CollegeBoard Pre-AP®

Pre-AP® Algebra 2

TEACHER RESOURCES

Unit 4M

Please visit Pre-AP online at **preap.org** for more information and updates about the course and program features.

ABOUT COLLEGE BOARD

College Board is a mission-driven not-for-profit organization that connects students to college success and opportunity. Founded in 1900, College Board was created to expand access to higher education. Today, the membership association is made up of over 6,000 of the world's leading educational institutions and is dedicated to promoting excellence and equity in education. Each year, College Board helps more than seven million students prepare for a successful transition to college through programs and services in college readiness and college success—including the SAT® and The Advanced Placement® Program (AP®). The organization also serves the education community through research and advocacy on behalf of students, educators, and schools.

For further information, visit **www.collegeboard.org**.

PRE-AP EQUITY AND ACCESS POLICY

College Board believes that all students deserve engaging, relevant, and challenging grade-level coursework. Access to this type of coursework increases opportunities for all students, including groups that have been traditionally underrepresented in AP and college classrooms. Therefore, the Pre-AP Program is dedicated to collaborating with educators across the country to ensure all students have the supports to succeed in appropriately challenging classroom experiences that allow students to learn and grow. It is only through a sustained commitment to equitable preparation, access, and support that true excellence can be achieved for all students, and the Pre-AP Course Designation requires this commitment.

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Introduction to Pre-AP Algebra 2

About Pre-AP

Introduction to Pre-AP

Every student deserves classroom opportunities to learn, grow, and succeed. College Board developed Pre-AP® to deliver on this simple premise. Pre-AP courses are designed to support all students across varying levels of readiness. They are not honors or advanced courses.

Participation in Pre-AP courses allows students to slow down and focus on the most essential and relevant concepts and skills. Students have frequent opportunities to engage deeply with texts, sources, and data as well as compelling higher-order questions and problems. Across Pre-AP courses, students experience shared instructional practices and routines that help them develop and strengthen the important critical thinking skills they will need to employ in high school, college, and life. Students and teachers can see progress and opportunities for growth through varied classroom assessments that provide clear and meaningful feedback at key checkpoints throughout each course.

DEVELOPING THE PRE-AP COURSES

Pre-AP courses are carefully developed in partnership with experienced educators, including middle school, high school, and college faculty. Pre-AP educator committees work closely with College Board to ensure that the course resources define, illustrate, and measure grade-level-appropriate learning in a clear, accessible, and engaging way. College Board also gathers feedback from a variety of stakeholders, including Pre-AP partner schools from across the nation who have participated in multiyear pilots of select courses. Data and feedback from partner schools, educator committees, and advisory panels are carefully considered to ensure that Pre-AP courses provide all students with grade-level-appropriate learning experiences that place them on a path to college and career readiness.

PRE-AP PROGRAM COMMITMENTS

The Pre-AP Program asks participating schools to make four commitments:

1. **Pre-AP for All:** Pre-AP frameworks and assessments serve as the foundation for all sections of the course at the school.
2. **Course Frameworks:** Teachers align their classroom instruction with the Pre-AP course frameworks.
    - Schools commit to provide the core resources to ensure Pre-AP teachers and students have the materials they need to engage in the course.

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3. **Assessments:** Teachers administer at least one learning checkpoint per unit, on Pre-AP Classroom, and four performance tasks.
4. **Professional Learning:** Teachers complete the foundational professional learning (Online Foundational Modules or Pre-AP Summer Institute) and at least one online performance task scoring module. The current Pre-AP coordinator completes the Pre-AP Coordinator Online Module.

PRE-AP EDUCATOR NETWORK

Similar to the way in which teachers of Advanced Placement® (AP®) courses can become more deeply involved in the program by becoming AP Readers or workshop consultants, Pre-AP teachers also have opportunities to become active in their educator network. Each year, College Board expands and strengthens the Pre-AP National Faculty—the team of educators who facilitate Pre-AP Professional Learning Workshops. Pre-AP teachers can also become curriculum and assessment contributors by working with College Board to design, review, or pilot the course resources.

HOW TO GET INVOLVED

Schools and districts interested in learning more about participating in Pre-AP should visit **preap.org/join** or contact us at **preap@collegeboard.org**.

Teachers interested in becoming members of Pre-AP National Faculty or participating in content development should visit **preap.org/national-faculty** or contact us at **preap@collegeboard.org**.

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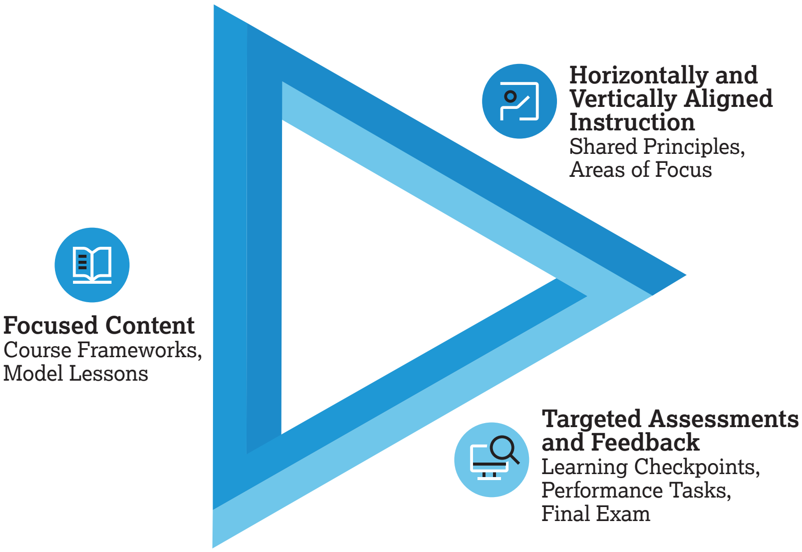
Pre-AP Approach to Teaching and Learning

Pre-AP courses invite all students to learn, grow, and succeed through focused content, horizontally and vertically aligned instruction, and targeted assessments for learning. The Pre-AP approach to teaching and learning, as described below, is not overly complex, yet the combined strength results in powerful and lasting benefits for both teachers and students. This is our theory of action.



FOCUSED CONTENT

Pre-AP courses focus deeply on a limited number of concepts and skills with the broadest relevance for high school coursework and college and career success. The course framework serves as the foundation of the course and defines these prioritized concepts and skills. Pre-AP model lessons and assessments are based directly on this focused framework. The course design provides students and teachers with intentional permission to slow down and focus.

HORIZONTALLY AND VERTICALLY ALIGNED INSTRUCTION

Shared principles cut across all Pre-AP courses and disciplines. Each course is also aligned to discipline-specific areas of focus that prioritize the critical reasoning skills and practices central to that discipline.

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SHARED PRINCIPLES

All Pre-AP courses share the following set of research-supported instructional principles. Classrooms that regularly focus on these cross-disciplinary principles allow students to effectively extend their content knowledge while strengthening their critical thinking skills. When students are enrolled in multiple Pre-AP courses, the horizontal alignment of the shared principles provides students and teachers across disciplines with a shared language for their learning and investigation, and multiple opportunities to practice and grow. The critical reasoning and problem-solving tools students develop through these shared principles are highly valued in college coursework and in the workplace.

Close Observation and Analysis

Students are provided time to carefully observe one data set, text image, performance piece, or problem before being asked to explain, analyze, or evaluate. This creates a safe entry point to simply express what they notice and what they wonder. It also encourages students to slow down and capture relevant details with intentionality to support more meaningful analysis, rather than rush to completion at the expense of understanding.

Higher-Order Questioning

Students engage with questions designed to encourage thinking that is elevated beyond simple memorization and recall. Higher-order questions require students to make predictions, synthesize, evaluate, and compare. As students grapple with these questions, they learn that being inquisitive promotes extended thinking and leads to deeper understanding.

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Evidence-Based Writing

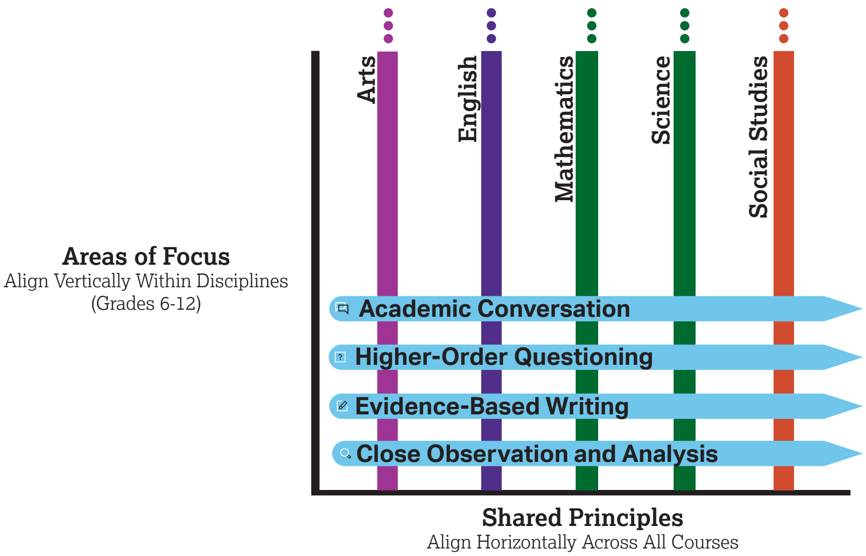
With strategic support, students frequently engage in writing coherent arguments from relevant and valid sources of evidence. Pre-AP courses embrace a purposeful and scaffolded approach to writing that begins with a focus on precise and effective sentences before progressing to longer forms of writing.

Academic Conversation

Through peer-to-peer dialogue, students' ideas are explored, challenged, and refined. As students engage in academic conversation, they come to see the value in being open to new ideas and modifying their own ideas based on new information. Students grow as they frequently practice this type of respectful dialogue and critique and learn to recognize that all voices, including their own, deserve to be heard.

AREAS OF FOCUS

The areas of focus are discipline-specific reasoning skills that students develop and leverage as they engage with content. Whereas the shared principles promote horizontal alignment across disciplines, the areas of focus provide vertical alignment within a discipline, giving students the opportunity to strengthen and deepen their work with these skills in subsequent courses in the same discipline.



For information about the Pre-AP mathematics areas of focus, see page 15.

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TARGETED ASSESSMENTS FOR LEARNING

Pre-AP courses include strategically designed classroom assessments that serve as tools for understanding progress and identifying areas that need more support. The assessments provide frequent and meaningful feedback for both teachers and students across each unit of the course and for the course as a whole. For more information about assessments in Pre-AP Algebra 2, see page 55.

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Pre-AP Professional Learning

As part of the program commitments, Pre-AP teachers agree to engage in two professional learning opportunities:

1. The first commitment is designed to help prepare teachers to teach their specific course. There are two options to meet this commitment: the Pre-AP Summer Institute (Pre-APSI) and the Online Foundational Modules. Both options provide continuing education units upon completion.
   - The Pre-AP Summer Institute provides a collaborative experience that empowers participants to prepare and plan for their Pre-AP course. While attending, teachers engage with Pre-AP course frameworks, shared principles, areas of focus, and sample model lessons. Participants are given supportive planning time where they work with peers to begin building their Pre-AP course plan.
   - Online Foundational Modules are available to all teachers of Pre-AP courses. In their 12- to 20-hour asynchronous course, teachers explore course materials and experience model lessons from the student's point of view. They also begin building their Pre-AP course plan.
2. The second professional learning opportunity helps teachers prepare for the performance tasks. As part of this commitment, teachers agree to complete at least one online performance task scoring module. Online scoring modules offer guidance and practice applying scoring guidelines and examining student work. Teachers may complete the modules independently or with teachers of the same course in their school's professional learning communities.

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About the Course

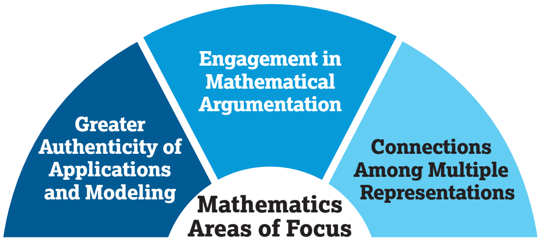
Introduction to Pre-AP Algebra 2

Pre-AP Algebra 2 is designed to optimize students' readiness for college-level mathematics classes. Rather than seeking to cover all topics traditionally included in a standard second-year algebra textbook, this course extends the conceptual understanding of and procedural fluency with functions and data analysis that students developed in their previous mathematics courses. It offers an approach that concentrates on the mathematical content and skills that matter most for college readiness. This approach creates more equitable opportunities for students to take AP STEM courses, especially for those students who are underrepresented in STEM courses and careers. The Pre-AP Algebra 2 Course Framework highlights how to guide students to connect core ideas within and across the units of the course, promoting a coherent understanding of functions.

The components of this course have been crafted to prepare not only the next generation of mathematicians, scientists, programmers, statisticians, and engineers, but also a broader base of mathematically informed citizens who are well equipped to respond to the array of mathematics-related issues that impact our lives at the personal, local, and global levels.

PRE-AP MATHEMATICS AREAS OF FOCUS

The Pre-AP mathematics areas of focus, shown below, are mathematical practices that students develop and leverage as they engage with content. They were identified through educator feedback and research about where students and teachers need the most curriculum support. These areas of focus are vertically aligned to the mathematical practices embedded in other mathematics courses in high school, including AP, and in college, giving students multiple opportunities to strengthen and deepen their work with these skills throughout their educational career. They also support and align to the AP Calculus Mathematical Practices, the AP Statistics Course Skills, and the mathematical practices listed in various state standards.



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Greater Authenticity of Applications and Modeling

Students create and use mathematical models to understand and explain authentic scenarios.

Mathematical modeling is a process that helps people analyze and explain the world. In Pre-AP Algebra 2, students explore real-world contexts where mathematics can be used to make sense of a situation. They engage in the modeling process by making choices about what function to use to construct a model, assessing how well the model represents the available data, refining their model as needed, drawing conclusions from their model, and justifying decisions they make through the process.

In addition to mathematical modeling, students engage in mathematics through authentic applications. Applications are similar to modeling problems in that they are drawn from real-world phenomena, but they differ because the applications dictate the appropriate mathematics to use to solve the problem. Pre-AP Algebra 2 balances these two types of real-world tasks.

Engagement in Mathematical Argumentation

Students use evidence to craft mathematical conjectures and prove or disprove them.

Conjecture, reasoning, and proof lie at the heart of the discipline of mathematics. Mathematics is both a way of thinking and a set of tools for solving problems. Pre-AP Algebra 2 students gain proficiency in constructing arguments with definitions of mathematical concepts, reasoning to solve equations, developing skills in using algebra to make sense of data, and crafting assertions using data as evidence and support. Through mathematical argumentation, students learn how to be critical of their own reasoning and the reasoning of others.

Connections Among Multiple Representations

Students represent mathematical concepts in a variety of forms and move fluently among the forms.

Pre-AP Algebra 2 students explore how to weave together multiple representations of function concepts. Every mathematical representation illuminates certain characteristics of a concept while also obscuring other aspects. Throughout the course, students continue to represent mathematical concepts using a variety of forms, allowing them to develop a nuanced understanding of which representations best serve a particular purpose.

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PRE-AP ALGEBRA 2 AND CAREER READINESS

The Pre-AP Algebra 2 course resources are designed to expose students to a wide range of career opportunities that depend on algebraic knowledge and skills. Examples include not only field-specific specialty careers such as mathematician and statistician, but also other endeavors where algebraic knowledge is relevant, such as accounting, economics, engineering, and programming.

Career clusters that involve algebra, along with examples of careers in mathematics or related to mathematics, are provided below. Teachers should consider discussing these with students throughout the year to promote motivation and engagement.

Career Clusters Involving Mathematics	
architecture and construction arts, A/V technology, and communications business management and administration finance government and public administration health science information technology marketing STEM (science, technology, engineering, and math) transportation, distribution, and logistics	
Examples of Mathematics Related Careers	**Examples of Algebra 2 Related Careers**
actuary financial analyst mathematician mathematics teacher professor programmer statistician	accountant computer programmer economist electrician engineer health science technician operations research analyst

Source for Career Clusters: "Advanced Placement and Career and Technical Education: Working Together." Advance CTE and the College Board. October 2018. **https://careertech.org/resource/ap-cte-working-together**.

For more information about careers that involve mathematics, teachers and students can visit and explore the College Board's Big Future resources: **https://bigfuture.collegeboard.org/majors/math-statistics-mathematics**.

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SUMMARY OF RESOURCES AND SUPPORTS

Teachers are strongly encouraged to take advantage of the full set of resources and supports for Pre-AP Algebra 2, which is summarized below. Some of these resources are part of the Pre-AP Program commitments that lead to Pre-AP Course Designation. To learn more about the commitments for course designation, see details below and on page 77.

COURSE FRAMEWORK

Included in this guide as well as in the *Pre-AP Algebra 2 Teacher Resources*, the course framework defines what students should know and be able to do by the end of the course. It serves as an anchor for model lessons and assessments, and it is the primary resource needed to plan the course. **Teachers commit to aligning their classroom instruction with the course framework**. *For more details see page 22.*

MODEL LESSONS

Teacher resources, available in print and online, include a robust set of model lessons that demonstrate how to translate the course framework, shared principles, and areas of focus into daily instruction. **Use of the model lessons is encouraged**. *For more details see page 53.*

LEARNING CHECKPOINTS

Accessed through Pre-AP Classroom, these short formative assessments provide insight into student progress. They are automatically scored and include multiple-choice and technology-enhanced items with rationales that explain correct and incorrect answers. **Teachers commit to administering one learning checkpoint per unit**. *For more details see page 55.*

PERFORMANCE TASKS

Available in the printed teacher resources as well as on Pre-AP Classroom, performance tasks allow students to demonstrate their learning through extended problem-solving, writing, analysis, and/or reasoning tasks. Scoring guidelines are provided to inform teacher scoring, with additional practice and feedback suggestions available in online modules on Pre-AP Classroom. **Teachers commit to using each unit's performance task**. *For more details see page 57.*

PRACTICE PERFORMANCE TASKS

Available in the student resources, with supporting materials in the teacher resources, these tasks provide an opportunity for students to practice applying skills and knowledge as they would in a performance task, but in a more scaffolded environment. **Use of the practice performance tasks is encouraged**. *For more details see page 57.*

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FINAL EXAM

Accessed through Pre-AP Classroom, the final exam serves as a classroom-based, summative assessment designed to measure students' success in learning and applying the knowledge and skills articulated in the course framework. **Administration of the final exam is encouraged**. *For more details see page 71.*

PROFESSIONAL LEARNING

Both the Pre-AP Summer Institute (Pre-APSI) and the Online Foundational Modules support teachers in preparing and planning to teach their Pre-AP course. **All Pre-AP teachers make a commitment to either attend the Pre-APSI (in person or virtually) or complete the Online Foundational Modules. In addition, teachers agree to complete at least one Online Performance Task Scoring module**. *For more details see page 11.*

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Course Map

PLAN

The course map shows how components are positioned throughout the course. As the map indicates, the course is designed to be taught over 140 class periods (based on 45-minute class periods), for a total of 28 weeks.

Model lessons are included for approximately 50% of the total instructional time, with the percentage varying by unit. Each unit is divided into key concepts.

TEACH

The model lessons demonstrate how the Pre-AP shared principles and mathematics areas of focus come to life in the classroom.

Shared Principles
Close observation and analysis
Higher-order questioning
Evidence-based writing
Academic conversation

Areas of Focus
Greater authenticity of applications and modeling
Engagement in mathematical argumentation
Connections among multiple representations

ASSESS AND REFLECT

Each unit includes two learning checkpoints and a performance task. These formative assessments are designed to provide meaningful feedback for both teachers and students.

Note: The final exam, offered during a six-week window in the spring, is not represented on the map.

