the next year, the $980 will be worth $980 \bullet 0.98 = \$960.40$.

- Make a table of values that will show how much the original $1,000 is worth over 5 years.

Number of Years	Value of $1,000
0	$1,000
1	$980
2	$960.40
3	$941.19
4	$922.37
5	$903.92

- Explain how you know this is an exponential function.

  There is a common ratio between successive years. To get the value of the money, you multiply by 0.98.

- Next, ask students to write a formula for the scenario so they can determine the value of $1,000 any given number of years from now. Invite students to share the formulas they have developed. If students disagree about how the formula should be constructed, encourage them to engage in academic conversation to support their views and challenge those of their classmates.

  This scenario can be modeled by the function $V(t)=1{,}000(0.98)^t$, where t stands for the years from now and $V(t)$ stands for the value of the initial $1,000.

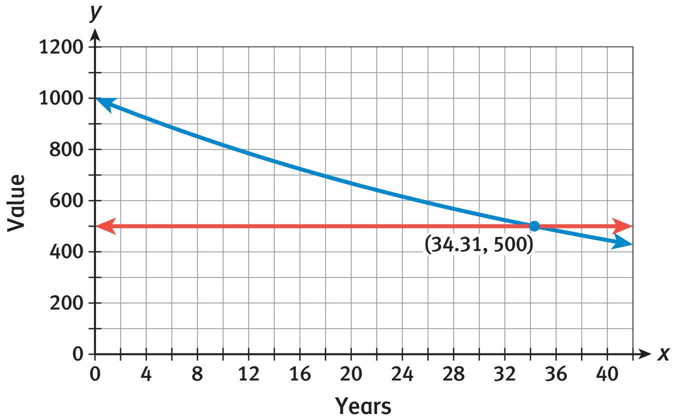
- Once students agree on a formula, prompt them to use it to answer these questions about the scenario:

  - Suppose we want to know how much our $1,000 will be worth in 20 years. Do we know an input or an output? How could we figure it out?

    We can use the function and input $t=20$ to determine the answer: $V(20)=1{,}000(0.98)^{20}\approx \$667.61$.

- Suppose we want to know approximately how many years it would take for the original $1,000 to be worth half as much. Do we know an input or an output? How could we figure it out?

  We need to know what input value will give an output of $500. We can use guess and check or we can look at a graph. As the graph shown indicates, it would take approximately 34 years for the value to depreciate to $500.



PART 2: MODELING COPYING GENES

In this part of the lesson, students apply their knowledge of exponential functions to explore a method that geneticists commonly use to generate many copies of a gene for their laboratory experiments. See **Handout 4.5.A: Copying Genes** for the student task.

- Students should be familiar with the general processes used in this part of the lesson, as the use of PCR to copy DNA is commonly taught in their biology courses. You could choose to work through the task as a whole class, or you could have students work in pairs or individually and discuss the answers when students have finished. The handout with answers is reproduced on the next two pages for your reference.

Copying Genes

Researchers working in the field of genetics perform experiments and observe specific genes and their effects on living organisms. Scientists often conduct multiple experimental trials using the same gene; therefore, they frequently need many copies of a gene. Labs usually have only a small number of copies of a gene, so researchers need a way to make a lot more copies in a small amount of time. To do this, scientists use a method called polymerase chain reaction (PCR) to turn a small number of copies of a gene into a lot of copies. In each PCR cycle, geneticists add polymerase (an enzyme) to the existing genes and quickly change the temperature, doubling the number of copies.

Suppose that a geneticist starts off with 5 copies of an important gene and wants to run several cycles of PCR.

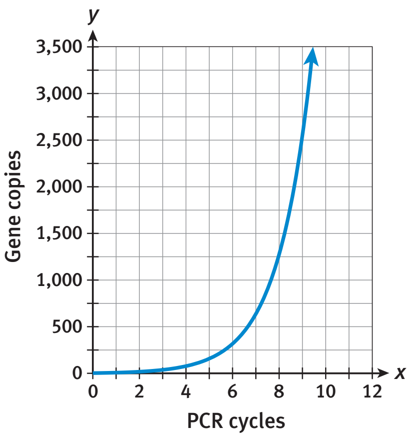
1. Write down the number of copies of the gene the scientist would have after 5 PCR cycles.