UNIT 1 Modeling with Functions

~35 Class Periods

Pre-AP model lessons provided for 40% of instructional time in this unit

KEY CONCEPT 1.1
Choosing Appropriate Function Models

Learning Checkpoint 1

KEY CONCEPT 1.2
Rate of Change

Performance Task for Unit 1

KEY CONCEPT 1.3
Piecewise-Defined Models

Learning Checkpoint 2

UNIT 2 The Algebra of Functions

~30 Class Periods

Pre-AP model lessons provided for approximately 40% of instructional time in this unit

KEY CONCEPT 2.1
Composing Functions

KEY CONCEPT 2.2
Transforming Functions

Learning Checkpoint 1

KEY CONCEPT 2.3
Inverting Functions

Learning Checkpoint 2

Performance Task for Unit 2

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UNIT 3 Function Families

~45 Class Periods

Pre-AP model lessons provided for approximately 20% of instructional time in this unit

KEY CONCEPT 3.1
Exponential and Logarithmic Functions

Learning Checkpoint 1

KEY CONCEPT 3.2
Polynomial and Rational Functions

Performance Task for Unit 3

KEY CONCEPT 3.3
Square Root and Cube Root Functions

Learning Checkpoint 2

UNIT 4T Trigonometric Functions

~30 Class Periods

Pre-AP model lessons provided for approximately 40% of instructional time in this unit

KEY CONCEPT 4T.1
Radian Measure and Sinusoidal Functions

Performance Task for Unit 4T

Learning Checkpoint 1

KEY CONCEPT 4T.2
The Tangent Function and Other Trigonometric Functions

KEY CONCEPT 4T.3
Inverting Trigonometric Functions

Learning Checkpoint 2

UNIT 4M Matrices and Their Applications

~30 Class Periods

Pre-AP model lessons provided for approximately 40% of instructional time in this unit

KEY CONCEPT 4M.1
Geometric Transformations

Learning Checkpoint 1

KEY CONCEPT 4M.2
Solving Systems of Equations with Matrices

KEY CONCEPT 4M.3
Applications of Matrix Multiplication

Learning Checkpoint 2

Performance Task for Unit 4M

Note: Schools can choose to complete either Unit 4T **or** Unit 4M, depending on which unit is the best fit for state or local standards.

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Pre-AP Algebra 2 Course Framework

INTRODUCTION

Based on the Understanding by Design® (Wiggins and McTighe) model, the Pre-AP Algebra 2 Course Framework is back mapped from AP expectations and aligned to essential grade-level expectations. The course framework serves as a teacher's blueprint for the Pre-AP Algebra 2 instructional resources and assessments.

The course framework was designed to meet the following criteria:

- **Focused:** The framework provides a deep focus on a limited number of concepts and skills that have the broadest relevance for later high school, college, and career success.
- **Measurable:** The framework's learning objectives are observable and measurable statements about the knowledge and skills students should develop in the course.
- **Manageable:** The framework is manageable for a full year of instruction, fosters the ability to explore concepts in depth, and enables room for additional local or state standards to be addressed where appropriate.
- **Accessible:** The framework's learning objectives are designed to provide all students, across varying levels of readiness, with opportunities to learn, grow, and succeed.

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COURSE FRAMEWORK COMPONENTS

The Pre-AP Algebra 2 Course Framework includes the following components:

Big Ideas

The big ideas are recurring themes that allow students to create meaningful connections between course concepts. Revisiting the big ideas throughout the course and applying them in a variety of contexts allow students to develop deeper conceptual understandings.

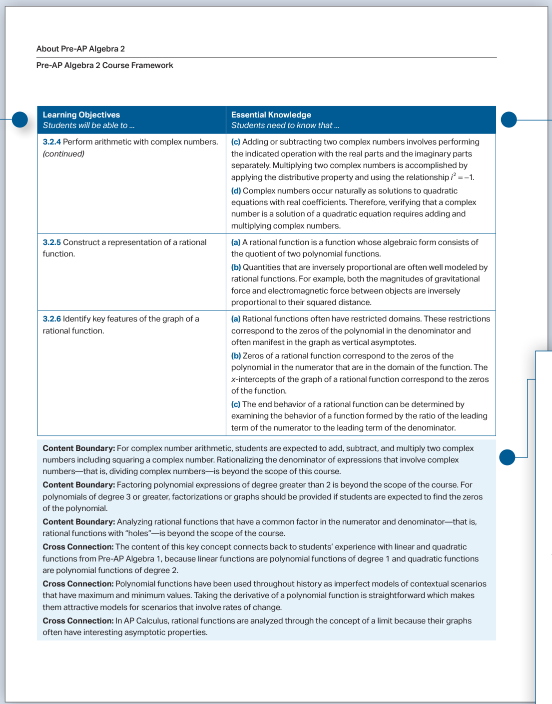
Enduring Understandings

Each unit focuses on a small set of enduring understandings. These are the long-term takeaways related to the big ideas that leave a lasting impression on students. Students build these understandings over time by exploring and applying course content throughout the year.

Key Concepts

To support teacher planning and instruction, each unit is organized by key concepts. Each key concept includes relevant **learning objectives** and **essential knowledge statements** and may also include **content boundary and cross connection statements**. These are illustrated and defined below.

Learning Objectives:
These objectives define what a student needs to be able to do with essential knowledge to progress toward the enduring understandings. The learning objectives serve as actionable targets for instruction and assessment.



About Pre-AP Algebra 2

Pre-AP Algebra 2 Course Framework

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
3.2.4 Perform arithmetic with complex numbers. *(continued)*	**(c)** Adding or subtracting two complex numbers involves performing the indicated operation with the real parts and the imaginary parts separately. Multiplying two complex numbers is accomplished by applying the distributive property and using the relationship $i^2 = -1$. **(d)** Complex numbers occur naturally as solutions to quadratic equations with real coefficients. Therefore, verifying that a complex number is a solution of a quadratic equation requires adding and multiplying complex numbers.
3.2.5 Construct a representation of a rational function.	**(a)** A rational function is a function whose algebraic form consists of the quotient of two polynomial functions. **(b)** Quantities that are inversely proportional are often well modeled by rational functions. For example, both the magnitudes of gravitational force and electromagnetic force between objects are inversely proportional to their squared distance.
3.2.6 Identify key features of the graph of a rational function.	**(a)** Rational functions often have restricted domains. These restrictions correspond to the zeros of the polynomial in the denominator and often manifest in the graph as vertical asymptotes. **(b)** Zeros of a rational function correspond to the zeros of the polynomial in the numerator that are in the domain of the function. The x-intercepts of the graph of a rational function correspond to the zeros of the function. **(c)** The end behavior of a rational function can be determined by examining the behavior of a function formed by the ratio of the leading term of the numerator to the leading term of the denominator.

Content Boundary: For complex number arithmetic, students are expected to add, subtract, and multiply two complex numbers including squaring a complex number. Rationalizing the denominator of expressions that involve complex numbers—that is, dividing complex numbers—is beyond the scope of this course.

Content Boundary: Factoring polynomial expressions of degree greater than 2 is beyond the scope of the course. For polynomials of degree 3 or greater, factorizations or graphs should be provided if students are expected to find the zeros of the polynomial.

Content Boundary: Analyzing rational functions that have a common factor in the numerator and denominator—that is, rational functions with "holes"—is beyond the scope of the course.

Cross Connection: The content of this key concept connects back to students' experience with linear and quadratic functions from Pre-AP Algebra 1, because linear functions are polynomial functions of degree 1 and quadratic functions are polynomial functions of degree 2.

Cross Connection: Polynomial functions have been used throughout history as imperfect models of contextual scenarios that have maximum and minimum values. Taking the derivative of a polynomial function is straightforward which makes them attractive models for scenarios that involve rates of change.

Cross Connection: In AP Calculus, rational functions are analyzed through the concept of a limit because their graphs often have interesting asymptotic properties.

Essential Knowledge Statements:
Each essential knowledge statement is linked to a learning objective. One or more essential knowledge statements describe the knowledge required to perform each learning objective.

Content Boundary and Cross Connection Statements:
When needed, content boundary statements provide additional clarity about the content and skills that lie within versus beyond the scope of this course.

Cross connection statements highlight important connections that should be made between key concepts within and across the units.

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BIG IDEAS IN PRE-AP ALGEBRA 2

While the Pre-AP Algebra 2 framework is organized into four core units of study, the content is grounded in three big ideas, which are cross-cutting concepts that build conceptual understanding and spiral throughout the course. Since these ideas cut across units, they serve as the underlying foundation for the enduring understandings, key concepts, and learning objectives that make up the focus of each unit. A deep and productive understanding in Pre-AP Algebra 2 relies on these three big ideas:

- **Function:** The mathematical concept of function describes a special kind of relationship where each input value corresponds to a single output value. Functions are among the most important objects in modern mathematics. Functions can be constructed to model phenomena that involve quantities such as time, force, and money, among others. These function models of real-world phenomena allow us to discover and investigate patterns among the related quantities. Studying patterns through the lens of a function model provides insights that lead to reasoned predictions and sound decision making.
- **Operations with Functions:** Two functions can be combined, composed, and transformed to form a new function that is a better model of a real-world phenomenon than the original function. Operations on functions include the arithmetic operations of addition, subtraction, multiplication, and division, as well as a special kind of operation, composition. Composition is the process of using the output of one function as the input of another function. When one of the functions in the composition is either a sum or product of a constant and a variable, the composition is referred to as a function transformation because the effect of such an operation on the graph of the function can be described in terms of geometric transformations. A thorough understanding of how to use function operations to construct more complex and nuanced function models is critical to the success of the mathematical modeling process.
- **Inverse Functions:** Solving an equation often relies on undoing an operation with its inverse operation. The processes of inverse operations are formalized as the concept of an inverse function, which expresses the idea that some mathematical relationships can be reversed. If a function defines a way to determine the output value from an input value, an inverse function defines a way to determine the input value from an output value. This implies that patterns observed in the output of one function can be seen in the input of the inverse function. Some of the functions covered in the course, such as exponential and logarithmic functions, are inverses of each other.

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OVERVIEW OF PRE-AP ALGEBRA 2 UNITS AND ENDURING UNDERSTANDINGS

Unit 1: Modeling with Functions	Unit 2: The Algebra of Functions
▪ Many bivariate data sets can be appropriately modeled by linear, quadratic, or exponential functions because the relationships between the quantities exhibit characteristics similar to those functions. ▪ Mathematical functions almost never perfectly fit a real-world context, but a function model can be useful for making sense of that context. ▪ Average rate of change allows us to understand multifaceted relationships between quantities by modeling them with linear functions.	▪ Composing functions allows simpler functions to be combined to construct a function model that more appropriately captures the characteristics of a contextual scenario. ▪ Transformations are a special kind of composition. When one of the functions being composed consists only of addition or multiplication, the effects on the other function are straightforward to determine. ▪ An inverse function defines the way to determine the input value that corresponds to a given output value.
Unit 3: Function Families	**Unit 4T: Trigonometric Functions**
▪ A function is a special mathematical relationship between two variables that can often be used to make sense of observable patterns in contextual scenarios. ▪ Functions in a family have similar properties, similar algebraic representations, and graphs that share key features.	▪ Trigonometry connects the study of circles and the study of right triangles. ▪ Real-world contexts that exhibit periodic behavior or circular motion can be modeled by trigonometric functions.
Unit 4M: Matrices and Their Applications	
▪ A matrix is a tool that can be used to efficiently organize mathematical information. ▪ Matrix multiplication provides efficient ways to evaluate multiple linear expressions simultaneously. ▪ Systems of linear equations can be solved using inverse matrices.	

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Unit 1: Modeling with Functions

Suggested timing: Approximately 7 weeks

In the first unit of the course, students build upon their previous experience with linear, quadratic, and exponential functions. These important functions form the foundation upon which other functions introduced in this course are built. Unit 1 focuses on using functions to model real-world data sets and contextual scenarios. This focus on modeling provides authentic opportunities for students to investigate and confirm the defining characteristics of linear, quadratic, and exponential functions while simultaneously reinforcing procedural fluency with these function families.

Throughout Pre-AP Algebra 2, students are expected to take ownership of the mathematics they use by crafting arguments for why one type of function is better than another for modeling a particular data set or contextual scenario. This allows students to develop a deeper understanding of these foundational functions as they drive the mathematical modeling process themselves. This requires a more thorough understanding of modeling than prior Pre-AP mathematics courses, in which students were asked to explain why a given function type was an appropriate model for a given data set or contextual scenario.

ENDURING UNDERSTANDINGS

Students will understand that …

- Many bivariate data sets can be appropriately modeled by linear, quadratic, or exponential functions because the relationships between the quantities exhibit characteristics similar to those functions.
- Mathematical functions almost never perfectly fit a real-world context, but a function model can be useful for making sense of that context.
- Average rate of change allows us to understand multifaceted relationships between quantities by modeling them with linear functions.

KEY CONCEPTS

- 1.1: Choosing Appropriate Function Models – Using linear, quadratic, and exponential functions to make sense of relationships between two quantities
- 1.2: Rate of Change – Using linear functions to make sense of complex relationships
- 1.3: Piecewise-Defined Models – Using functions defined over discrete intervals to make sense of contexts with varied characteristics

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KEY CONCEPT 1.1: CHOOSING APPROPRIATE FUNCTION MODELS

Using linear, quadratic, and exponential functions to make sense of relationships between two quantities

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
1.1.1 Identify a function family that would appropriately model a data set or contextual scenario.	**1.1.1a** Linear functions often appropriately model data sets that exhibit a roughly constant rate of change. **1.1.1b** Quadratic functions often appropriately model data sets that exhibit roughly linear rates of change, are roughly symmetric, and have a unique maximum or minimum output value. **1.1.1c** Exponential functions often appropriately model data sets that exhibit roughly constant ratios of output values over equal intervals of input values.
1.1.2 Use residual plots to determine whether a function model appropriately models a data set.	**1.1.2a** The residual for a data point is the deviation between the observed data value and the value predicted by a function model. Graphically, this can be thought of as the vertical segment between the data point and the graph of the function model. **1.1.2b** A residual plot is a scatterplot of values of the independent variable and their corresponding residuals. **1.1.2c** The sign of the residual indicates whether the function model is an overestimate or underestimate of the observed data value. **1.1.2d** An appropriate function model for a data set produces a residual plot with no discernible pattern. Residual plots that display some systematic pattern indicate that there is variation in the data not accounted for in the function model.
1.1.3 Construct a representation of a linear, quadratic, or exponential function both with and without technology.	**1.1.3a** A linear function can be expressed in slope-intercept form to reveal the constant rate of change and the initial value, or in point-slope form to reveal the constant rate of change and one ordered pair that satisfies the relationship. **1.1.3b** A quadratic function can be expressed in vertex form to reveal its maximum or minimum value; in factored form to reveal the zeros of the function, which often correspond to the boundaries of the contextual domain; or in standard form to reveal the initial value. **1.1.3c** An exponential function can be expressed in the form $f(x)=a(1+r)^x$ to reveal the percent change in the output, r, for a one-unit change in the input, or in the form $f(x)=a\bullet b^{\left(\frac{x}{n}\right)}$ to reveal the growth/decay factor, b, over an n-unit change in the input. **1.1.3d** A function within a function family that best fits a data set minimizes the error of the function model, which is often quantified by the sum of the squares of the residuals.

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Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
1.1.4 Use a function that models a data set or contextual scenario to predict values of the dependent variable.	**1.1.4a** An appropriate model for a bivariate data set can be used to predict values of the dependent variable from a value of the independent variable. **1.1.4b** Functions that model a data set are frequently only useful over their contextual domain.

Content Boundary: A primary focus for this key concept is the use of functions as models for data sets and contextual scenarios. Calculating a function model from a large data set by hand is beyond the scope of the course. The use of technology to determine a function model for a data set is strongly encouraged; however, the analysis arising from a function model is best done by the student.

Cross Connection: In this key concept, students build upon their experience with scatterplots and trend lines from Pre-AP Algebra 1. Through this unit, they see that some data sets are best modeled by linear functions, while other data sets are more appropriately modeled by quadratic or exponential functions.

Cross Connection: Because linear, quadratic, and exponential functions are the most broadly useful functions for making sense of real-world phenomena, developing deep conceptual understanding and procedural fluency supports student success on SAT and access to advanced mathematics courses.

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KEY CONCEPT 1.2: RATE OF CHANGE

Using linear functions to make sense of complex relationships

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
1.2.1 Interpret the average rate of change of a function over a given interval, including contextual scenarios.	**1.2.1a** The average rate of change of a function over an interval can be interpreted as the constant rate of change of a linear function with the same change in output for the given change in the input. This constant rate of change can be interpreted graphically as the slope of the line between the points on the ends of the interval. **1.2.1b** The average rate of change of a function f over the interval $[a, b]$ is the ratio $\frac{f(b)-f(a)}{b-a}$. That is, the average rate of change is $\frac{\Delta f(x)}{\Delta x}$.
1.2.2 Predict values of a function using the average rate of change and an input-output pair of a function model.	**1.2.2a** The average rate of change of a function over the interval $[a, b]$ can be used to estimate values of the function within or near the interval. **1.2.2b** The change in the value of $f(x)$ over an interval of width Δx can be determined by the product of the average rate of change of f and Δx.

Cross Connection: This key concept builds directly on students' understanding from Pre-AP Algebra 1 that linear functions have a constant rate of change. That prior knowledge can be leveraged here as students come to see how linear functions are used to make sense of more complex scenarios.

Cross Connection: The concept of average rate of change over increasingly smaller intervals is the basis for understanding the derivative of a function. In a calculus class, students will learn that the average rate of change of a function f over the interval $[x, x+\Delta x]$ is determined by the difference quotient, $\frac{f(x+\Delta x)-f(x)}{\Delta x}$.

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KEY CONCEPT 1.3: PIECEWISE-DEFINED MODELS

Using functions defined over discrete intervals to make sense of contexts with varied characteristics

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
1.3.1 Construct a representation of a piecewise-defined function.	**1.3.1a** A piecewise-defined function is a function that is defined on a set of nonoverlapping intervals. **1.3.1b** Data sets or contextual scenarios that demonstrate different characteristics, such as rates of change, over different intervals of the domain would be appropriately modeled by a piecewise-defined function. **1.3.1c** An algebraic representation of a piecewise-defined function consists of multiple algebraic expressions that describe the function over nonoverlapping intervals of the domain. **1.3.1d** The graph of a piecewise-defined function is the set of input-output coordinate pairs that satisfy the function relationship.
1.3.2 Evaluate a piecewise-defined function at specified values of the domain.	**1.3.2a** The output value of a piecewise-defined function for a specific input is determined by applying the algebraic rule for which the input value is defined. **1.3.2b** Output values of a piecewise-defined function can be estimated, and sometimes determined, from a graph of the function.
1.3.3 Construct a representation of an absolute value function.	**1.3.3a** The absolute value function is algebraically denoted as $f(x)=\|x\|$. **1.3.3b** The function $f(x)=\|x\|$ is a piecewise-defined function. If x is nonnegative, then $\|x\|=x$; if x is negative, then $\|x\|=-x$. The graph of $y=f(x)$ consists of $y=x$ for $x\geq 0$ and $y=-x$ for $x<0$.

Content Boundary: Intervals of real numbers can be expressed in interval notation or in inequality notation. Students are expected to be familiar with reading and writing intervals using both notations. The particular context in which an interval of real numbers is used should determine which notation is more appropriate.

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Unit 2: The Algebra of Functions

Suggested timing: Approximately 6 weeks

In Unit 2, students develop a conceptual understanding of the algebra of functions and build procedural fluency with function notation. Students tend to think about transformations of functions and composition of functions as unrelated topics. In this unit, students connect these important concepts to develop a more coherent understanding of functions by first exploring function composition, a new operation that chains functions together in a sequence. Once students understand the power of function composition, they work to see how function transformations are a special case of composition in which a given function is composed with a linear function.

The unit culminates in an exploration of inverses—the mathematical concept of undoing—through inverse operations and inverse functions. Students develop familiarity with inverse operations through their elementary school experiences with addition and multiplication, and their respective inverses, subtraction and division. In this unit, the inverse operation of exponentiating—taking a logarithm—is introduced. From prior coursework, students know that a function associates each input with one output. In this course, students learn that if a function has an inverse function, it associates an output back to its input. By considering inverses as both operations and functions, students develop a deep understanding of this critical concept.

ENDURING UNDERSTANDINGS

Students will understand that …

- Composing functions allows simpler functions to be combined to construct a function model that more appropriately captures the characteristics of a contextual scenario.
- Transformations are a special kind of composition. When one of the functions being composed consists only of addition or multiplication, the effects on the other function are straightforward to determine.
- An inverse function defines the way to determine the input value that corresponds to a given output value.

KEY CONCEPTS

- 2.1: Composing Functions – Chaining functions together in a sequence to construct better function models
- 2.2: Transforming Functions – Exploring how addition and multiplication affect the input or output of a function
- 2.3: Inverting Functions – Making sense of doing and undoing through inverse operations and inverse functions

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KEY CONCEPT 2.1: COMPOSING FUNCTIONS

Chaining functions together in a sequence to construct better function models

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
2.1.1 Determine the output value of the composition of two or more functions for a given input value when the functions have the same or different representations.	**2.1.1a** Composing functions is a process in which the output of one function is used as the input of another function. **2.1.1b** Composing functions is generally not a commutative operation. That is, for most functions, the value of $f(g(x))$ is not equal to the value of $g(f(x))$ for a given value of x.
2.1.2 Construct a representation of a composite function when the functions being composed have the same or different representations.	**2.1.2a** Composing two functions, f and g, results in a new function, called the composite function, that can be notated $f \circ g$ where $f \circ g(x) = f(g(x))$. **2.1.2b** An algebraic representation of $f \circ g$ is constructed by substituting every instance of x in the algebraic representation of $f(x)$ with the algebraic representation of $g(x)$. **2.1.2c** A graphical representation of $f \circ g$ can be constructed from the algebraic representation of $f \circ g$ or approximated by plotting some ordered pairs of the form $(x, f(g(x)))$. **2.1.2d** A numerical representation of $f \circ g$ consists of a subset of the ordered pairs that satisfy the relationship and is constructed by directly calculating values of $f(g(x))$ from values of x that are in the domain of g.
2.1.3 Express a given algebraic representation of a function in an equivalent form as the composition of two or more functions.	**2.1.3a** Any function can be expressed as the composition of two or more functions. One of these functions can be the identity function, $f(x) = x$. **2.1.3b** Algebraic techniques, such as factoring, can be used to express an algebraic representation of a function as a composition of functions.

Cross Connection: Students used function composition in Pre-AP Geometry with Statistics when they used sequences of multiple rigid motion and/or similarity transformations to associate one figure with another. In those experiences, the output of one transformation was treated as the input of another transformation. Students learned that changing the order in which the same transformations are applied to a preimage often yields different images. That understanding is reinforced through this key concept when students learn the importance of the order of composing two or more functions.

Cross Connection: Function composition often appears in problems in which the frame of reference for a function model is specified. For example, the output of a function could be the height as measured from the roof instead of the height as measured from the ground, or the input of a function could be the time that has elapsed since noon instead of the time that has elapsed since midnight. Using function composition to change the frame of reference is a valuable technique with applications across mathematics and science courses.

Cross Connection: The work of Learning Objective 2.1.3, understanding how one function can be written as a composition of two or more functions, is a critical first step in understanding the chain rule in AP Calculus. The chain rule defines how to take the derivative of a composite function.

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KEY CONCEPT 2.2: TRANSFORMING FUNCTIONS

Exploring how addition and multiplication affect the input or output of a function

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
2.2.1 Compare a function f with an additive transformation of f, that is, $f(x+k)$ or $f(x)+k$.	**2.2.1a** An additive transformation of f is a composition of f with $g(x)=x+k$. That is, $f\circ g(x)=f(x+k)$ and $g\circ f(x)=f(x)+k$. **2.2.1b** The graph of an additive transformation of f is either a vertical or horizontal translation of the graph of f. **2.2.1c** If (a, b) is an input-output pair of f, then $(a-k, b)$ must be an input-output pair of $f(x+k)$. Therefore, the graph of $f(x+k)$ is a horizontal translation of the graph of f by $-k$ units. **2.2.1d** If (a, b) is an input-output pair of f, then $(a, b+k)$ must be an input-output pair of $f(x)+k$. Therefore, the graph of $f(x)+k$ is a vertical translation of the graph of f by k units.
2.2.2 Compare a function f with a multiplicative transformation of f, that is, $f(kx)$ or $k \cdot f(x)$.	**2.2.2a** A multiplicative transformation of f is a composition of f with $g(x)=kx$, where $k\neq 0$. That is, $f\circ g(x)=f(kx)$ and $g\circ f(x)=k\cdot f(x)$. **2.2.2b** The graph of a multiplicative transformation of f is either a vertical or horizontal dilation of the graph of f. When $k<0$, the graph of the multiplicative transformation is also a reflection of the graph of f over one of the axes. **2.2.2c** If (a, b) is an input-output pair of f, then $\left(\frac{1}{k}\cdot a, b\right)$ must be an input-output pair of $f(kx)$. Therefore, the graph of $f(kx)$ is a horizontal dilation of the graph of f by a factor of $\left\|\frac{1}{k}\right\|$. **2.2.2d** If (a, b) is an input-output pair of f, then (a, kb) must be an input-output pair of $k\cdot f(x)$. Therefore, the graph of $k\cdot f(x)$ is a vertical dilation of the graph by a factor of $\|k\|$.
2.2.3 Construct a representation of the transformation of a function.	**2.2.3a** A function transformation is a sequence of additive and multiplicative transformations of f. The order in which the transformations are applied matters. **2.2.3b** Changing the reference point for the input or output quantity of a function can be achieved with an additive transformation. **2.2.3c** Converting the unit of measure for an input or output quantity of a function can be achieved with a multiplicative transformation.

Cross Connection: Students used additive and multiplicative transformations in Pre-AP Geometry with Statistics in the form of translations and dilations, respectively. Referring back to related geometric concepts in this key concept will help students make deep connections among the various transformations with which they have worked.

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KEY CONCEPT 2.3: INVERTING FUNCTIONS

Making sense of doing and undoing through inverse operations and inverse functions

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
2.3.1 Determine all input values that correspond to a specified output value given a function model on a specified domain.	**2.3.1a** For algebraic representations of an equation, inverse operations, such as squaring/square rooting and cubing/cube rooting, can be used to determine the input values that correspond to a specified output value. **2.3.1b** For algebraic representations of an equation involving an exponential expression, the inverse operation of exponentiating, called taking a logarithm, can be used to determine the input values that correspond to a specified output value. The solution to the equation $b^x = c$ where b and c are both positive and $b \neq 1$, is expressed by $x = \log_b(c)$. **2.3.1c** For graphical representations of an equation, identifying all ordered pairs that lie on the intersection of the line $y = k$ and the graph of $y = f(x)$ provides all input values that correspond with the output value k.
2.3.2 Express exactly or approximate the value of a logarithmic expression as a rational number.	**2.3.2a** An exact value of a logarithm can be determined using laws of exponents. **2.3.2b** If the logarithmic expression $\log_b(c)$ can be expressed exactly as a rational number, then its value is the rational number, x, that makes the equation $b^x = c$ true. **2.3.2c** If the logarithmic expression $\log_b(c)$ cannot be written exactly as a rational number, then its value can be approximated by a rational number x, for which $b^x \approx c$.
2.3.3 Determine a domain over which the inverse function of a specified function is defined.	**2.3.3a** A function f has an inverse function on a specified domain if each output value of f corresponds to exactly one input value in that domain. **2.3.3b** A function f is called invertible on a specified domain if there exists an inverse function, f^{-1}, such that $f(a) = b$ implies $f^{-1}(b) = a$. **2.3.3c** There are multiple ways to restrict the domain of a function so that the function is invertible. The appropriate domain restrictions for making a function invertible may depend on the context.

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Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
2.3.4 Construct a representation of the inverse function given a function that is invertible on its domain.	**2.3.4a** A table of values for the inverse function of f consists of all input-output ordered pairs (b, a) such that (a, b) is an input-output ordered pair of f. **2.3.4b** The graph of the inverse function of f is a reflection of the graph of $y = f(x)$ across the line $y = x$. **2.3.4c** The algebraic representation of the inverse function of $y = f(x)$ is determined using inverse operations to express x in terms of y. **2.3.4d** The domain and range of f^{-1} are the range and domain of f, respectively.
2.3.5 Verify that one function is an inverse of another function using composition.	**2.3.5a** If f is an inverse function of g, then g is an inverse function of f. **2.3.5b** If f is an inverse function of g, then composing f and g in either order will map each input onto itself. **2.3.5c** The function f is the inverse of the function g if and only if their composition in either order is the identity function, x. That is, $f(g(x)) = x$ and $g(f(x)) = x$.

Content Boundary: In this key concept, students are expected to solve quadratic and exponential functions using their associated inverse operations, taking the square root and taking a logarithm. Students are expected to develop an intuition for exact and approximate values of square roots of real numbers and of logarithms of real numbers by understanding their relationship to their respective inverses.

Content Boundary: The problems and exercises that address Learning Objective 2.3.4 are limited to linear functions and quadratic functions, with their domains appropriately constrained. Determining an inverse function for an exponential function is beyond the scope of this unit but will be expected in Unit 3.

Cross Connection: In Pre-AP Algebra 1, students learned that the expression $\sqrt{a}$ is a notation for the number whose square is a. Similarly, in this key concept, students learn that the expression $\log_b(c)$ is a notation for the exponent of b, such that b raised to that exponent has a value of c. That is, the equations $x = \log_b(c)$ and $b^x = c$ convey the same information about the relationship between the numbers b and c.

Cross Connection: The concepts of doing and undoing are central to many mathematical concepts. Inverse operations allow equations to be solved methodically rather than through an inefficient guess-and-check process. The existence of an inverse function shows that a function relationship between two quantities can be associated in two directions: from input to output and from output to input. In AP Calculus, the Fundamental Theorem of Calculus establishes an inverse relationship between differentiating and integrating a function.

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Unit 3: Function Families

Suggested timing: Approximately 9 weeks

Explorations of function families are an important component of any Algebra 2 course because they expand the repertoire of functions students can draw upon to model real-world phenomena. Not all phenomena are appropriately modeled by a linear, quadratic, or exponential function. For example, the gravitational force between two objects is inversely proportional to the square of their distance apart. This relationship would be best modeled with a rational function, one of the functions introduced in Unit 3. Throughout this unit, students learn that a parent function and its transformations form a function family. All functions in the same function family share some properties with each other.

Because function families are traditionally taught as isolated topics, students rarely have time to thoroughly investigate which properties of a function family are maintained by transformations and which are not. Therefore, the key concepts in this unit intentionally focus students' thinking on how function families are related in meaningful ways. The structure of the unit is intended to help students construct a network of connections among these function families. As with all explorations of functions throughout Pre-AP, the emphasis is on contextual scenarios that can be effectively modeled by each function family.

ENDURING UNDERSTANDINGS

Students will understand that …

- A function is a special mathematical relationship between two variables that can often be used to make sense of observable patterns in contextual scenarios.
- Functions in a family have similar properties, similar algebraic representations, and graphs that share key features.

KEY CONCEPTS

- 3.1: Exponential and Logarithmic Functions – Using functions to make sense of multiplicative patterns of change
- 3.2: Polynomial and Rational Functions – Using functions to make sense of sums and quotients of powers
- 3.3: Square Root and Cube Root Functions – Using functions to make sense of inverting quadratic and cubic relationships

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KEY CONCEPT 3.1: EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Using functions to make sense of multiplicative patterns of change

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
3.1.1 Construct a representation of an exponential function using the natural base, *e*.	**3.1.1a** The natural base *e*, which is approximately 2.718, is often used as the base in exponential functions that model contextual scenarios involving continuously compounded interest.
3.1.2 Express an exponential function in an equivalent form to reveal properties of the graph and/or the contextual scenario.	**3.1.2a** Any exponential model can be expressed in any base, including the natural base, using properties of exponents and/or function composition and properties of logarithms. **3.1.2b** A horizontal translation of the graph of an exponential function can also be thought of as a vertical dilation of the graph because $f(x)=b^{(x+k)}$ can be expressed as $f(x)=b^x \cdot b^k$, where b^k is a constant. **3.1.2c** A horizontal dilation of the graph of an exponential function is equivalent to a change of the base of the function, because $f(x)=b^{kx}$ can be expressed as $f(x)=\left(b^k\right)^x$, where b^k is a constant.
3.1.3 Construct a representation of a logarithmic function.	**3.1.3a** The graph of a logarithmic function exhibits vertical asymptotic behavior. **3.1.3b** An algebraic representation of a logarithmic function with base *b* is a transformation of $f(x)=\log_b(x)$, where $b \neq 1$ and $b>0$. **3.1.3c** A verbal representation of a logarithmic function describes additive changes in the output corresponding to multiplicative changes in the input. **3.1.3d** Data sets and/or contextual scenarios that exhibit a roughly constant ratio of the inputs for equal additive changes in the outputs are appropriately modeled by logarithmic functions.
3.1.4 Express a logarithmic function in an equivalent form to reveal properties of the graph and/or the contextual scenario.	**3.1.4a** A logarithmic function can be expressed in any base using properties of logarithms. Logarithmic functions are often expressed in base *e*, which is called the natural logarithm and is notated as “ln” rather than $\log_e$. **3.1.4b** A horizontal dilation is equivalent to a vertical translation because $f(x)=\log_b(kx)$ can be rewritten as $f(x)=\log_b(k)+\log_b(x)$, where $\log_b(k)$ is a constant. **3.1.4c** Raising the input of a logarithmic function to a power of *k* results in a vertical dilation of the graph because $f(x)=\log_b\left(x^k\right)$ can be rewritten as $f(x)=k \cdot \log_b(x)$.
3.1.5 Construct a representation of the inverse function of an exponential or logarithmic function.	**3.1.5a** The inverse of the exponential function $f(x)=b^x$ is the logarithmic function $g(x)=\log_b(x)$. **3.1.5b** The inverse of the logarithmic function $f(x)=\log_b(x)$ is the exponential function $g(x)=b^x$.

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Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
3.1.6 Solve equations involving exponential or logarithmic functions, including those arising from contextual scenarios.	**3.1.6a** Equations involving exponential functions can be solved algebraically by taking a logarithm or have solutions that can be estimated by examining a graph of the function. **3.1.6b** Equations involving logarithmic functions can be solved algebraically by exponentiating or have solutions that can be estimated by examining a graph of the function.

Content Boundary: Because students have familiarity with exponential functions from Pre-AP Algebra 1 and Unit 1 of this course, the exponential functions in this key concept should primarily include the natural base, *e*.

Cross Connection: This key concept provides an opportunity for students to use what they learned about exponential functions in Pre-AP Algebra 1 and function transformations, composition, and inverses in Unit 2 to deepen their understanding of exponential functions, a critically important function family.

Cross Connection: Logarithms are an essential tool in advanced mathematics courses. They are used in AP Statistics to transform a data set into a set that displays a more linear trend than the original one, which makes the data set easier to model. In AP Calculus, logarithms are often used in solving differential equations, especially those involving population growth, which is often exponential.

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KEY CONCEPT 3.2: POLYNOMIAL AND RATIONAL FUNCTIONS

Using functions to make sense of sums and quotients of powers

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
3.2.1 Construct a representation of a polynomial function.	**3.2.1a** A polynomial function is a function whose algebraic representation can be expressed as $f(x)=a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_n \neq 0$. **3.2.1b** Scenarios involving areas of figures and surface areas and volumes of solids are often well modeled by polynomial functions. **3.2.1c** Data sets and/or contextual scenarios that exhibit maximum or minimum values are sometimes well modeled by polynomial functions. **3.2.1d** Data sets and/or contextual scenarios where equal changes in the input values correspond to approximately constant nth differences in the output values are often well modeled by polynomial functions of degree n.
3.2.2 Express a polynomial function in an equivalent algebraic form to reveal properties of the function.	**3.2.2a** The standard form of a polynomial function reveals the degree of the polynomial, n, which is the highest power of all the terms. **3.2.2b** A linear factor, $(x-a)$, of a polynomial function, p, corresponds to a zero (or root) of p at $x=a$ because $p(a)=0$. **3.2.2c** A polynomial function factored into a product of linear factors reveals the x-intercepts of the graph of the function, which are the real zeros of the polynomial. The total number of real zeros is at most equal to the degree of the polynomial function.
3.2.3 Identify key features of the graph of a polynomial function.	**3.2.3a** A local maximum or minimum of a nonconstant polynomial function corresponds to the output value of the point at which the function switches between increasing to decreasing in either order. **3.2.3b** Between every two real zeros of a nonconstant polynomial function, there must be at least one input value corresponding to a local maximum or minimum. **3.2.3c** The end behavior of a polynomial function can be determined visually from its graph or by examining the degree of the polynomial and the sign of its leading coefficient. **3.2.3d** If a linear factor $(x-a)$ of a polynomial function has an even power, then the signs of the output values are the same for input values near $x=a$. For these polynomials, the graph will be tangent to the x-axis at $x=a$.
3.2.4 Perform arithmetic with complex numbers.	**3.2.4a** Every complex number has the form $a+bi$ where a and b are real numbers and $i^2=-1$. The real part of the complex number is a and the imaginary part of the complex number is b. **3.2.4b** The real numbers are a subset of the complex numbers since every real number a is equivalent to the complex number $a+0i$.

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Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
3.2.4 Perform arithmetic with complex numbers. *(continued)*	**3.2.4c** Adding or subtracting two complex numbers involves performing the indicated operation with the real parts and the imaginary parts separately. Multiplying two complex numbers is accomplished by applying the distributive property and using the relationship $i^2 = -1$. **3.2.4d** Complex numbers occur naturally as solutions to quadratic equations with real coefficients. Therefore, verifying that a complex number is a solution of a quadratic equation requires adding and multiplying complex numbers.
3.2.5 Construct a representation of a rational function.	**3.2.5a** A rational function is a function whose algebraic form consists of the quotient of two polynomial functions. **3.2.5b** Quantities that are inversely proportional are often well modeled by rational functions. For example, both the magnitudes of gravitational force and electromagnetic force between objects are inversely proportional to their squared distance.
3.2.6 Identify key features of the graph of a rational function.	**3.2.6a** Rational functions often have restricted domains. These restrictions correspond to the zeros of the polynomial in the denominator and often manifest in the graph as vertical asymptotes. **3.2.6b** Zeros of a rational function correspond to the zeros of the polynomial in the numerator that are in the domain of the function. The x-intercepts of the graph of a rational function correspond to the zeros of the function. **3.2.6c** The end behavior of a rational function can be determined by examining the behavior of a function formed by the ratio of the leading term of the numerator to the leading term of the denominator.

Content Boundary: For complex number arithmetic, students are expected to add, subtract, and multiply two complex numbers including squaring a complex number. Rationalizing the denominator of expressions that involve complex numbers—that is, dividing complex numbers—is beyond the scope of this course.

Content Boundary: Factoring polynomial expressions of degree greater than 2 is beyond the scope of the course. For polynomials of degree 3 or greater, factorizations or graphs should be provided if students are expected to find the zeros of the polynomial.

Content Boundary: Analyzing rational functions that have a common factor in the numerator and denominator—that is, rational functions with "holes"—is beyond the scope of the course.

Cross Connection: The content of this key concept connects back to students' experience with linear and quadratic functions from Pre-AP Algebra 1, because linear functions are polynomial functions of degree 1 and quadratic functions are polynomial functions of degree 2.

Cross Connection: Polynomial functions have been used throughout history as imperfect models of contextual scenarios that have maximum and minimum values. Taking the derivative of a polynomial function is straightforward which makes them attractive models for scenarios that involve rates of change.

Cross Connection: In AP Calculus, rational functions are analyzed through the concept of a limit because their graphs often have interesting asymptotic properties.

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KEY CONCEPT 3.3: SQUARE ROOT AND CUBE ROOT FUNCTIONS

Using functions to make sense of inverting quadratic and cubic relationships

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
3.3.1 Construct a representation of a square root function.	**3.3.1a** The square root function, $f(x)=\sqrt{x}$, is the inverse of the quadratic function $g(x)=x^2$ over the restricted domain $[0, \infty)$. Therefore, the graph of $y=\sqrt{x}$ resembles the graph of $y=x^2$ for $x \geq 0$ reflected across the line $y=x$. **3.3.1b** The domain of a square root function, a function transformation of $f(x)=\sqrt{x}$, corresponds to the set of input values for which the expression under the radical is nonnegative. **3.3.1c** Real-world scenarios involving distance traveled and elapsed time for free-falling objects are well modeled by square root functions.
3.3.2 Construct a representation of a cube root function.	**3.3.2a** The cube root function, $f(x)=\sqrt[3]{x}$, is the inverse of the cubic function $g(x)=x^3$. Thus, the graph of $y=\sqrt[3]{x}$ resembles the graph of $y=x^3$ reflected across the line $y=x$. **3.3.2b** The domain of a cube root function, a function transformation of $f(x)=\sqrt[3]{x}$, is all real numbers. **3.3.2c** Real-world scenarios involving side lengths of solids with a known volume are well modeled by cube root functions.
3.3.3 Construct a representation of the inverse function of a given quadratic function.	**3.3.3a** The inverse of a quadratic function is a square root function. **3.3.3b** The algebraic representation of the inverse of a quadratic function $y=f(x)$ is determined by first expressing f in vertex form and then using inverse operations to express x in terms of y. **3.3.3c** Since many output values of quadratic functions are each associated with multiple input values, constructing an inverse of a quadratic function requires restricting the domain of the function so the function is invertible. For values of x in the constrained domain of the quadratic function f, $f^{-1}(f(x))=x$.
3.3.4 Solve equations involving square root or cube root functions, including those arising from contextual scenarios.	**3.3.4a** Equations involving square roots and cube roots arising from contextual scenarios can be solved algebraically using inverse operations, such as squaring or cubing, or have solutions that can be estimated by examining an associated graph of the function models. **3.3.4b** Solving equations by squaring can introduce values called extraneous solutions, which are not actual solutions of the equation.

Cross Connection: The square root function appears in a variety of applications, most notably in scenarios involving distance or velocity as a function of time. These types of problems could involve finding the distance between two points in the plane using the Pythagorean theorem or finding a typical distance between a data value and the mean of the data set, called the standard deviation of a data set. In both cases, taking the square root is used to undo squaring a quantity.