   After 5 PCR cycles, the scientist would have 10, 20, 40, 80, 160 copies of the gene.

2. Write a function that could be used to determine the number of copies of the gene the scientist would have after any number of PCR cycles. What kind of function did you use and why?

   An exponential function would be a good choice to model the number of copies of the gene after any number of PCR cycles because the quantity of copies will double after each cycle. The function is $G(c)=5(2)^c$, where c stands for the number of cycles and $G(c)$ stands for the number of gene copies created after c cycles.

3. Graph the function that you wrote for question 2.



Handout 4.5.A

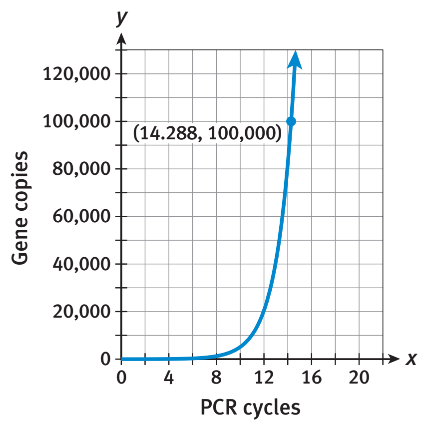
UNIT 4

4. How many copies of the gene would the scientist have after 10 PCR cycles? Show your work.

   The scientist would have $G(10) = 5(2)^{10} = 5{,}120$ copies of the gene after 10 PCR cycles.

5. Suppose the scientist needed about 100,000 copies of the gene for her experiment. How many PCR cycles should she run? How did you get your answer?

   Since the scientist needed 100,000 copies of the gene, I found $G(c) = 100{,}000$ on the graph and looked for the minimum number of cycles needed. After 14 cycles, the scientist will have 81,920 copies. After the 15th cycle, the scientist will have 163,840 copies. She would have to run 15 cycles to have enough copies of the gene.



PART 3: SUMMARY AND PRACTICE

In this part of the lesson, students have an opportunity to use exponential functions to practice modeling various contextual scenarios. See **Handout 4.5.B: Modeling with Exponential Functions** for the student task.

- In the problem set provided, there are two growth problems and two decay problems. Students can work on the problems collaboratively or individually, either in class or outside class. Answers are provided in the Assess and Reflect section.
- After students complete the problem set, lead a whole-class discussion to generate a list of important features of exponential functions. Students should be able to identify these key ideas:

- The algebraic form of an exponential function is $f(x) = a \bullet c^x$, where $(0, a)$ are the coordinates of the y-intercept and c is the common ratio.
- When the value of c is greater than 1, it is a growth function. When the value of c is less than 1 but greater than 0, it is a decay function.
- Exponential functions grow by equal factors over equal intervals.
- One half of the graph of an exponential function will approach the x-axis, while the other side gets farther away from the x-axis.
- Exponential functions are a good choice to model scenarios where there is a percent change.

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

The population of African elephants has been declining over the past century due to both expansion of human settlements and changing climate conditions. The best estimates suggest that the number of African elephants has been reduced by 60 percent over the past 100 years. In 2015, there were about 400,000 African elephants in the world.

(a) Write a function that you could use to estimate the population of African elephants in terms of hundreds of years from 2015.

A function is $P(t) = 400{,}000(0.40)^t$, where t represents the time since 2015 in hundreds of years and $P(t)$ represents the population of elephants. A reduction of 60 percent means there are only 40 percent of the elephants remaining in consecutive years.

(b) Based on the information in the problem, approximately how many African elephants were there in 1915? Explain how you calculated your answer.

There were approximately 1,000,000 elephants in 1915. I determined the answer by calculating $P(-1) = 400{,}000(0.40)^{-1}$ for the 100-year period prior to 2015.

(c) If the decrease in the African elephant population continues without any changes, in approximately how many hundreds of years would you expect to see a population of less than 1,000 elephants? Explain how you calculated your answer.

I examined a graph of the function to determine when the graph intersected the line $y = 1{,}000$. The intersection happened between $x = 6$ and $x = 7$. This means that if the population continues to decrease by 60 percent every 100 years, there will be population of 1,000 elephants in 600 to 700 years.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 4.5.B: Modeling with Exponential Functions

1. (a) A function for the amount of medication in Jasper's system is $M(t) = 200\left(\frac{1}{2}\right)^{\frac{t}{3}}$, where t is the elapsed time in hours and $M(t)$ is the milligrams of medication.