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Unit 4T: Trigonometric Functions

Suggested timing: Approximately 6 weeks

This unit provides an exploration of trigonometric functions. Trigonometry is the branch of mathematics that connects two fundamental geometric objects: triangles and circles. In Pre-AP Geometry with Statistics, students learned that the trigonometric ratios relate acute angle measures to ratios of side lengths in right triangles. Algebra 2 extends those relationships to include all real numbers. When the domains of trigonometric functions include angle measures greater than 90°, including greater than 360°, these functions are far more useful in modeling contextual scenarios that involve periodic phenomena, such as the rotation of a ceiling fan, the height of a Ferris wheel car, or the ebb and flow of tides. A sound understanding of trigonometric functions demystifies these common real-world contexts, putting the power of mathematical reasoning at students' command.

Beginning with the introduction of radians as units of angle measure, the unit continues with an investigation of the sine and cosine functions and their transformations, collectively referred to as sinusoidal functions. Students then use what they know about sinusoidal functions and properties of quotients of functions to understand the properties of the tangent function and the three reciprocal trigonometric functions. Finally, students use inverse trigonometric functions to solve problems related to circular and periodic motion. Please note that the Pre-AP three-year mathematics sequence includes trigonometry in Algebra 2 to create a more equitable pathway for students who take Algebra 1 in 9th grade to potentially enroll in AP Calculus AB in 12th grade. If a state's standards do not require trigonometry in Algebra 2, then the school may use Unit 4M: Matrices and Their Applications to fulfill their Pre-AP commitments.

ENDURING UNDERSTANDINGS

Students will understand that …

- Trigonometry connects the study of circles and the study of right triangles.
- Real-world contexts that exhibit periodic behavior or circular motion can be modeled by trigonometric functions.

KEY CONCEPTS

- 4T.1: Radian Measure and Sinusoidal Functions – Using circles and triangles to make sense of periodic phenomena
- 4T.2: The Tangent Function and Other Trigonometric Functions – Using quotients of trigonometric functions to define new functions
- 4T.3: Inverting Trigonometric Functions – Using trigonometry to solve equations involving circular motion and periodic phenomena

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KEY CONCEPT 4T.1: RADIAN MEASURE AND SINUSOIDAL FUNCTIONS

Using circles and triangles to make sense of periodic phenomena

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
4T.1.1 Use the radian measure of an angle to relate the radius of a circle to the length of the arc subtended by that angle.	**4T.1.1a** The radian measure of an angle expresses the ratio of the subtended arc length to the radius of the circle in which it is a central angle. **4T.1.1b** An angle that has a measure of 1 radian cuts off an arc length equal to the length of the radius. **4T.1.1c** There is a proportional relationship between the radian measure of an angle, the subtended arc length, and the radius length. For a given angle, the ratio of the subtended arc length to the radius length is constant for any radius of the circle in which the angle is a central angle.
4T.1.2 Determine when two angles in the coordinate plane are coterminal.	**4T.1.2a** Angle measures can be expressed in radians or in degrees. The relationship between these units is proportional such that $\frac{\text{measure in degrees}}{\text{measure in radians}} = \frac{360}{2\pi}$. **4T.1.2b** In the coordinate plane, an angle is in standard position when its vertex is at the origin and one of its rays lies along the positive x-axis. Its other ray is called the terminal ray. **4T.1.2c** Two angles in standard position are coterminal if their terminal rays coincide. The amount of rotation of coterminal angles may differ by an integer number of revolutions. **4T.1.2d** Positive angle measures indicate that the terminal ray of the angle is constructed by the counterclockwise rotation of the ray about the origin. Negative angle measures indicate the terminal ray of the angle is constructed by a clockwise rotation of the ray about the origin.
4T.1.3 Construct a representation of a sinusoidal function.	**4T.1.3a** Periodic phenomena have repeating patterns of output values. Aspects of these phenomena can often be appropriately modeled by sinusoidal functions. **4T.1.3b** A unit circle has a radius of 1 unit of measure. The unit of measure should be determined by the context. **4T.1.3c** In the context of circular motion, the function $f(\theta) = \sin(\theta)$ relates the measure of an angle in standard position to the vertical displacement from the origin of a point on the unit circle and the function $f(\theta) = \cos(\theta)$ relates the measure of an angle in standard position to the horizontal displacement from the origin of a point on the unit circle. **4T.1.3d** Sinusoidal functions include the functions $f(x) = \sin(x)$ and $f(x) = \cos(x)$ and their transformations.

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Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
4T.1.4 Determine the exact coordinates of any point on a circle centered at the origin.	**4T.1.4a** The coordinates of the point at which the terminal ray of an angle in standard position intersects a unit circle with radius r are uniquely determined by the measure of that angle, θ, where $(x, y) = (r\cos(\theta), r\sin(\theta))$. For points on a unit circle, the coordinates are given by $(x, y) = (\cos(\theta), \sin(\theta))$. **4T.1.4b** The reference triangle for a given point on a circle is a right triangle whose three vertices are the origin, the point itself, and a point on the x-axis. The reference triangle can be useful in determining the exact coordinates of the given point.
4T.1.5 Identify key characteristics of a sinusoidal function.	**4T.1.5a** The amplitude of a sinusoidal function is half the difference between its maximum and minimum values. **4T.1.5b** The period of a sinusoidal function is the length of the interval of the input values over which it completes one rotation. The frequency of a sinusoidal function is the number of periods within an interval of length 2π. **4T.1.5c** Additive transformations of sinusoidal functions result in vertical or horizontal translations of the graph. A horizontal translation of a sinusoidal function is called a phase shift. **4T.1.5d** Multiplicative transformations of sinusoidal functions result in vertical or horizontal dilations of the graph. These transformations can impact the amplitude, period, or frequency of the sinusoidal function.
4T.1.6 Construct a sinusoidal function to model a periodic phenomenon that has a specified frequency, period, amplitude, and phase shift.	**4T.1.6a** The smallest interval of the input values over which the maximum or minimum output values start to repeat can be used to determine or estimate the period and frequency of the sinusoidal function model. **4T.1.6b** The maximum and minimum output values can be used to determine or estimate the amplitude for a sinusoidal function model. **4T.1.6c** The reference point for the input quantity can be used to determine or estimate a phase shift for the sinusoidal function model.
4T.1.7 Solve problems involving trigonometric identities.	**4T.1.7a** The Pythagorean theorem for trigonometric functions, $\sin^2(\theta) + \cos^2(\theta) = 1$, can be deduced from the fact that a circle of radius r centered at the origin is the solution set to the equation $x^2 + y^2 = r^2$ and that the coordinates of a point on that circle are given by $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

Content Boundary: The double-angle and half-angle formulas are beyond the scope of the course because of their limited usefulness in developing an understanding of trigonometric functions. The reciprocal identities are beyond the scope of this key concept but are introduced in Key Concept 4T.2. Students are expected to use the Pythagorean identity to solve problems involving squared trigonometric expressions, but verifying trigonometric identities is beyond the scope of the course.

Cross Connection: Embedding an angle in the coordinate plane allows students to connect their Pre-AP Geometry with Statistics experience with the concept of an angle, a union of two rays, to the concept of rotation. The measure of an angle in the coordinate plane quantifies the amount of the rotation about the vertex required of the first ray to coincide with the second ray, while the sign of the angle measure indicates the direction of this rotation.

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KEY CONCEPT 4T.2: THE TANGENT FUNCTION AND OTHER TRIGONOMETRIC FUNCTIONS

Using quotients of trigonometric functions to define new functions

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
4T.2.1 Construct a representation of a tangent function.	**4T.2.1a** For an angle in standard position, the tangent of the angle's measure is the slope of the terminal ray. **4T.2.1b** When the terminal ray is vertical, the slope of the ray is undefined. Therefore, the domain of the tangent function is restricted to exclude all values that correspond to vertical terminal rays. In other words, the domain of $f(\theta)=\tan(\theta)$ excludes all values of θ such that $\theta=\frac{\pi}{2}+k\pi$ for integer values of k. **4T.2.1c** When two angles in standard position have measures that differ by π radians, their terminal rays form a line and the rays have the same slope. As such, the tangent function has a period of π. **4T.2.1d** Contextual scenarios involving the height of a rising or falling object or slopes of lines are often appropriately modeled with a tangent function.
4T.2.2 Identify key characteristics and values of functions that are defined by quotients of sinusoidal functions.	**4T.2.2a** The tangent of angle θ is the slope of the terminal ray, which passes through the points (0, 0) and $(r\cos(\theta), r\sin(\theta))$. Thus, $\tan(\theta)=\frac{\sin(\theta)}{\cos(\theta)}$. **4T.2.2b** Secant, cosecant, and cotangent are the names given to trigonometric functions formed by quotients of sinusoidal functions, defined as $\sec(\theta)=\frac{1}{\cos(\theta)}$, $\csc(\theta)=\frac{1}{\sin(\theta)}$, and $\cot(\theta)=\frac{\cos(\theta)}{\sin(\theta)}$.

Content Boundary: Students develop an understanding of the properties of tangent, secant, cosecant, and cotangent functions and their relationships to the sinusoidal functions through this key concept. Verifying trigonometric identities is beyond the scope of the course.

Cross Connection: This key concept connects the slope of a line to the tangent of an angle that the line forms with the horizontal axis. This connection provides an opportunity for students to build a deeper understanding of linear relationships and link trigonometry to the content of Pre-AP Algebra 1. A comprehensive understanding of slope supports students in accessing the concept of the derivative of a function in AP Calculus.

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KEY CONCEPT 4T.3: INVERTING TRIGONOMETRIC FUNCTIONS

Using trigonometry to solve equations involving circular motion and periodic phenomena

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
4T.3.1 Construct a representation of an inverse trigonometric function.	**4T.3.1a** The inputs and outputs of inverse trigonometric functions are switched from their corresponding trigonometric functions, so the output of an inverse trigonometric function is often interpreted as an angle measure and the input is a value in the range of the corresponding trigonometric function. **4T.3.1b** The inverse trigonometric functions arcsine, arccosine, and arctangent (also represented as $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$) are defined by restricting the domain of the sine function to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, restricting the domain of the cosine function to $[0, \pi]$, and restricting the domain of the tangent function to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ so the trigonometric functions are invertible on their restricted domains.
4T.3.2 Solve equations involving trigonometric functions.	**4T.3.2a** For algebraic equations involving trigonometric expressions, inverse trigonometric functions can be used to determine the inputs corresponding to a specified output. **4T.3.2b** There could be multiple solutions to a trigonometric equation. The exact number of solutions is determined by the context.

Content Boundary: Solving a trigonometric equation for all possible solutions is beyond the scope of this course. Problems and exercises involving trigonometric equations used to model periodic phenomena should be limited to solutions over a finite interval.

Cross Connection: The names of inverse trigonometric functions include the prefix *arc-* because the output of each of these functions is an arc length measured in radians. Knowing why the prefix *arc-* is used can help students connect inverse trigonometric functions to circles, as explored in Key Concept 4T.1.

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Unit 4M: Matrices and Their Applications

Suggested timing: Approximately 6 weeks

The final unit in this course provides an exploration of matrices and their applications. Matrices are useful tools for organizing information and are used in career fields as diverse as engineering, business, and statistics. The goal of the unit is for students to understand how to construct and use matrices to apply certain geometric transformations and solve systems of linear equations. It is important to note that the operations performed on matrices in this course are limited to multiplication and finding the *n*th power of a square matrix. Addition and subtraction of matrices are not included in this unit because these operations are not necessary for applying geometric transformations or solving systems of linear equations using matrices. Also, most of the operations performed on matrices in this unit are expected to be performed with technology.

In this unit, students build upon their previous experience with functions, inverse functions, and transformations in the plane to support their understanding of how certain geometric transformations can be represented with matrix multiplication. Students use the inverse of a transformation matrix to identify the preimage of a specified point under a given linear transformation. Students use the inverse of a matrix to solve a system of linear equations.

The unit culminates in an exploration of recursive processes that model the relationship between the sizes of subgroups of a partitioned population. Using real-world scenarios, students apply matrix operations to determine the previous and future sizes of these subgroups and observe that their sizes stabilize over time.

ENDURING UNDERSTANDINGS

Students will understand that …

- A matrix is a tool that can be used to efficiently organize mathematical information.
- Matrix multiplication provides efficient ways to evaluate multiple linear expressions simultaneously.
- Systems of linear equations can be solved using inverse matrices.

KEY CONCEPTS

- 4M.1: Geometric Transformations – Using matrices to determine images and preimages of points in the coordinate plane
- 4M.2: Solving Systems of Equations with Matrices – Solving systems of equations with inverse matrices
- 4M.3: Applications of Matrix Multiplication – Modeling how subgroups of a population move in a predictable way

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KEY CONCEPT 4M.1: GEOMETRIC TRANSFORMATIONS

Using matrices to determine images and preimages of points in the coordinate plane

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
4M.1.1 Identify whether a geometric transformation is a linear transformation.	**4M.1.1a** Given a geometric transformation, T, the notation $T(x, y)$ represents the image of point (x, y). **4M.1.1b** A geometric transformation that can be expressed in the form $T(x, y) = (ax + by, cx + dy)$ is called a linear transformation. **4M.1.1c** A linear transformation is a function that maps a point (x, y) to its image. **4M.1.1d** In any linear transformation, the origin maps to the origin, and lines map to lines.
4M.1.2 Express a system of linear expressions using multiple representations.	**4M.1.2a** The sum of terms consisting of the product of a constant and an unknown raised to the first power is called a linear expression (e.g., $ax + by$ or $ax + by + cz$). **4M.1.2b** The coefficients of a system of linear expressions can be organized as entries in a matrix. The entries in each column describe constraints on one of the variables and the entries in each row correspond to the coefficients of each expression. **4M.1.2c** An algebraic representation of a linear transformation is a system of linear expressions that defines the coordinates of the preimage. This is expressed as $T(x, y) = (ax + by, cx + dy)$. **4M.1.2d** The linear transformation $T(x, y) = (ax + by, cx + dy)$ can be expressed using matrix multiplication as $T(x, y) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.
4M.1.3 Multiply two matrices when the product is defined, both with and without technology.	**4M.1.3a** A matrix with m rows and n columns is said to have dimension $m \times n$. If the number of rows is equivalent to the number of columns, then the matrix is called a square matrix. **4M.1.3b** Given two matrices, A and B, the product AB is defined if the number of columns in matrix A is equal to the number of rows in matrix B. The resulting matrix, AB, has the same number of rows as matrix A and the same number of columns as matrix B. **4M.1.3c** Given the product of two matrices, where $AB = C$, the entry in the ith row and jth column of matrix C is determined by the sum of the products of the corresponding entries in the ith row of matrix A and the jth column of matrix B. **4M.1.3d** Given two matrices, A and B, the product AB is not necessarily equal to BA. That is, matrix multiplication is not commutative.

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Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
4M.1.4 Determine the image of a set of points under a linear transformation or a sequence of linear transformations.	**4M.1.4a** For a linear transformation defined as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the product $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ is equivalent to $\begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$. The product can be interpreted as the coordinate pair $(ax+by, cx+dy)$, which is the image of the point (x, y). **4M.1.4b** The set of n points $(x_1, y_1)\ldots(x_n, y_n)$ can be represented as the $2 \times n$ matrix $X = \begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{bmatrix}$. The image of these n points is given by AX where A is a 2×2 matrix that represents a linear transformation. **4M.1.4c** If matrices A and B represent two linear transformations and X is the $2 \times n$ coordinate matrix, then these transformations can be composed by first applying linear transformation A, which is given by AX, and then linear transformation B. The image of this sequence of transformations is given by $B \bullet (AX)$. This composition is described by the product $B \bullet (AX)$ since matrix multiplication is not commutative. **4M.1.4d** A square matrix with entries of 1 along the main diagonal and all other entries of 0 is called the identity matrix. The 2×2 identity matrix defines the identity transformation and maps all points in the plane onto themselves.
4M.1.5 Interpret the determinant of a 2×2 matrix geometrically.	**4M.1.5a** The determinant of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $ad - bc$. The determinant of matrix A can be represented as $\det(A)$. **4M.1.5b** Under a linear transformation represented by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the image of the square with vertices at (0, 0), (0, 1), (1, 0), and (1, 1) is a parallelogram with vertices at (0, 0), (a, c), (b, d), and $(a+b, c+d)$. This parallelogram has an area of $\lvert ad - bc \rvert$ square units, which is the absolute value of the determinant of the transformation matrix. **4M.1.5c** A linear transformation will map a polygon in the plane to another polygon in the plane. The area of the image can be determined by scaling the area of the preimage by the absolute value of the determinant of the matrix that represents the linear transformation.
4M.1.6 Express a sequence of linear transformations as a single linear transformation.	**4M.1.6a** Matrix multiplication is associative; that is, $A \bullet (BX) = (AB) \bullet X$. **4M.1.6b** If two linear transformations are given by matrices A and B, then the linear transformation AB is equivalent to first applying linear transformation B and then applying linear transformation A.

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Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
4M.1.7 Determine the inverse of a square matrix, including matrices describing linear transformations, both with and without technology.	**4M.1.7a** Given a square matrix, A, its inverse, if it exists, is another square matrix, A^{-1}. The products AA^{-1} and $A^{-1}A$ are both equal to the identity matrix. **4M.1.7b** For a 2×2 matrix A, if $\det(A)$ is nonzero then the inverse matrix A^{-1} exists. If $\det(A)$ is nonzero then the inverse of A is given by $\frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
4M.1.8 Determine the preimage of a specified point under a given linear transformation.	**4M.1.8a** If T and T^{-1} are inverse linear transformations, then $T(x, y) = (a, b)$ implies $T^{-1}(a, b) = (x, y)$. The matrix that describes linear transformation T^{-1} is the inverse of the matrix that describes the linear transformation T. **4M.1.8b** Matrix multiplication on the left by A^{-1} is the inverse operation of matrix multiplication on the left by A. **4M.1.8c** If matrix A represents a linear transformation and maps point (x_1, y_1) to (x_2, y_2), then solving the equation $A\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ for $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ will give the preimage of the point (x_2, y_2). This equation can be solved by multiplying each side of the equation on the left by A^{-1}, assuming A^{-1} exists.

Content Boundary: Geometric transformations are limited to two-dimensional transformations. Students focus on characteristics of linear transformations and how these transformations act on the entire plane.

Content Boundary: Students should be able to perform matrix multiplication by hand for matrices with dimensions $(n \times n)(n \times 1)$ and $(2 \times 2)(2 \times 2)$ and use technology to multiply matrices of other dimensions.

Cross Connection: This key concept builds directly on students' understanding from Pre-AP Geometry with Statistics of performing transformations on points in the plane and expressing transformations using function notation.

Cross Connection: In AP Precalculus, students revisit topics from this key concept as they make sense of matrices acting as functions on vectors. In this course, the focus is on students understanding how matrices act as functions on points.

Cross Connection: In AP Computer Science A, students apply basic understandings of this key concept when they represent matrices as two-dimensional arrays in a programming language.

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KEY CONCEPT 4M.2: SOLVING SYSTEMS OF EQUATIONS WITH MATRICES

Solving systems of equations with inverse matrices

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
4M.2.1 Express a system of linear equations using multiple representations.	**4M.2.1a** A linear equation is a linear expression that is equal to a constraint (e.g., $ax+by+cz=d$). Linear equations can have more than two unknowns. **4M.2.1b** A verbal representation of a system of linear equations describes the relationship between the unknowns and the constraints on how the unknown values can vary. Verbal representations often describe contextual scenarios. **4M.2.1c** The system of linear equations given by $ax+by=e$ and $cx+dy=f$ can be expressed in an equivalent form as the matrix equation $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} e \\ f \end{bmatrix}$.
4M.2.2 Solve a system of linear equations using inverse matrices.	**4M.2.2a** The solution to the system of linear equations represented by $AX=B$ is $X=A^{-1}B$, assuming A^{-1} exists. **4M.2.2b** The matrix equation $AX=B$ has a unique solution when det(A) is nonzero because the inverse, A^{-1}, is defined. If det(A)$=0$, then A^{-1} is not defined and the matrix equation $AX=B$ has either no solution or infinitely many solutions. **4M.2.2c** The solution to a system of linear equations expressed with matrices derived from a context must be interpreted through that context for its meaning to be understood.
4M.2.3 Construct the algebraic representation of a polynomial function that passes through a set of given points using technology.	**4M.2.3a** Given n points, a polynomial of at most degree $n-1$ that passes through those points can be found by solving for coefficients of that polynomial. **4M.2.3b** Substituting the coordinates of n points into a polynomial of degree $n-1$ of the form $y=a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\ldots+a_1x+a_0$ produces a system of n linear equations where the coefficients are the unknowns. **4M.2.3c** A system of n linear equations, where the coefficients are the unknowns, can be solved by writing the system of n linear equations as a matrix equation and then using technology to solve that matrix equation.

Content Boundary: Students are expected to calculate the determinant of a 2×2 matrix by hand. The process for evaluating the determinant of a higher-order square matrix by hand is out of scope and should be done using technology.

Cross Connection: This key concept builds on students' experiences with solving systems of equations and modeling with systems of equations from Pre-AP Algebra 1. Through this unit, students extend their problem-solving options by using matrices to solve systems of equations.

Cross Connection: In AP Precalculus, students are expected to use matrices to solve systems of linear equations to minimize the use of algebraic techniques commonly used in Pre-AP Algebra 1.

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KEY CONCEPT 4M.3: APPLICATIONS OF MATRIX MULTIPLICATION

Modeling how subgroups of a population move in a predictable way

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
4M.3.1 Express a recursive process that models the relationship between two or three related subgroups of a population using multiple representations.	**4M.3.1a** Populations that are partitioned into subgroups whose members move among these subgroups in predictable ways are well modeled by matrices and transition diagrams. **4M.3.1b** A transition diagram represents the subgroups of a population and uses arrows to describe how the members of the population move between these subgroups in predictable ways. **4M.3.1c** In the equation $AX_0 = X_1$, the entries of matrix A describe how members of a population move between subgroups over 1 unit of time, matrix X_0 describes the size of each subgroup at a specified moment, and matrix X_1 describes the size of each subgroup 1 time-unit away. **4M.3.1d** Matrix multiplication can be used to determine the sizes of subgroups of a population at specific moments of time.
4M.3.2 Determine the future sizes of subgroups of a population given the current size of each subgroup.	**4M.3.2a** The entries of the matrix A^n quantify how members of the population move between its subgroups over an n-unit period of time. **4M.3.2b** The matrix A^n is determined by calculating the product of n copies of matrix A. The matrix A^n does not result from raising each entry of matrix A to the nth power. **4M.3.2c** Given an $m \times 1$ matrix containing the size of each subgroup of the population at time k, X_k, the matrix expression $A^n X_k$ gives the size of each subgroup n time-units from time k. This matrix expression is equivalent to the $m \times 1$ matrix X_{k+n}.
4M.3.3 Identify how the sizes of related subgroups of a population stabilize over time.	**4M.3.3a** For a given population partitioned into two or three subgroups, each subgroup will approach a constant size over time even as members of the population continue to move between subgroups. **4M.3.3b** For a population partitioned into two subgroups, the ratio of the sizes of pairs of subgroups at a particular moment will approach a constant value over time. This ratio is independent of size of the population.
4M.3.4 Determine the previous sizes of subgroups of a population given the current size of each subgroup.	**4M.3.4a** Solving the equation $AX_0 = X_1$ for X_0 will give the previous size of each subgroup of a population.

Content Boundary: A primary focus for this key concept is to use random processes to generate values so that the entries in the coefficient matrix of the matrix equation are probabilities.

Content Boundary: Although not all systems with three subpopulations stabilize, the systems selected as examples in for this unit will stabilize.

Cross Connection: Matrices are used as transformations in both this course and AP Precalculus. In this course, these matrices act on points in the plane, while in AP Precalculus, they act on vectors.

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Pre-AP Algebra 2 Model Lessons

Model lessons in Pre-AP Algebra 2 are developed in collaboration with Algebra 2 educators across the country and are rooted in the course framework, shared principles, and areas of focus. Model lessons are carefully designed to illustrate on-grade-level instruction. Pre-AP strongly encourages teachers to internalize the lessons and then offer the supports, extensions, and adaptations necessary to help all students achieve the lesson goals.

The purpose of these model lessons is twofold:

- **Robust instructional support for teachers:** Pre-AP Algebra 2 model lessons are comprehensive lesson plans that, along with accompanying student resources, embody the Pre-AP approach to teaching and learning. Model lessons provide clear and substantial instructional guidance to support teachers as they engage students in the shared principles and areas of focus.
- **Key instructional strategies:** Commentary and analysis embedded in each lesson highlight not just what students and teachers do in the lesson, but also how and why they do it. This educative approach provides a way for teachers to gain unique insight into key instructional moves that are powerfully aligned with the Pre-AP approach to teaching and learning. In this way, each model lesson works to support teachers in the moment of use with students in their classroom.

Teachers have the option to use any or all model lessons alongside their own locally developed instructional resources. Model lessons target content areas that tend to be challenging for teachers and students. While the lessons are distributed throughout all four units, they are concentrated more heavily in the beginning of the course to support teachers and students in establishing a strong foundation in the Pre-AP approach to teaching and learning.

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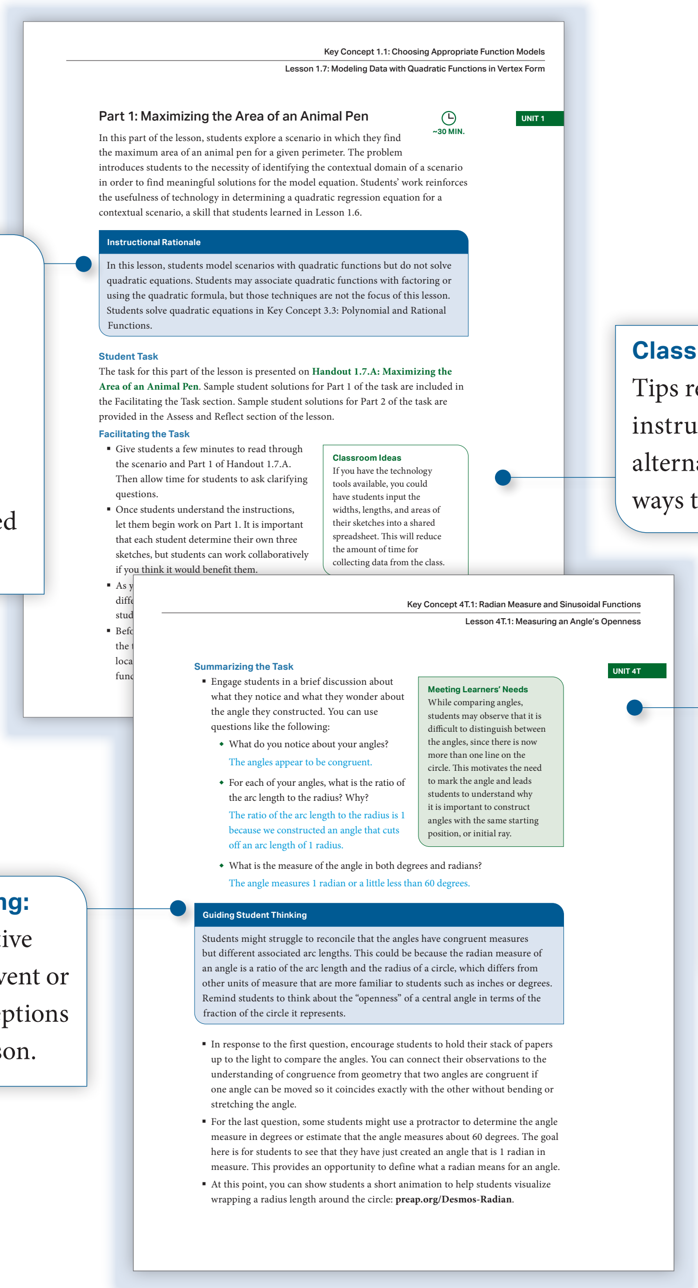
SUPPORT FEATURES IN MODEL LESSONS

The following support features recur throughout the Pre-AP Algebra 2 lessons to promote teacher understanding of the lesson design and provide direct-to-teacher strategies for adapting lessons to meet their students' needs:

- Instructional Rationale
- Guiding Student Thinking
- Classroom Ideas
- Meeting Learners' Needs

Instructional Rationale: Insight into the strategic design and purpose of the instructional choices, flow, and scaffolding within the model lesson. Rationales often describe how a concept is continued later in the lesson or unit.



Key Concept 1.1: Choosing Appropriate Function Models
Lesson 1.7: Modeling Data with Quadratic Functions in Vertex Form

UNIT 1

Part 1: Maximizing the Area of an Animal Pen ~30 MIN.

In this part of the lesson, students explore a scenario in which they find the maximum area of an animal pen for a given perimeter. The problem introduces students to the necessity of identifying the contextual domain of a scenario in order to find meaningful solutions for the model equation. Students' work reinforces the usefulness of technology in determining a quadratic regression equation for a contextual scenario, a skill that students learned in Lesson 1.6.

Instructional Rationale

In this lesson, students model scenarios with quadratic functions but do not solve quadratic equations. Students may associate quadratic functions with factoring or using the quadratic formula, but those techniques are not the focus of this lesson. Students solve quadratic equations in Key Concept 3.3: Polynomial and Rational Functions.

Student Task

The task for this part of the lesson is presented on **Handout 1.7.A: Maximizing the Area of an Animal Pen.** Sample student solutions for Part 1 of the task are included in the Facilitating the Task section. Sample student solutions for Part 2 of the task are provided in the Assess and Reflect section of the lesson.

Facilitating the Task

- Give students a few minutes to read through the scenario and Part 1 of Handout 1.7.A. Then allow time for students to ask clarifying questions.
- Once students understand the instructions, let them begin work on Part 1. It is important that each student determine their own three sketches, but students can work collaboratively if you think it would benefit them.

Classroom Ideas

If you have the technology tools available, you could have students input the widths, lengths, and areas of their sketches into a shared spreadsheet. This will reduce the amount of time for collecting data from the class.

Key Concept 4T.1: Radian Measure and Sinusoidal Functions
Lesson 4T.1: Measuring an Angle's Openness

UNIT 4T

Summarizing the Task

- Engage students in a brief discussion about what they notice and what they wonder about the angle they constructed. You can use questions like the following:
  - What do you notice about your angles?
    The angles appear to be congruent.
  - For each of your angles, what is the ratio of the arc length to the radius? Why?
    The ratio of the arc length to the radius is 1 because we constructed an angle that cuts off an arc length of 1 radius.
  - What is the measure of the angle in both degrees and radians?
    The angle measures 1 radian or a little less than 60 degrees.

Meeting Learners' Needs

While comparing angles, students may observe that it is difficult to distinguish between the angles, since there is now more than one line on the circle. This motivates the need to mark the angle and leads students to understand why it is important to construct angles with the same starting position, or initial ray.

Guiding Student Thinking

Students might struggle to reconcile that the angles have congruent measures but different associated arc lengths. This could be because the radian measure of an angle is a ratio of the arc length and the radius of a circle, which differs from other units of measure that are more familiar to students such as inches or degrees. Remind students to think about the "openness" of a central angle in terms of the fraction of the circle it represents.

- In response to the first question, encourage students to hold their stack of papers up to the light to compare the angles. You can connect their observations to the understanding of congruence from geometry that two angles are congruent if one angle can be moved so it coincides exactly with the other without bending or stretching the angle.
- For the last question, some students might use a protractor to determine the angle measure in degrees or estimate that the angle measures about 60 degrees. The goal here is for students to see that they have just created an angle that is 1 radian in measure. This provides an opportunity to define what a radian means for an angle.
- At this point, you can show students a short animation to help students visualize wrapping a radius length around the circle: **preap.org/Desmos-Radian**.

Classroom Ideas: Tips related to the logistics of the instruction, such as suggestions for alternative presentation methods, or ways to alleviate pacing concerns.

Meeting Learners' Needs: Optional differentiation strategies to address diverse learning needs, such as ideas for just-in-time skill building during a lesson or ways to break a task into smaller tasks, if needed, to make it more accessible.

Guiding Student Thinking: Ways to facilitate productive student thinking and prevent or address student misconceptions in critical areas of the lesson.

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Pre-AP Algebra 2 Assessments for Learning

Pre-AP Algebra 2 assessments function as components of the teaching and learning cycle. Progress is not measured by performance on any single assessment. Rather, Pre-AP Algebra 2 offers a place to practice, to grow, and to recognize that learning takes time. The assessments are updated and refreshed periodically.

LEARNING CHECKPOINTS

Based on the Pre-AP Algebra 2 Course Framework, the learning checkpoints require students to examine data, models, diagrams, and short texts—set in authentic contexts—in order to respond to a targeted set of questions that measure students' application of the key concepts and skills from the unit. All eight learning checkpoints are automatically scored, with results provided through feedback reports that contain explanations of all questions and answers as well as individual and class views for educators. Teachers also have access to assessment summaries on Pre-AP Classroom, which provide more insight into the question sets and targeted learning objectives for each assessment.

The following tables provide a synopsis of key elements of the Pre-AP Algebra 2 learning checkpoints.

Format	Two learning checkpoints per unit Digitally administered with automated scoring and reporting Questions target both concepts and skills from the course framework
Time Allocated	Designed for one 45-minute class period per assessment
Number of Questions	10–12 questions per assessment ▪ 7–9 four-option multiple choice ▪ 3–5 technology-enhanced questions

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Domains Assessed	
Learning Objectives	Learning objectives within each key concept from the course framework
Areas of Focus	Three skill categories aligned to the Pre-AP mathematics areas of focus are assessed regularly across all eight learning checkpoints: ▪ greater authenticity of applications and modeling ▪ engagement in mathematical argumentation ▪ connections among multiple representations

Question Styles	Question sets consist of two or three questions that focus on a single stimulus, such as a diagram, graph, or table. Questions embed mathematical concepts in real-world contexts. *Please see page 72 for a sample question set that illustrates the types of questions included in Pre-AP learning checkpoints and the Pre-AP final exam.*

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PERFORMANCE TASKS

Each unit includes one performance-based assessment designed to evaluate the depth of student understanding of key concepts and skills that are not easily assessed in a multiple-choice format.

These tasks, developed for high school students across a broad range of readiness levels, are accessible while still providing sufficient challenge and the opportunity to practice the analytical skills that will be required in AP mathematics courses and for college and career readiness. Teachers participating in the official Pre-AP Program will receive access to online learning modules to support them in evaluating student work for each performance task.

Format	One performance task per unit May be administered online or in print ▪ If administered online, then a score report is available. Educator scored using scoring guidelines
Time Allocated	Approximately 45 minutes or as indicated
Number of Questions	An open-response task with multiple parts

Domains Assessed	
Key Concepts	Key concepts and prioritized learning objectives from the course framework
Skills	Three skill categories aligned to the Pre-AP mathematics areas of focus: ▪ greater authenticity of applications and modeling ▪ engagement in mathematical argumentation ▪ connections among multiple representations

PRACTICE PERFORMANCE TASKS

A practice performance task in each unit provides students with the opportunity to practice applying skills and knowledge in a context similar to a performance task, but in a more scaffolded environment. These tasks include strategies for adapting instruction based on student performance and ideas for modifying or extending tasks based on students’ needs.

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SAMPLE PERFORMANCE TASK AND SCORING GUIDELINES

The following task and set of scoring guidelines are representative of what students and educators will encounter on the performance tasks. (The example below is a practice performance task in Unit 2.)

PRACTICE PERFORMANCE TASK

Using Transformations to Model a Lion's Location

LEARNING OBJECTIVES

2.2.1 Compare a function f with an additive transformation of f, that is, $f(x+k)$ or $f(x)+k$.

2.2.2 Compare a function f with a multiplicative transformation of f, that is, $f(kx)$ or $k \cdot f(x)$.

2.2.3 Construct a representation of the transformation of a function.

PRACTICE PERFORMANCE TASK DESCRIPTION

In this practice performance task, students explore a real-world scenario in which scientists track the distance a lion travels in a day. Students write a function to model the lion's distance from a radio receiver and then use function transformation to write a new function that converts the units of measure for both the independent and dependent variables. This practice performance task provides students with an opportunity to explore additive and multiplicative transformations in a real-world context. Transformations that are applied to the output of a function can be thought of as a composition of functions in which the output of the original function becomes the input of the transformation function. In transformations that are applied to the input of a function, by contrast, the output of the transformation function becomes the input of the original function.

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Engagement in Mathematical Argumentation
- Connections Among Multiple Representations

SUGGESTED TIMING

~45 minutes

MATERIALS

- calculator (optional)

HANDOUT

- Unit 2 Practice Performance Task: Using Transformations to Model a Lion's Location

AP Connections

This performance task supports AP preparation through alignment to the following AP Calculus Course Skills:

- **2.B** Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.
- **3.E** Provide reasons or rationales for solutions and conclusions.

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ELICITING PRIOR KNOWLEDGE

The goal of this task is for students to demonstrate their understanding of using transformations and composition to construct a function that models a contextual scenario.

- To begin, introduce students to the practice performance task. An excerpt from the student handout of the task is shown in the Scoring Student Work section.

- To prepare students to engage in the task, you could ask the following warm-up problems:
  - Consider a general function $y = f(x)$. Compare the transformation $f(x)+1$ to the function $y = f(x)$.

    The transformation $f(x)+1$ is an additive transformation of *f*. It has the effect of adding 1 to the output of *f*. It can also be interpreted as a composition of *f* and $g(x) = x+1$ such that $f(x)+1 = g(f(x))$. The graph of $y = f(x)+1$ looks like the graph of $y = f(x)$ translated vertically up by 1 unit.

  - Compare the transformation $f(x+1)$ to the function $y = f(x)$.

    The transformation $f(x+1)$ is an additive transformation of *f*. It has the effect of adding 1 to each input value before *f* is applied. It can also be interpreted as a composition of *f* and $g(x) = x+1$ such that $f(x+1) = f(g(x))$. The graph of $y = f(x+1)$ looks like the graph of $y = f(x)$ translated horizontally to the left by 1 unit.

  - Compare the transformation $2f(x)$ to the function $y = f(x)$.

    The transformation $2f(x)$ is a multiplicative transformation of *f*. It has the effect of multiplying each output of *f* by 2. It can also be interpreted as a composition of *f* and $g(x) = 2x$ such that $2f(x) = g(f(x))$. The graph of $y = 2f(x)$ looks like the graph of $y = f(x)$ vertically dilated by a factor of 2.

  - Compare the transformation $f(2x)$ to the function $y = f(x)$.

    The transformation $f(2x)$ is a multiplicative transformation of *f*. It has the effect of multiplying each input value by 2 before *f* is applied. It can also be interpreted as a composition of *f* and $g(x) = 2x$ such that $f(2x) = f(g(x))$. The graph of $y = f(2x)$ looks like the graph of $y = f(x)$ dilated horizontally by a factor of $\frac{1}{2}$.

Meeting Learners' Needs

If you find that students need more support to engage with the warm-up problems, you can provide them with a simple, specific function like $f(x) = x^2$ or $f(x) = |x|$.

Meeting Learners' Needs

Students may find it helpful to make a table of values for each function to make sense of the order in which the transformation functions are composed and to keep track of the relationships between the inputs and outputs.

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- If students struggle with the warm-up problems, it could indicate that they are not yet fully prepared to engage in the practice performance task. You may find it beneficial to provide a just-in-time review of the concepts critical for success with this task: function composition and transformations of functions.

SUPPORTING STUDENTS

Here are some possible implementation strategies you could use to help support students in engaging with the task.

- **Previewing the Task:** To support students in identifying key features of the problem, you could display the introductory text and graph and read them with the entire class. This particular practice performance task includes a great deal of information that students must read and analyze. Allow some time for students to ask clarifying questions about anything they read or observe, which may decrease the likelihood that they start working on the task with incorrect interpretations of the given information.
- **Collaboration:** To encourage students to engage in academic conversations in mathematics, you could have them work in pairs to complete the task. It is not recommended that students work in groups of three or more. While there is ample work and enough potential discussion areas for two students, some students in groups of more than two may not have an opportunity to engage meaningfully in all parts of the task.
- **Chunking the Task:** To support students who struggle with time management or may be overwhelmed by large tasks, you could chunk the task into several parts. For this task, parts (a) and (b) could be completed together or separately. Parts (c) and (d) would be best completed together, and part (e) could be completed separately from the other parts.
- **Iteration Support:** If you choose to chunk the task, then be sure to spend a few moments discussing the components of the solution with students during each teacher-facilitated check. Focus on what revisions, if any, they could make to their solution to craft a more complete response. Parts (a) and (b) of the task require a written explanation, which students sometimes forget or neglect to include in their response.

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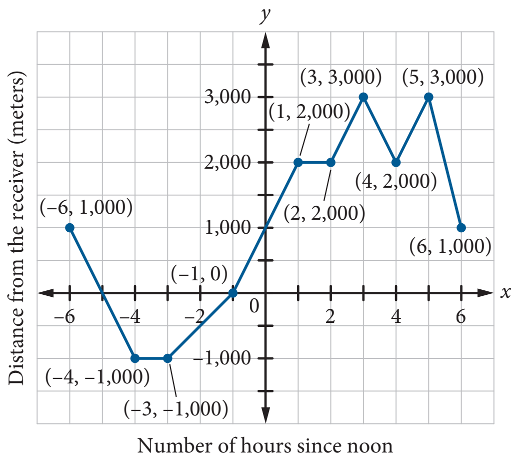
SCORING STUDENT WORK

Whether you decide to have students score their own solutions, have students score their classmates' solutions, or score the solutions yourself, you should use the results of the practice performance task to inform further instruction.