(b) There is $M(8)=200\left(\frac{1}{2}\right)^{\frac{8}{3}}\approx 31.5$ mg of medication in Jasper's system after 8 hours.

(c) By examining the graph, I looked for the hours that corresponded to 10 mg of medication in Jasper's system. This occurs about 13 hours after he took the medicine.

2. (a) There will be $15{,}000(1.01)^{12}\approx \$16{,}902$ in Minnie's account after 1 year.

(b) A function to model the account balance at the end of every month is $A(m)=15{,}000(1.01)^{m}$, where m is the months since the initial deposit and $A(m)$ is the account balance.

(c) By examining the graph of the function, I looked for what month has a balance close to $30,000. This will happen around 69 months, which is 5 years and 9 months.

3. (a) The value of the car is decreasing at a constant ratio, because $\frac{20{,}000}{15{,}000}=\frac{15{,}000}{11{,}250}=0.75$.

(b) A function that Billy could use to determine the value of his car n years after he bought the car is $V(n)=20{,}000(0.75)^{n}$.

(c) The value of the car after 10 years would be $V(10)=20{,}000(0.75)^{10}\approx \$1{,}126$. Billy should sell the car for a price close to the value of the car.

4. (a) A function that can be used to model the growth of the bacteria population in the Petri dish is $P(t)=1(3)^{\frac{t}{5}}$, where t is the time in minutes and $P(t)$ is the quantity of bacteria in the dish.

(b) If the dish was full at time $t=60$ minutes, then there would be $P(60)=1(3)^{\frac{60}{5}}=531{,}441$ bacteria in the Petri dish.

(c) The Petri dish is half full when there are about 265,720 bacteria in the dish. By examining the graph, I can find the minutes that corresponds to that amount of bacteria. This happens when $t\approx 46$, so the Petri dish is half full at 8:46 a.m.

Unit 4

Performance Task

PERFORMANCE TASK

UNIT 4

Computer-Aided Drawing

OVERVIEW

DESCRIPTION

This performance task allows students an opportunity to transfer the knowledge they have developed in recent lessons to a novel context. The task focuses on modeling a contextual scenario involving a computer-aided drawing program with an exponential function. Students can complete this task at the end of Unit 4.

CONTENT FOCUS

In this performance task, students apply their understanding of geometric sequences and exponential functions to answer questions about a novel context. The context in the task involves a percent increase in the size of an image displayed on a screen, which can be modeled by an exponential function.

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Connections Among Multiple Representations

SUGGESTED TIMING

~45 minutes

HANDOUT

- Unit 4 Performance Task: Computer-Aided Drawing

MATERIALS

- graphing utility
- calculator
- graph paper

UNIT 4

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ Exponential functions can be used to model contextual scenarios that involve constant multiplicative growth or decay.	
Learning Objectives	**Essential Knowledge**
4.4.2 Create graphical or numerical representations of an exponential function using the common growth or decay factor.	**4.4.2b** Given any input-output pair from an exponential function, the common growth or decay factor can be used to generate all other pairs of values that satisfy the relationship.
4.4.3 Convert a given representation of an exponential function to another representation of the exponential function.	**4.4.3a** The graphical representation of an exponential function displays ordered pairs that satisfy the relationship. The exact coordinates of the ordered pairs may or may not be evident from the graph of the function. **4.4.3b** A numerical representation of an exponential function consists of only a subset of the ordered pairs that satisfy the relationship, but can be used to compute the common ratio of growth or decay and determine all other aspects of the exponential function. **4.4.3c** An algebraic representation of an exponential function contains the complete information about the function because any output value can be determined from any given input value.