Using Transformations to Model a Lion's Location

African wildlife ecologists study lion populations' behaviors, movements, and interactions in order to develop appropriate conservation plans to better protect these declining populations. This work can be challenging as these large carnivores utilize vast territories across many African countries, such as Botswana, Namibia, Zambia, and Zimbabwe. To monitor the lions' movements, ecologists tag the animals with high-tech collars equipped with radio transmitters that send signals to a stationary receiver, which allows their movements to be tracked and plotted.

One particular ecologist tracks the location of a lion relative to the receiver over the course of one day. From her data, she assigns the number of hours since noon (12 p.m.) as the independent variable and the lion's distance from the receiver (in meters) as the dependent variable. She records the lion's distance from the receiver at different times and constructs a graph of a function model, *f*, shown in the following figure:



Her colleague also constructs a model based on observations of the same lion over the same time period but using different units of measure. So that the two ecologists can compare their models, the first ecologist needs to construct transformations of her function so that the unit of measure for the independent variable is the number of hours since midnight (12 a.m.) and the unit of measure for the dependent variable is kilometers.

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There are 12 possible points for this practice performance task.

Student Stimulus and Part (a)

(a) Consider the transformation of the unit of measure for the dependent variable described in the third paragraph. Use the points on the graph of *f* to write an algebraic representation of a transformation of *f* such that the dependent variable (the output) of the transformation of *f* is the distance of the lion from the receiver in kilometers. Explain whether the transformation is an additive or multiplicative transformation. What effect would this transformation have on the graph of *f*?

Sample Solution	Points Possible
Constructing a transformation of *f* so that the unit of measure of its dependent variable is kilometers requires dividing the output of *f* by 1,000. As a result, the output of the transformation yields values in kilometers rather than in meters. The algebraic representation of the transformation of *f* is $\frac{1}{1{,}000}f(x)$. It is a multiplicative transformation because the transformation involves multiplying the values of the dependent variable by $\frac{1}{1{,}000}$ (or, equivalently, dividing by 1,000). This transformation would have the effect of vertically dilating the graph of *f* by a factor of $\frac{1}{1{,}000}$. *Scoring note:* Students can receive the third point if they correctly describe the effect on the graph of their incorrect transformation.	**3 points maximum** 1 point for providing a correct algebraic representation of a transformation of *f* 1 point for providing a correct explanation of why the transformation is a multiplicative transformation 1 point for providing a correct description of the effect the transformation would have on the graph of *f*

Student Stimulus and Part (b)

(b) Consider the transformation of the unit of measure for the independent variable described in the third paragraph. Use the points given on the graph of *f* to write an algebraic representation of a transformation of *f* such that the independent variable (the input) of the transformation is the number of hours since midnight. Explain whether the transformation is an additive or multiplicative transformation. What effect would this transformation have on the graph of *f*?

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Sample Solution	Points Possible
Constructing a transformation of f so that the unit of measure of its independent variable is the number of hours since midnight requires subtracting 12 from the input of the transformation so that it corresponds with the input of f. The algebraic representation of the transformation of f is $f(x-12)$. For example, the ordered pair $(0, 1{,}000)$ for f means that at noon, the lion is 1,000 meters from the receiver. For the transformation of function f, the independent variable should be the number of hours since midnight. Because noon is 12 hours since midnight, an input of 12 should be associated with an output of 1,000 so that the ordered pair (12, 1,000) should be a solution to the transformation of f. Evaluating the function $f(x-12)$ at $x=12$ yields the same output as $x=0$: $f(12-12)=f(0)=1{,}000$. This is an additive transformation because the independent variable is transformed by addition. The transformation would have the effect of horizontally translating the graph of f to the right 12 units. *Scoring note:* Students can receive the third point if they correctly describe the effect on the graph of their incorrect transformation.	**3 points maximum** 1 point for providing a correct algebraic representation of a transformation of f 1 point for providing a correct explanation of why the transformation is an additive transformation 1 point for providing a correct description of the effect the transformation would have on the graph of f

Student Stimulus and Part (c)

(c) Let g be a function that models the relationship between the number of hours since midnight and the lion's distance from the receiver in kilometers. Use your transformations from parts (a) and (b) to write an algebraic representation of g in terms of f.

Sample Solution	Points Possible
An algebraic representation of g that uses the transformations of f from parts (a) and (b) is $g(x)=\frac{1}{1{,}000}f(x-12)$. *Scoring note:* Students can receive this point if they correctly combine the transformations they wrote in parts (a) and (b), even if one or both of those transformations are incorrect.	**1 point maximum** 1 point for combining the transformations of f from parts (a) and (b) to form function g

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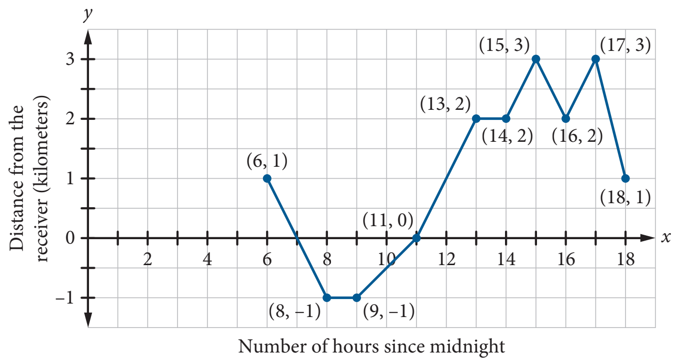
Student Stimulus and Part (d)

(d) Sketch a graph of the function g that you constructed in part (c) using the coordinates given on the graph of f.

Sample Solution	Points Possible
 A graph of g is shown in the figure. *Scoring note:* Students can receive full credit for this part if their graph for function g corresponds to their algebraic representation of g from part (c), even if the algebraic representation is incorrect.	**2 points maximum** 1 point for representing the horizontal translation 1 point for representing the vertical dilation

Student Stimulus and Part (e)

(e) Identify the intercepts and interpret their meaning in the graph of g. How does this meaning compare to the meaning of the x-intercepts of the graph of f?

Sample Solution	Points Possible
The x-intercepts of the graph of g are $(7,\ 0)$ and $(11,\ 0)$. This means that at 7 hours since midnight, or 7 a.m., and at 11 hours since midnight, or 11 a.m., the lion is 0 kilometers from the receiver. The values of the x-intercepts are different for the graph of f, but the meanings are the same. The x-intercepts for the graph of f are $(-5,\ 0)$ and $(-1,\ 0)$. These correspond to 5 hours before noon, or 7 a.m., and one hour before noon, or 11 a.m. The lion is 0 meters from the receiver at these times.	**3 points maximum** 1 point for identifying the x-intercepts of the graph of g 1 point for providing a correct interpretation of the meaning of the x-intercepts of g with correct units 1 point for providing a correct comparison of the x-intercepts for the graphs of f and g

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PROVIDING FEEDBACK ON STUDENT WORK

After scoring your students' work, it is important to identify trends in their responses to inform further instruction. These trends should include topics that students consistently displayed mastery of, as well as conceptual errors that students commonly made. Possible trends and suggested guidance for each part of the task follow, although the patterns you observe in your classroom may differ.

(a) If students have difficulty identifying whether they should multiply or divide by 1,000 to convert from meters to kilometers, it could be helpful to have them make a table of values with equivalent measurements in both units to look for a pattern. Also, students may find a graphing utility to be a useful tool for identifying the effect a multiplicative transformation has on the graph of a function.

Teacher Notes and Reflections

(b) If students have difficulty identifying whether they should add or subtract 12 to the input of the function transformation, it may be helpful to have them make a table of values with different times of day represented using both units of measure. Also, students may find a graphing utility to be a useful tool for identifying the effect of an additive transformation on the graph of a function.

Teacher Notes and Reflections

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(c) Students may have difficulty using function composition to combine the two transformed functions into a single transformed function model. They may benefit from creating a table of values with columns for hours since midnight, hours since noon, distance in meters, and distance in kilometers to see how the desired input (number of hours since midnight) can be associated with the desired output (distance in kilometers) using the given data.

Teacher Notes and Reflections

(d) If students cannot produce a correct graph by analyzing the graph of f and their algebraic representation of the function g, they may benefit from using a dynamic graphing utility to look for patterns in the effect that different transformations have on the graph of function f.

Teacher Notes and Reflections

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(e) Some students could experience difficulty interpreting the meaning of the x-intercepts of the graphs because they do not see the connections between the two representations. These students may benefit from some additional practice interpreting a graph generated from a different contextual scenario.

Teacher Notes and Reflections

Assure students that converting their score into a percentage does not provide an accurate measure of how they performed on the task. You can use the following suggested score interpretations with students to discuss their performance.

Points Received	How Students Should Interpret Their Score
11 or 12 points	"I know all of these algebraic concepts really well. This is top-level work. (A)"
8 to 10 points	"I know all of these algebraic concepts well, but I made a few mistakes. This is above-average work. (B)"
5 to 7 points	"I know some of these algebraic concepts well, but not all of them. This is average-level work. (C)"
2 to 4 points	"I know only a little bit about these algebraic concepts. This is below-average work. (D)"
0 or 1 point	"I don't know much about these algebraic concepts at all. This is not passing work. (F)"

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Using Transformations to Model a Lion's Location

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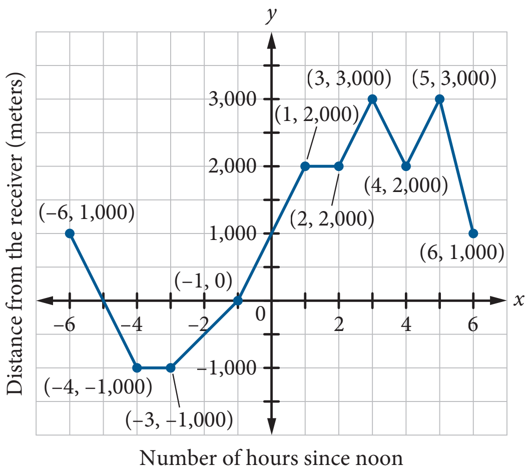

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(a) Consider the transformation of the unit of measure for the dependent variable described in the third paragraph. Use the points on the graph of f to write an algebraic representation of a transformation of f such that the dependent variable (the output) of the transformation of f is the distance of the lion from the receiver in kilometers. Explain whether the transformation is an additive or multiplicative transformation. What effect would this transformation have on the graph of f?

(b) Consider the transformation of the unit of measure for the independent variable described in the third paragraph. Use the points given on the graph of f to write an algebraic representation of a transformation of f such that the independent variable (the input) of the transformation is the number of hours since midnight. Explain whether the transformation is an additive or multiplicative transformation. What effect would this transformation have on the graph of f?

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(c) Let *g* be a function that models the relationship between the number of hours since midnight and the lion's distance from the receiver in kilometers. Use your transformations from parts (a) and (b) to write an algebraic representation of *g* in terms of *f*.

(d) Sketch a graph of the function *g* that you constructed in part (c) using the coordinates given on the graph of *f*.

(e) Identify the intercepts and interpret their meaning in the graph of *g*. How does this meaning compare to the meaning of the *x*-intercepts of the graph of *f*?

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FINAL EXAM

Starting in the school year of 2023–24, Pre-AP Algebra 2 will include a final exam featuring multiple-choice and technology-enhanced questions as well as an open-response question. The final exam will be a summative assessment designed to measure students' success in learning and applying the knowledge and skills articulated in the Pre-AP Algebra 2 Course Framework. The final exam's development will follow best practices such as multiple levels of review by educators and experts in the field for content accuracy, fairness, and sensitivity. The questions on the final exam will be pretested, and the resulting data will be collected and analyzed to ensure that the final exam is fair and represents an appropriate range of the knowledge and skills of the course.

The final exam will be delivered on a secure digital platform in a classroom setting. Educators will have the option of administering the final exam in a single extended session or in two shorter consecutive sessions to accommodate a range of final exam schedules.

Multiple-choice and technology-enhanced questions will be delivered digitally and scored automatically with detailed score reports available to educators. This portion of the final exam will build on the question styles and formats of the learning checkpoints; thus, in addition to their formative purpose, the learning checkpoints provide practice and familiarity with the final exam. The open-response question, modeled after the performance tasks, will be delivered as part of the digital final exam but scored locally by educators.

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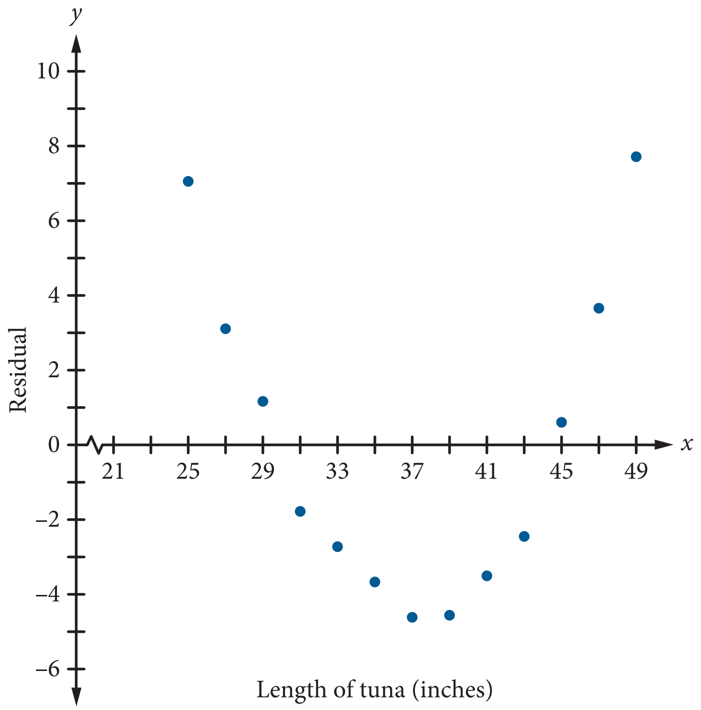
SAMPLE ASSESSMENT QUESTIONS

The following questions are representative of what students and educators will encounter on the learning checkpoints and final exam.



A particular fishing company records the length, in inches, and the weight, in pounds, of the yellowfin tuna that they catch. To determine if there is a linear relationship between the length and weight, the company constructs a regression equation, whose residual plot is shown in the figure. Which of the following conclusions about the relationship between the length and weight of the tuna is best supported using the information presented in the residual plot?

(A) A linear model is appropriate for the relationship between the length and weight because the residual plot displays a quadratic pattern.

(B) A linear model is appropriate for the relationship between the length and weight because the residual plot displays no observable pattern.

(C) A linear model is not appropriate for the relationship between the length and weight because the residual plot displays a pattern.

(D) A linear model is not appropriate for the relationship between the length and weight because the residual plot displays no observable pattern.

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Assessment Focus

In problem 1, students evaluate the appropriateness of using a linear equation to model the relationship between two variables by determining if the associated residual plot for the function model displays a discernable pattern.

Correct Answer: C

Learning Objective:

1.1.2 Use residual plots to determine whether a function model appropriately models a data set.

Area of Focus: Engagement in Mathematical Argumentation

The function h is defined as $h(x) = \sqrt{x^2 + 1}$. If h can be formed by the composition of functions f and g such that $h(x) = f(g(x))$, then which TWO of the following pairs of functions could be f and g?

(A) $f(x) = x^2 + 1$ and $g(x) = \sqrt{x}$

(B) $f(x) = \sqrt{x + 1}$ and $g(x) = x^2$

(C) $f(x) = x^2$ and $g(x) = \sqrt{x + 1}$

(D) $f(x) = \sqrt{x - 1}$ and $g(x) = x^2 + 2$

Assessment Focus

In problem 2, students decompose a composite function into two individual functions. The multiple responses required to correctly answer the question reinforces for students that there are multiple correct ways to decompose a function.

Correct Answer: B and D

Learning Objective:

2.1.3 Express a given algebraic representation of a function in an equivalent form as the composition of two or more functions.

Area of Focus: Connections Among Multiple Representations

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Let the functions f and g be defined as follows: $f(x) = \log_b(x)$ and $g(x) = \log_b(x - c)$, where c is a positive real number. Which of the following statements correctly describes the relationship between the graphs of $y = f(x)$ and $y = g(x)$?

(A) The graph of $y = g(x)$ coincides with the graph of $y = f(x)$ for all values of $x > c$.

(B) The graph of $y = g(x)$ is a dilation of the graph of $y = f(x)$ by a factor of c.

(C) The graph of $y = g(x)$ is a vertical translation of the graph of $y = f(x)$ down c units.

(D) The graph of $y = g(x)$ is a horizontal translation of the graph of $y = f(x)$ to the right c units.

Assessment Focus

In problem 3, students compare the algebraic forms of two related logarithmic functions to determine the relationship between their graphs.

Correct Answer: D

Learning Objective:

3.1.3 Construct a representation of a logarithmic function.

Area of Focus: Connections Among Multiple Representations

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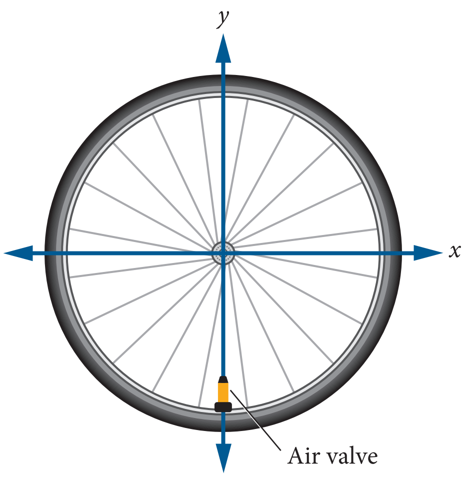
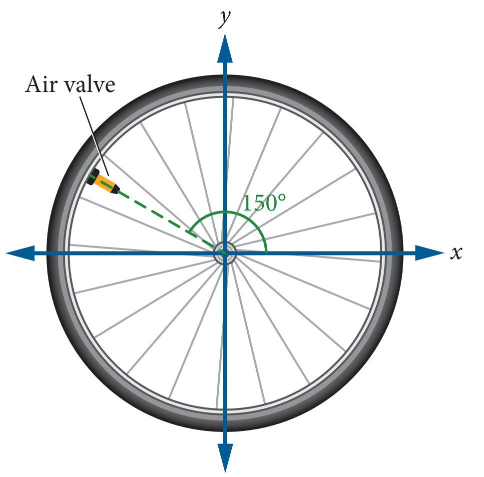
Starting Position	Final Position





A bicycle tire with a diameter of 700 millimeters is shown at its starting and final positions. At the start of a bicycle ride, the front tire's air valve is positioned at the bottom of the tire. After riding the bicycle, the final position of the air valve forms an angle in standard position, relative to the horizontal axis of the bicycle tire, that measures 150°. Which of the following expressions represents the height of the air valve from the ground?

(A) $350\cos\left(\frac{5\pi}{6}\right)$

(B) $350+350\cos\left(\frac{5\pi}{6}\right)$

(C) $350\sin\left(\frac{5\pi}{6}\right)$

(D) $350+350\sin\left(\frac{5\pi}{6}\right)$

Assessment Focus

In problem 4T, students are expected to apply their knowledge of the relationship between the coordinates of a point on a circle and a related central angle to answer a question in a real-world context.

Correct Answer: D

Learning Objective:

4T.1.4 Determine the exact coordinates of any point on a circle centered at the origin.

Area of Focus: Greater Authenticity of Applications and Modeling

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$$T(x, y) = \begin{bmatrix} -2 & 1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

In the coordinate plane, the transformation T maps point A to a point with coordinates (–10, 20). Which of the following ordered pairs are the coordinates of point A?

(A) $(-25, -15)$

(B) $(5, 0)$

(C) $(40, -100)$

(D) $(100, -70)$

Assessment Focus

In problem 4M, students are expected to multiply the column matrix that represents the image point, $\begin{bmatrix} -10 \\ 20 \end{bmatrix}$, by the inverse of the transformation matrix, $\begin{bmatrix} -1.5 & -0.5 \\ -2 & -1 \end{bmatrix}$, to determine the matrix form of the preimage point A, $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

Correct Answer: B

Learning Objective:

4M.1.8 Determine the preimage of a specified point under a given linear transformation.

Area of Focus: Connections Among Multiple Representations

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Pre-AP Algebra 2 Course Designation

Schools can earn an official Pre-AP Algebra 2 Course Designation by meeting the Pre-AP Program commitments summarized below. Pre-AP Course Audit Administrators and teachers will complete a Pre-AP Course Audit process to attest to these commitments. All schools offering courses that have received a Pre-AP Course Designation will be listed in the Pre-AP Course Ledger, in a process similar to that used for listing authorized AP courses.

PROGRAM COMMITMENTS

- Teachers have read the most recent *Pre-AP Algebra 2 Course Guide*.
- The school ensures that Pre-AP course frameworks and assessments serve as the foundation for all sections of the course at the school. This means that the school must not establish any barriers (e.g., test scores, grades in prior coursework, teacher or counselor recommendation) to student access and participation in the Pre-AP Algebra 2 coursework.
- Teachers administer at least one of two learning checkpoints per unit on Pre-AP Classroom and one performance task per unit.
- Teachers complete the foundational professional learning (Online Foundational Modules or Pre-AP Summer Institute) and at least one online performance task scoring module. The current Pre-AP coordinator completes the Pre-AP Coordinator Online Module.
- Teachers align instruction to the Pre-AP Algebra 2 Course Framework and ensure their course meets the curricular commitments summarized below.
- The school ensures that the resource commitments summarized below are met.
- Please note if a state's standards do not include trigonometry in Algebra 2, then the school may fulfill the Pre-AP course commitments with Unit 4M rather than Unit 4T. Schools have to choose only one fourth unit.

CURRICULAR COMMITMENTS

- The course provides opportunities for students to develop an understanding of the Pre-AP Algebra 2 key concepts and skills articulated in the course framework through the four units of study.
- The course provides opportunities for students to engage in the Pre-AP shared instructional principles.
    - close observation and analysis
    - evidence-based writing

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- higher-order questioning
- academic conversation

- The course provides opportunities for students to engage in the three Pre-AP mathematics areas of focus. The areas of focus are:
  - greater authenticity of applications and modeling
  - engagement in mathematical argumentation
  - connections among multiple representations
- The instructional plan for the course includes opportunities for students to continue to practice and develop disciplinary skills.
- The instructional plan reflects time and instructional methods for engaging students in reflection and feedback based on their progress.
- The instructional plan reflects making responsive adjustments to instruction based on student performance.

RESOURCE REQUIREMENTS

- The school ensures that participating teachers and students are provided computer and internet access.
- Teachers should have consistent access to a video projector for sharing web-based instructional content and short web videos.

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Accessing the Digital Materials

Pre-AP Classroom is the online application through which teachers and students can access Pre-AP instructional resources and assessments. The digital platform is similar to AP Classroom, the online system used for AP courses.

Pre-AP coordinators receive access to Pre-AP Classroom via an access code delivered after orders are processed. Teachers receive access after the Pre-AP Course Audit process has been completed.

Once teachers have created course sections, students can enroll in them via a join code. When both teachers and students have access, teachers can share instructional resources with students, assign and score assessments, and complete online learning modules; students can view resources shared by the teacher, take assessments, and receive feedback reports to understand progress and growth.

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Unit 4M

Unit 4M
Matrices and Their Applications

Overview

SUGGESTED TIMING: APPROXIMATELY 6 WEEKS

The final unit in this course provides an exploration of matrices and their applications. Matrices are useful tools for organizing information and are used in career fields as diverse as engineering, business, and statistics. The goal of the unit is for students to understand how to construct and use matrices to apply certain geometric transformations and solve systems of linear equations. It is important to note that the operations performed on matrices in this course are limited to multiplication and finding the nth power of a square matrix. Addition and subtraction of matrices are not included in this unit because these operations are not necessary for applying geometric transformations or solving systems of linear equations using matrices. Also, most of the operations performed on matrices in this unit are expected to be performed with technology.

In this unit, students build upon their previous experience with functions, inverse functions, and transformations in the plane to support their understanding of how certain geometric transformations can be represented with matrix multiplication. Students use the inverse of a transformation matrix to identify the preimage of a specified point under a given linear transformation. Students use the inverse of a matrix to solve a system of linear equations.

The unit culminates in an exploration of recursive processes that model the relationship between the sizes of subgroups of a partitioned population. Using real-world scenarios, students apply matrix operations to determine the previous and future sizes of these subgroups and observe that their sizes stabilize over time.

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UNIT 4M

ENDURING UNDERSTANDINGS

This unit focuses on the following enduring understandings:

- A matrix is a tool that can be used to efficiently organize mathematical information.
- Matrix multiplication provides efficient ways to evaluate multiple linear expressions simultaneously.
- Systems of linear equations can be solved using inverse matrices.

KEY CONCEPTS

This unit focuses on the following key concepts:

- 4M.1: Geometric Transformations—Using matrices to determine images and preimages of points in the coordinate plane
- 4M.2: Solving Systems of Equations with Matrices—Solving systems of equations with inverse matrices
- 4M.3: Applications of Matrix Multiplication—Modeling how subgroups of a population move in a predictable way

UNIT RESOURCES

The tables below outline the resources provided by Pre-AP for this unit.

Lessons for Key Concept 4M.1: Geometric Transformations			
Lesson Title	**Learning Objectives Addressed**	**Suggested Timing**	**Areas of Focus**
4M.1: Introduction to Linear Transformations	4M.1.1	~95 minutes	Connections Among Multiple Representations
4M.2: Expressing Linear Transformations with Matrix Multiplication	4M.1.2, 4M.1.3	~120 minutes	Connections Among Multiple Representations
4M.3: Determining Images of Multiple Points Simultaneously	4M.1.3, 4M.1.4	~90 minutes	Connections Among Multiple Representations, Engagement in Mathematical Argumentation

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Lessons for Key Concept 4M.1: Geometric Transformations			
Lesson Title	**Learning Objectives Addressed**	**Suggested Timing**	**Areas of Focus**
4M.4: Area and the Determinant of a Matrix	4M.1.5	~115 minutes	Connections Among Multiple Representations
4M.5: Sequences of Linear Transformations	4M.1.6	~75 minutes	Connections Among Multiple Representations
4M.6: Undoing Transformations and Finding Preimages	4M.1.7, 4M.1.8	~135 minutes	Connections Among Multiple Representations
	All learning objectives from this key concept are addressed with the provided materials.		

Learning Checkpoint 1: Key Concept 4M.1 (~45 minutes)
This learning checkpoint assesses learning objectives from Key Concept 4M.1. For sample items and learning checkpoint details, visit Pre-AP Classroom.

Lessons for Key Concept 4M.2: Solving Systems of Equations with Matrices	
There are no provided Pre-AP lessons for this key concept. As with all key concepts, this key concept is addressed in a learning checkpoint.	
	All Key Concept 4M.2 learning objectives should be addressed with teacher-developed materials.

Practice Performance Task for Unit 4M (~45 minutes)
This practice performance task assesses learning objectives from Key Concept 4M.2.

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Lessons for Key Concept 4M.3: Applications of Matrix Multiplication			
Lesson Title	**Learning Objectives Addressed**	**Suggested Timing**	**Areas of Focus**
4M.7: Introduction to Recursive Processes	4M.3.1, 4M.3.2	~110 minutes	Greater Authenticity of Applications and Modeling, Connections Among Multiple Representations
4M.8: Stabilized Recursive Processes	4M.3.2, 4M.3.3	~110 minutes	Greater Authenticity of Applications and Modeling, Connections Among Multiple Representations
	The following Key Concept 4M.3 learning objective is not addressed in Pre-AP lessons. It should be addressed in teacher-developed materials. ▪ Learning Objective: 4M.3.4		

Learning Checkpoint 2: Key Concepts 4M.2 and 4M.3 (~45 minutes)
This learning checkpoint assesses learning objectives from Key Concepts 4M.2 and 4M.3. For sample items and learning checkpoint details, visit Pre-AP Classroom.

Performance Task for Unit 4M (~60 minutes)
This performance task assesses learning objectives from Key Concept 4M.3.

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LESSON 4M.1

Introduction to Linear Transformations

LEARNING OBJECTIVE

4M.1.1 Identify whether a geometric transformation is a linear transformation.

LESSON OVERVIEW

CONTENT FOCUS

In this lesson, students learn to classify geometric transformations based on their mapping properties. Through their work, students define a linear transformation as one that maps the origin to the origin and lines to lines. Students discover that geometric transformations that can be expressed in the form $T(x, y) = (ax + by,\ cx + dy)$ satisfy both conditions and are therefore linear transformations. Students begin to apply linear transformations to individual points as well as to figures.

AREA OF FOCUS

- Connections Among Multiple Representations

SUGGESTED TIMING

~95 minutes

MATERIALS

- access to **Desmos.com**
- graph paper

HANDOUTS

Lesson

- 4M.1.A: Exploring Geometric Transformations
- 4M.1.B: Identifying Linear Transformations

Practice

- 4M.1.C: Exploring Linear Transformations

LESSON DESCRIPTION

Part 1: Exploring Geometric Transformations

In this part of the lesson, students explore the geometric transformations (rotations, reflections, translations, and dilations) that they encountered in geometry. Students apply given geometric transformations to points in the coordinate plane to determine their images. Students conclude this part with an understanding of how to represent a geometric transformation as a coordinate function. Students leverage their understanding of the algebraic structure of coordinate function representations when they encounter geometric transformations defined as linear transformations in Parts 2 and 3.

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Part 2: Defining Linear Transformations
In this part of the lesson, students explore the image of the letter A in the coordinate plane under different geometric transformations. They use their work to discover that transformations of the form $T(x, y) = (ax + by, cx + dy)$ always map the origin to the origin and lines to lines. Transformations that can be expressed in this form are called linear transformations.

Part 3: Identifying Linear Transformations
In Part 3, students identify linear transformations that are expressed as coordinate functions. They examine algebraic expressions in coordinate function form to determine which characteristic(s) of linear transformations are or are not satisfied.

FORMATIVE ASSESSMENT GOAL

This lesson prepares students to complete the following formative assessment activity.

Consider the following algebraic representations of geometric transformations:

(a) $T(x, y) = (x - 5y + x, y - 3x + y)$

(b) $T(x, y) = \left(2\left(x^2 + y\right) - x^2, -\left(y^2 - x\right) + y^2\right)$

(c) $T(x, y) = (x(y + 2), 3(x + 2))$

1. Which of the given geometric transformations is a linear transformation? Give a reason for your answer.
2. Rewrite the geometric transformation you identified as linear in the form $T(x, y) = (ax + by, cx + dy)$.

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Part 1: Exploring Geometric Transformations

~45 MIN.

UNIT 4M

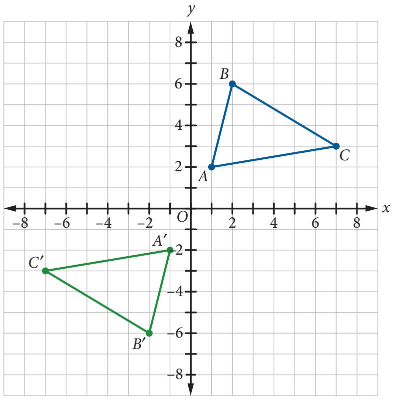
In this part of the lesson, students explore the geometric transformations (rotations, reflections, translations, and dilations) that they encountered in geometry. Students apply given geometric transformations to points in the coordinate plane to determine their images. Students conclude this part with an understanding of how to represent a geometric transformation as a coordinate function. Students leverage their understanding of the algebraic structure of coordinate function representations when they encounter geometric transformations defined as linear transformations in Parts 2 and 3.

Warm-Up

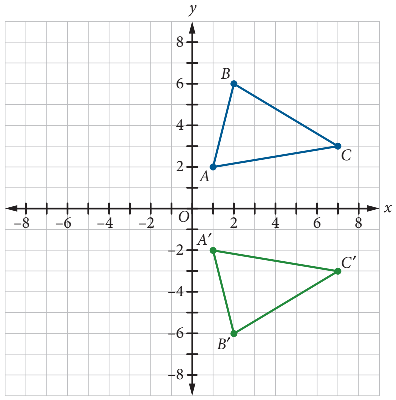
1. Using your own words, define geometric transformations.
2. For each of the following graphs, identify the corresponding geometric transformation from the given list.

   A. Dilation
   B. Horizontal translation
   C. Rotation 180° about the origin
   D. Reflections over x-axis
   E. Vertical translation
   F. Reflections over y-axis
   G. Reflection across the line $y = x$
   H. Rotation counterclockwise 90° about the origin

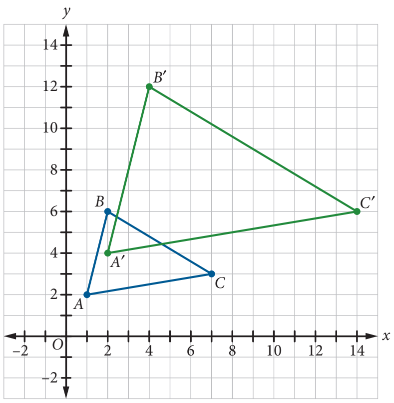
(a)



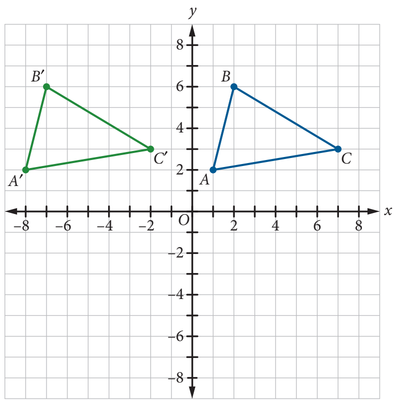
(b)



(c)



(d)



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Guiding Student Thinking

To support students with using their own words to describe geometric transformations in written form, you can present paragraph frames to get them started. An example of a paragraph frame is: "Geometric transformations map each preimage to___________. Geometric transformations are functions that describe ___________. The input of a geometric transformation is a ___________, and the output is ___________."

Student Task

The task is provided on **Handout 4M.1.A: Exploring Geometric Transformations**. In this task, students are given three points in the coordinate plane and are asked to apply a geometric transformation to determine their images. Next, students analyze the coordinates of the preimage and image points in tables and/or in the coordinate plane to write the coordinate function representation.

Facilitating the Task

Instructional Rationale

In this part of the lesson, students actively work to reactivate their geometric knowledge about transformations as functions. Specifically, they should recall that functions of the form $T(x, y)$ can have any point in the plane as an input, and the output is another point in the plane.

- Begin by providing students with Handout 4M.1.A and graph paper. Arrange students in small groups of 3 or 4 to complete this task. Allow students some time to closely observe the task and ask clarifying questions.
- Before students begin working on the handout, the term *coordinate function* shown in the fourth column of the handout may need further clarification and an explanation of its meaning.
- Students learned in Pre-AP Geometry with Statistics that algebra can be used to express how a transformation maps one set of coordinate pairs to another set of coordinate pairs. All transformations can be represented using function notation, but some transformations are difficult to define as algebraic expressions. Encouraging students to recall this information can help them make meaning of a coordinate function.
- To help students develop the meaning of coordinate functions, start by asking the following questions:

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- What is a function?

  A function is a type of relationship between two quantities where each input is related to one (and only one) value of the output.

- What are the main characteristics of a function?

  The main characteristics of a function are the domain (input values), the range (output values), and the one-to-one relationship between the input and output values.

- What is a coordinate? What are coordinates?

  A coordinate is the location of a point along one axis in the coordinate plane. Coordinates are the location of a point along both axes in the coordinate plane.

- If we call something a coordinate function, what do you think it could mean?

  A coordinate function is an algebraic representation that shows how to map input coordinates to output coordinates.

- Looking at the last column on the handout, how are coordinate functions similar to other functions you know? How are they different?

  Like other functions I have worked with, coordinate functions relate two equivalent mathematical statements.

- As you circulate around the room, you may find that it is challenging for some students to apply geometric transformations. For students who struggle to apply geometric transformations, have them graph the preimage points and then apply the geometric transformation to find the image of the points. Students can analyze their tables and graphs to write the coordinate function representations for each geometric transformation.

Meeting Learners' Needs

To see the effect of a rotation on the preimage of a figure, students can plot the points $(1,\ 2), (2,\ 6)$, and $(7,\ 3)$ and then rotate their graph paper as specified in the table to see the images of the points.

Summarizing the Task

- Summarize the student task by eliciting responses from groups of students about the coordinate function representations for selected geometric transformations. For example, you could highlight reflection over the y-axis, rotation counterclockwise 90° about the origin, and/or translation 3 units down and 2 units to the right.
- In addition to reviewing some of the geometric transformations on the student task, you could ask the following questions to formatively assess if students are

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able to articulate what happens to a set of points in the coordinate plane when the transformations are applied.

- Given the transformation $T(x, y) = (2x - 1, 2y - 1)$, what do you expect to happen when T is applied to a point?

  I expect a dilation by a scale factor of 2 and then a vertical and horizontal shift.

- Compare and contrast the transformations $T(x, y) = (2(x - 1), 2(y - 1))$ and $T(x, y) = (2x - 1, 2y - 1)$. Are there any differences? Why or why not?

  Yes. If the first coordinate function representation is rewritten as $T(x, y) = (2x - 2, 2y - 2)$, the differences in the vertical and horizontal shifts between it and $T(x, y) = (2x - 1, 2y - 1)$ are clear.

- Before moving on, explain to students that the transformations they reviewed map all points of the preimage to the corresponding points of the image. For example, if they think of the triangles in the student task, all the points of the figure move simultaneously after the transformations are applied, not just vertices A, B, and C.
- Explain that these transformations all have algebraic representations that can be used to evaluate the transformations of specific points of the preimage.
- Tell students that they will later explore other algebraic representations of geometric transformations that will be defined in the next lesson as linear transformations.

Guiding Student Thinking

Students might visualize these transformations as affecting only the points in the table or the preimage to the image. It is important to emphasize that the entire coordinate plane has shifted when geometric transformations are applied to points in the coordinate plane.

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Part 2: Defining Linear Transformations

~35 MIN.

In this part of the lesson, students explore the image of the letter A in the coordinate plane under different geometric transformations. They use their work to discover that transformations of the form $T(x, y) = (ax + by,\ cx + dy)$ always map the origin to the origin and lines to lines. Transformations that can be expressed in this form are called linear transformations.

Student Task

In this task, students use Desmos to explore the image of the letter A on a coordinate plane under four geometric transformations. Students start the task by making predictions about the image under each of the four geometric transformations before observing the actual image in Desmos. In the task, students also record what they notice and what they wonder as they analyze these transformations. Students end the task with an understanding that certain geometric transformations can be classified as linear transformations based on their properties and their algebraic structures.

Facilitating the Task

- Begin by providing students with **Handout 4M.1.B: Identifying Linear Transformations**. Students need to access the Desmos link in the student task to complete problem 2. Arrange students in small groups of 3 or 4 to complete this task. Allow students some time to closely observe the task and ask clarifying questions.
- Explain to students that they will explore four transformations in which they will focus on the properties of these transformations. Assign half the groups to form Group A and half the groups to form Group B.

Instructional Rationale

Each group will complete the same activity but will use different geometric transformations. This will ensure that there are sufficient examples for students to generalize from later in the lesson.

- Give students about ten minutes to complete problem 1 of the student task in which they are asked to sketch their predictions for the image of the letter A under each transformation. Be sure that students add their sketches to the first column of the table on the handout.

Meeting Learners' Needs

For students who would benefit from an extension, you can ask them to write a verbal description that characterizes their prediction of the transformation.

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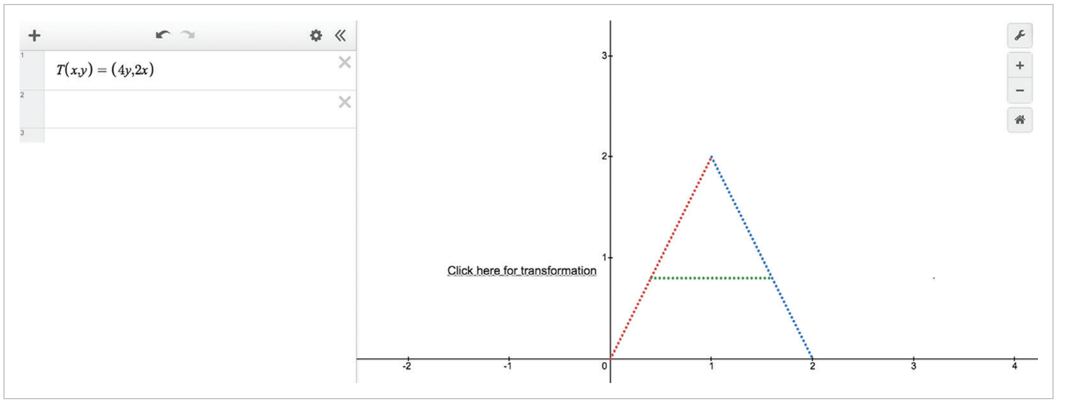
UNIT 4M

- Students may approach the prediction task in a variety of ways.
  - Some students may determine the images of the five main points that define the letter A (i.e., $(0,\ 0)$, $(1,\ 2)$, $(2,\ 0)$, $(0.4,\ 0.8)$, and $(1.6,\ 0.8)$) and connect these points with line segments.
  - Some students may determine the images of five points and notice that the image of the points along the sides of the letter A are not necessarily colinear.
  - Some students may try to describe general characteristics of the transformation based on the corresponding algebraic rule.