4.4.4 Approximate input and output values of an exponential function using representations of the exponential function.	**4.4.4a** An algebraic rule for an exponential function can be used to determine an exact output for a specified input and can be used to approximate an input for a specified output. **4.4.4b** A table of values for an exponential function can often be used to determine an exact or approximate output for a specified input or an exact or approximate input for a specified output. **4.4.4c** A graph of an exponential function can be used to approximate an output for a specified input and to approximate an input for a specified output.
4.4.5 Model a contextual scenario with an exponential function.	**4.4.5a** A contextual scenario that exhibits constant multiplicative growth or decay in its outputs over equal differences in the corresponding inputs can be modeled by an exponential function. **4.4.5b** Estimated inputs and outputs for exponential functions derived from a contextual scenario can be determined free of context, but the values must be interpreted in context to be correctly understood. **4.4.5c** The estimated inputs and outputs for an exponential function derived from a context should involve the same units as the variables in the contextual scenario.

SCORING GUIDELINES

There are 12 possible points for this performance task.

Part (a)

<table>
<tr><th>Sample Solutions</th><th>Points Possible</th></tr>
<tr><td>The display size of the door after each of the first four clicks are shown in the table:

<table>
<tr><th>Click</th><th>Display Width of Door (inches)</th></tr>
<tr><td>0</td><td>3.000</td></tr>
<tr><td>1</td><td>3.300</td></tr>
<tr><td>2</td><td>3.630</td></tr>
<tr><td>3</td><td>3.993</td></tr>
<tr><td>4</td><td>4.392</td></tr>
</table>
</td><td>3 points maximum

1 point for correctly identifying the quantities, number of clicks, and display width of the door (in inches)

1 point for correct values of first and second clicks

1 point for correct values of third and fourth clicks

Scoring note: Students do not have to include the original size (click 0) in the table of values. Students can begin their table at click 1. If students list click 1 as 3 inches, then they should receive a maximum of 2 points.

Scoring note: For the third and fourth click, if students perform calculations correctly but use incorrect values from previous table rows, award the full point.</td></tr>
<tr><th colspan="2">Targeted Feedback for Student Responses</th></tr>
<tr><td colspan="2">If students have difficulty with part (a), they may not understand that an increase of 10 percent means that the amount the length increases with each subsequent click will not be constant, because each time they zoom in, the reference amount (the width of the door) is different. Students may find that 0.3 inches is 10 percent of the initial width and then want to add 0.3 inches to the width for each subsequent click. Encourage them to determine the change in length after each subsequent click.</td></tr>
</table>

TEACHER NOTES AND REFLECTIONS

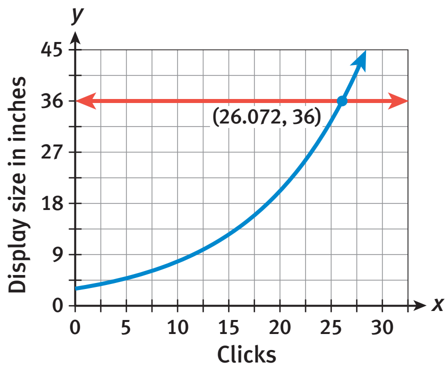
Part (b)

Sample Solutions	Points Possible
A function that will give the display size of the door for any number of clicks is $f(x)=3(1.1)^x$, where x is the number of clicks and $f(x)$ is the display width of the door in inches.	**3 points maximum** 1 point for the correct multiplier: 1.1 1 point for correct use of function notation 1 point for correctly identifying the quantities, number of clicks, and display width of the door (in inches)

Targeted Feedback for Student Responses
If students have difficulty with part (b), it could be because they are focusing on the change of 10 percent instead of the constant multiplier, 1.1. Try having students study the pattern of the outputs in part (a) to see that the constant multiplier is 1.1 and not 0.1.

TEACHER NOTES AND REFLECTIONS

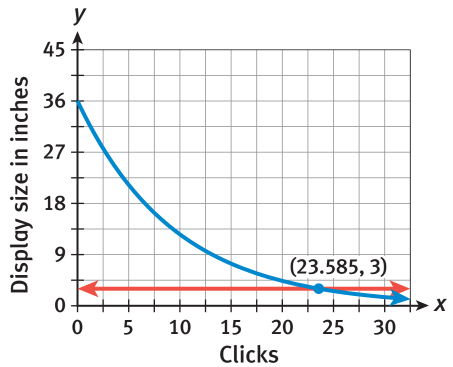
Part (c)

<table>
<tr><th>Sample Solutions</th><th>Points Possible</th></tr>
<tr><td>It would take 26 or 27 clicks to zoom in to approximately 3 feet.


I constructed a graph of the function $f(x)$ and looked to see when it would intersect with the line for 36 inches. Because the x-value that corresponds to exactly 36 inches is not a whole number, I tested how many whole-number clicks get the display closest to 36 inches. It will take 26 clicks for the display size of the door to reach 35.75 inches and 27 clicks for the display size to reach 39.330 inches. Twenty-six clicks produce a display size closer to 36 inches.</td><td>3 points maximum
1 point for identifying 26 and 27 clicks as potential solutions
1 point for a clear explanation of method
1 point for final answer of 26 clicks
Scoring note: Students should receive a maximum of 1 point for providing the answer 26.072 clicks.
Scoring note: Students should receive a maximum of 1 point for extending the table of values from part (a) by hand without explanation. Students can receive full credit for using technology to determine a numerical solution, with a correct and clear explanation.</td></tr>
<tr><th colspan="2">Targeted Feedback for Student Responses</th></tr>
<tr><td colspan="2">If students have difficulty with part (c), it could be because they have not recognized 36 as an output value (a number of inches for the width) and instead determined the width when they clicked to zoom in 36 times.</td></tr>
</table>

TEACHER NOTES AND REFLECTIONS

Part (d)

<table>
<tr><th>Sample Solutions</th><th>Points Possible</th></tr>
<tr><td>
It would take 23 or 24 clicks to zoom out from 36 inches to approximately 3 inches.


I wrote a new function for the zoom-out button, $g(x) = 36(0.9)^x$, and graphed it. I looked to see when the graph would intersect with the line for 3 inches. Because the x-value that corresponds to exactly 3 inches is not a whole number, I tested how many whole-number clicks get the display closest to 3 inches. Twenty-three clicks produce a display size of 3.191 inches, and 24 clicks produce a display size of 2.872 inches. Twenty-four clicks produce a display size closer to the original size of 3 inches.
</td><td>
3 points maximum
1 point for identifying that the multiplier is not 1.1 or for identifying the correct multiplier of 0.9
1 point for a clear and correct explanation of method
1 point for correctly identifying 23 or 24 clicks as potential solutions
</td></tr>
<tr><th colspan="2">Targeted Feedback for Student Responses</th></tr>
<tr><td colspan="2">If students have difficulty with part (d), it could be because they assume the same number of clicks needed to zoom in (36) will be needed to return to the original size. Encourage students to write a new algebraic rule to model zooming out and predict what the output of their new rule will be when the input is 36 (they will likely predict 3 inches). Then have students test this prediction to see that the width after zooming out 36 times is not 3 inches.</td></tr>
</table>

TEACHER NOTES AND REFLECTIONS

Suggested point conversion, if assigning a grade to this problem:

Points Received	Appropriate Letter Grade (if Graded)	How Students Should Interpret Their Score
11 or 12 points	A	"I know all of this algebra really well."
8 to 10 points	B	"I know all of this algebra well, but I made a few mistakes."
5 to 7 points	C	"I know some of this algebra well, but not all of it."
2 to 4 points	D	"I only know a little bit of this algebra."
0 or 1 point	F	"I don't know much of this algebra at all."

Computer-Aided Drawing

PERFORMANCE TASK

A house designer uses computer-aided drawings to illustrate new houses. She likes to show her drawings to clients on a large screen. She often uses the zoom-in function to enlarge the drawings so that clients can see certain features better.

Each click of the zoom-in button results in a 10 percent increase in the size of a drawing.

(a) On a certain drawing of a house, the width of the front door is 3 inches on screen, using the default settings. Make a table of values to show the width of the door on screen after each of the first four clicks of the zoom-in button. These values should be accurate to the thousandths place.

(b) Write an algebraic rule for the function that will give the display size of the door for any number of clicks.

(c) To show clients a detail on the front door, she needs to zoom in so the door is approximately 3 feet wide on screen. How many clicks of the zoom-in button will be needed to make this enlargement? Explain how you got your answer.

PERFORMANCE TASK

(d) Suppose one click of the zoom-out button results in a 10 percent decrease in the size of the drawing. How many clicks of the zoom-out button would it take to transform the display of the door from 3 feet wide back to a width of approximately 3 inches? Explain how you got your answer.