Instructional Rationale

While it is important that students have an opportunity to sketch their conjectures for the image of the letter A under each transformation, it is not important that students make a correct conjecture. Instead, students should have the chance to be surprised by the actual image under each transformation. Therefore, limit this activity to about ten minutes since the goal is to provide the necessary foundation for students to later have an academic conversation about what they noticed in applying the transformations.

- Once students have completed their predictions, they should complete problem 2 and test their predictions using the Desmos applet Transforming A at **preap.org/Desmos-LinearTransformations**.
- Explain to students that they will enter their assigned geometric transformation in the first row and then click “Click here for transformation” on the graph next to “Image of A” to see the image appear. The following image shows what students will see after clicking the Desmos link. The preimage of the letter A is dotted so that students can see the difference between the preimage and image.



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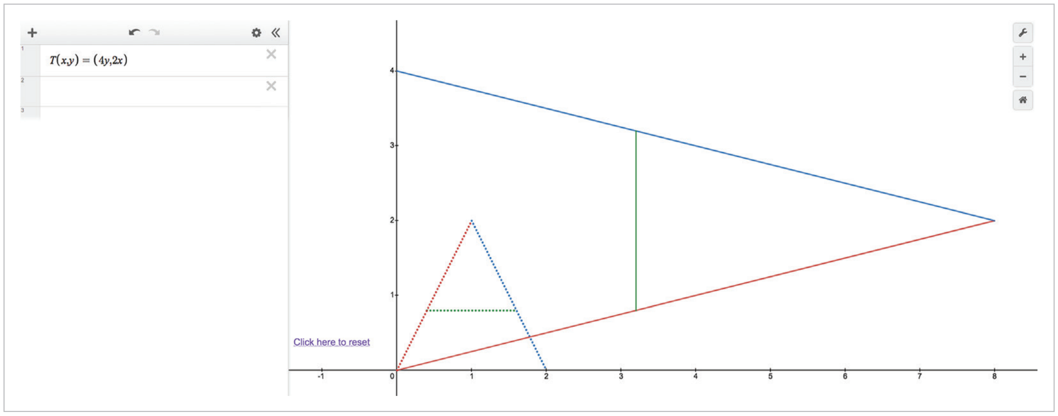
- Once students click “Click here for transformation,” their transformation will appear. The image of the transformation follows:



- Instruct students to sketch the image of the letter A on their handouts for all four transformations in column 2 for their assigned group. Make sure students label the axes in addition to the shape of the letter A.
- For each transformation, students should record what they notice and what they wonder in column 3, and include ways in which their prediction was correct or incorrect. Encourage students to describe characteristics of the transformation such as the shapes of the segments or curves that connect the vertices.
- As students work, circulate around the room and ask them what they notice about the transformations. Consider asking questions such as:
  - Are there any transformations where the actual image was different from your conjectured image?

    Answers may vary. After applying the geometric transformation to the letter A, curves connect some vertices instead of lines.

  - Are there any similarities that you notice amongst the different transformations?

    Answers may vary. For geometric transformations with squared x and/or y terms, the vertices are connected by curves. For geometric transformations with linear x and y terms, the vertices are connected by line segments.

- Students may notice that sometimes the image of the letter A is not symmetrical, that sometimes the sides of the image of the letter A are curved while the middle line remains straight, or that sometimes the image is rotated. While not the focus of this lesson, these are all mathematically accurate observations and should be validated in the discussion.

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- Ask students to think about which features of a transformation would produce an image in which both side segments of the letter A are straight and the origin is mapped to the origin. Help students see that transformations of the form $T(x, y) = (ax + by,\ cx + dy)$ satisfy these conditions.
- Guide students to understand that a transformation with this structure will always satisfy two conditions: lines will map to lines and the origin will map to the origin. This structure also allows us to determine some properties of transformations that cannot be efficiently described by a verbal description, such as $T(x, y) = (2x - 3y,\ -x + y)$.
- Tell students that just as we gave a special name to rigid motion transformations that preserve congruence, we give a special name to transformations in which the origin maps to the origin and lines map to lines. They are called *linear transformations* and can always be expressed in the form $T(x, y) = (ax + by,\ cx + dy)$. You can add linear transformations to your classroom's word wall, if you have one.

Instructional Rationale

The approach to defining a linear transformation presented in this unit is intended to make the concept accessible to Algebra 2 students. Linear transformations are often defined algebraically as T if and only if $T(a\mathbf{v}) = aT(\mathbf{v})$ and $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ where **u** and **v** are vectors. In the Cartesian coordinate system, these two definitions are equivalent. It is outside the scope of this course for students to know the algebraic definition of a linear transformation.

Summarizing the Task

- Engage students in an academic conversation to discuss what they noticed about the geometric transformations they explored. You can use the questions below to initiate a whole-class conversation. Listen for two main takeaways: lines map to lines and the origin maps to the origin.
  - What do you notice about the coordinate function representations of transformations that produce curved sides on the image of the letter A?

    The geometric transformations where the function rules include terms with factors of x^2, y^2, or the product xy result in images with curved sides.
  - What do you notice about the coordinate function representations of transformations that produce straight sides on the image of the letter A?

    The exponent of x and y needs to be 1 for the image to have straight sides.

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- What do you notice about the coordinate function representations of the transformations of the letter A so that the image of the preimage point at the origin is not at the origin?

  The image of the preimage point at the origin is not at the origin when a constant is added to either or both coordinates (e.g., $x+2$, $3x-y+4$, …).

- What do you notice about the coordinate function representations of transformations for which the image of the point at the origin remains at the origin?

  The image of the point at the origin remains at the origin when the expression defining the x- and y-coordinate has an x and/or y in each term.

- Before moving on, be sure to help students generalize that when one of the coordinates includes a product of two variables (e.g., xx, yy, xy, $x(xx)$, ...), the image of the sides of the letter A will be curved. The variable for which this is the case does not impact whether the sides of the image of the letter A are curved: the image of the x- and y-coordinates can be defined either in terms of only x or only y or as sums of x and y (e.g., x, y, $2x$, $2y$, $3x+2y$, ...).
- Explain to students that this idea of keeping the letter A anchored at the origin is more formally described as mapping the origin to the origin so that $T(0,\ 0)=(0,\ 0)$.
- To help students summarize the properties of a linear transformation, have them construct a vocabulary graphic organizer to organize the coordinate function representation, characteristics, examples, and nonexamples of linear transformations like the one shown here:

Classroom Ideas

If copies of Handout 1.1: Vocabulary Graphic Organizer are not available, you can have students write the term *linear transformation* in the center of a standard piece of paper and then fold it in half lengthwise and widthwise to create four areas for the coordinate function representation, characteristics, examples, and nonexamples.

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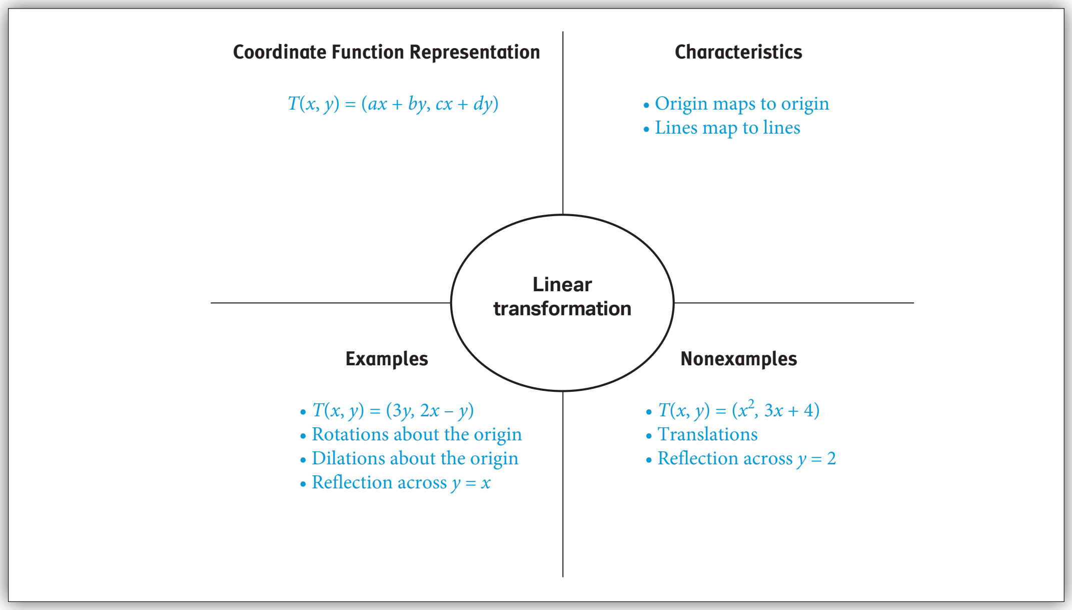

Handout 1.1 used for linear transformations

Part 3: Identifying Linear Transformations

~15 MIN.

In Part 3, students identify linear transformations that are expressed as coordinate functions. They examine algebraic expressions in coordinate function form to determine which characteristic(s) of linear transformations are or are not satisfied.

Student Task

Determine whether each of the following transformations is a linear transformation. If the transformation is not a linear transformation, determine which characteristic(s) are not satisfied.

1. $T(x, y) = (y - x,\ 2x)$
2. $T(x, y) = (x(3+2y),\ 4x - y)$
3. $T(x, y) = (3x,\ 2y)$
4. $T(x, y) = (3,\ 2x)$
5. $T(x, y) = (3(x+y),\ 2y)$
6. $T(x, y) = (3(x+y),\ 2(x+1))$

Facilitating the Task

- This task gives students practice with writing geometric transformations in coordinate function form to determine if they are linear transformations.

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- Begin by allowing students some time to closely observe the task and ask clarifying questions.
- Have students work with a partner to complete the task. Students should be focused on whether the transformations can be expressed in the form $T(x,\ y) = (ax + by,\ cx + dy)$. As you circulate around the room, remind students that they might need to apply the distributive property and perform other operations to rewrite each coordinate function in the form $T(x,\ y) = (ax + by,\ cx + dy)$.
- Students should be able to identify that Transformations 1, 3, and 5 are linear transformations because they can be expressed in the form $T(x,\ y) = (ax + by,\ cx + dy)$.
- Students should be able to identify that Transformations 2, 4, and 6 are not linear transformations, and be able to explain why. They should observe that Transformation 2 is equivalent to $T(x,\ y) = (3x + 2xy,\ 4x - 1y)$. Because the image of the x-coordinate includes the product xy, lines do not map to lines. Students should recognize that for Transformation 4, the image of the x-coordinate is 3, which is not of the form $ax + by$. This means this transformation does not map the origin to the origin. They should also recognize that Transformation 6 is equivalent to $T(x,\ y) = (3x + 3y,\ 2x + 2)$, so the image of the y-coordinate includes the addition of a constant. This also means the origin does not map to the origin.

Summarizing the Task

- To summarize this lesson, engage students in an academic conversation about how they determined whether each transformation was or was not linear.
  - What are you looking at to determine whether a geometric transformation is a linear transformation?

    Students should focus on whether the transformation can be expressed in the form $T(x,\ y) = (ax + by,\ cx + dy)$. Encourage them to apply the distributive property before trying to rewrite each coordinate in the specified form.
  - Do you have to apply the transformation to a point or a figure to determine whether it is a linear transformation?

    No, you can just look at the coordinate function representation to determine whether the geometric transformation is a linear transformation.

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Guiding Student Thinking

Many students will think the transformation $T(x,\ y) = (3(x + y),\ 2(x + 1))$ is an example of a linear function because of the expressions defining each coordinate $(3x + 3y,\ 2x + 2)$. The y-coordinate is of the form $mx + b$, which is a representation of a linear function but not a linear transformation because it includes a constant term. The addition of the constant means the origin does not map to the origin. Explain to students that each coordinate expression relates three variables: the x-coordinate of the preimage, the y-coordinate of the preimage, and one coordinate of the image. Help students see that the meaning of *linear* in the term *linear transformation* is different from the meaning of *linear* in *linear function.*

- For additional practice in identifying linear transformations by rewriting them as coordinate functions, have students complete **Handout 4M.1.C: Exploring Linear Transformations**.

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Assess and Reflect on the Lesson

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

Consider the following algebraic representations of geometric transformations:

(a) $T(x, y) = (x - 5y + x,\ y - 3x + y)$

(b) $T(x, y) = \left(2\left(x^2 + y\right) - x^2,\ -\left(y^2 - x\right) + y^2\right)$

(c) $T(x, y) = (x(y + 2),\ 3(x + 2))$

1. Which of the given geometric transformations is a linear transformation? Give a reason for your answer.

   Geometric transformation (a) is a linear transformation because it maps lines to lines and the origin to the origin.

2. Rewrite the geometric transformation you identified as linear in the form $T(x, y) = (ax + by,\ cx + dy)$.

   $T(x, y) = (2x - 5y,\ -3x + 2y)$

HANDOUTS

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 4M.1.A: Exploring Geometric Transformations

Geometric Transformation	Preimage	Image	Coordinate Function Representation
Reflection across the line $y = x$	(1, 2)	(2, 1)	$T(x, y) = (y, x)$
	(2, 6)	(6, 2)	
	(7, 3)	(3, 7)	
Rotation counterclockwise 90° about the origin	(1, 2)	(−2, 1)	$T(x, y) = (-y, x)$
	(2, 6)	(−6, 2)	
	(7, 3)	(−3, 7)	
Rotation clockwise 90° about the origin	(1, 2)	(2, −1)	$T(x, y) = (y, -x)$
	(2, 6)	(6, −2)	
	(7, 3)	(3, −7)	
Translation 3 units down and 2 units to the right	(1, 2)	(3, −1)	$T(x, y) = (x + 2, y - 3)$
	(2, 6)	(4, 3)	
	(7, 3)	(9, 0)	
Dilation centered at the origin with a scale factor of 2	(1, 2)	(2, 4)	$T(x, y) = (2x, 2y)$
	(2, 6)	(4, 12)	
	(7, 3)	(14, 6)	
Reflection across the x-axis	(1, 2)	(1, −2)	$T(x, y) = (x, -y)$
	(2, 6)	(2, −6)	
	(7, 3)	(7, −3)	
Reflection across the y-axis	(1, 2)	(−1, 2)	$T(x, y) = (-x, y)$
	(2, 6)	(−2, 6)	
	(7, 3)	(−7, 3)	
Rotation 180° counterclockwise about the origin	(1, 2)	(−1, −2)	$T(x, y) = (-x, -y)$
	(2, 6)	(−2, −6)	
	(7, 3)	(−7, −3)	

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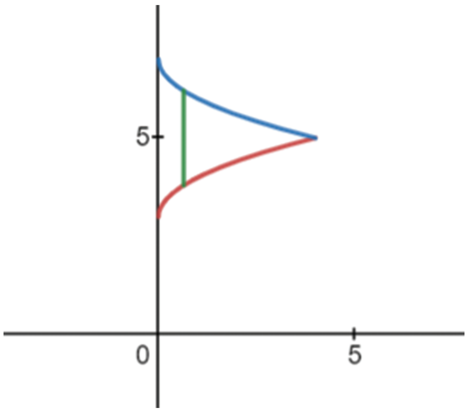
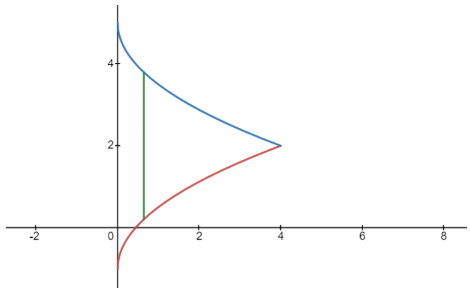
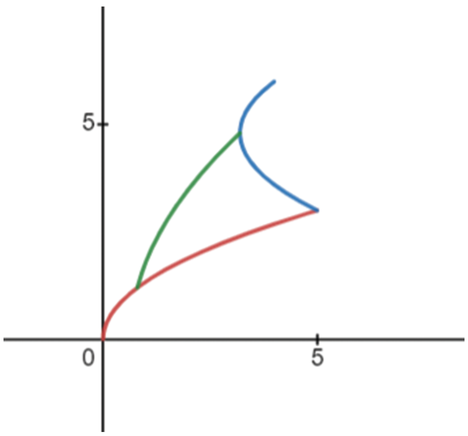
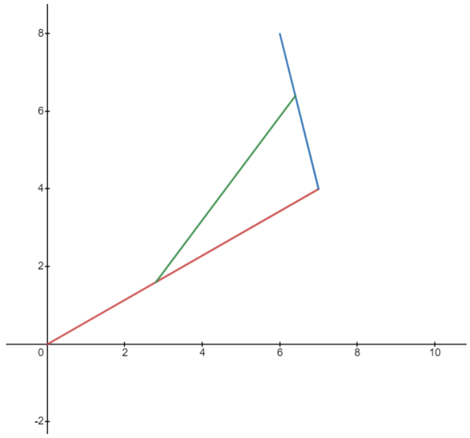
UNIT 4M

Handout 4M.1.B: Identifying Linear Transformations

Group A Transformations Actual Image	Group B Transformations Actual Image
$T(x, y) = (y^2, 2x + 3)$ 	$T(x, y) = (y^2, 3x - 1)$ 
$T(x, y) = (x^2 + y^2, 3x)$ 	$T(x, y) = (3x + 2y, 4x)$ 

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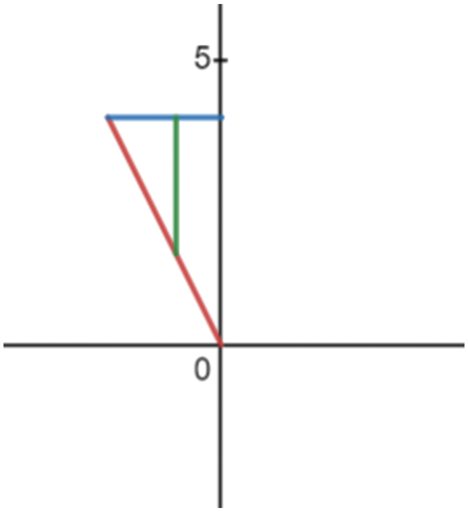
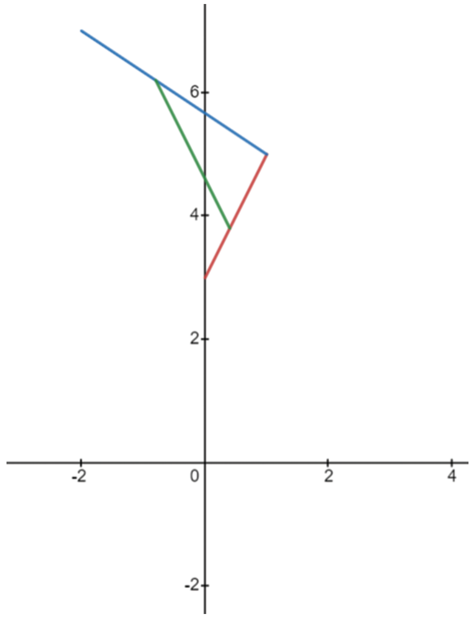
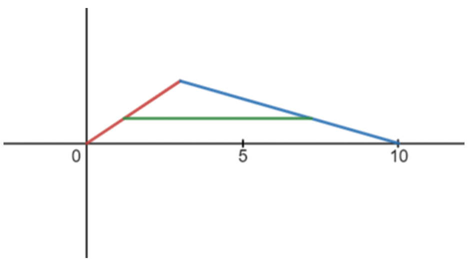
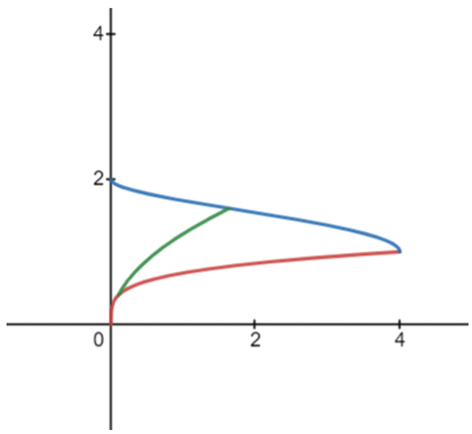
Group A Transformations Actual Image	Group B Transformations Actual Image
$T(x,\ y) = (-y,\ 2x + y)$  	$T(x,\ y) = (y - x,\ 2x + 3)$ 
$T(x,\ y) = (5x - y,\ y)$  	$T(x, y) = (x^2 y^2,\ x)$ 

Handout 4M.1.C: Exploring Linear Transformations

1. (a) This geometric transformation is not a linear transformation because point B is at the origin, but its image B' is not at the origin. Under this transformation, the origin does not map to the origin, so this geometric transformation is not a linear transformation.

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(b) This geometric transformation is not a linear transformation because lines do not map to lines.

(c) This geometric transformation is a linear transformation because lines map to lines and the origin to the origin.

2. (a) This maps with geometric transformation 1(b) where lines do not map to lines.

(b) This maps with geometric transformation 1(c) because this is the algebraic structure of a linear transformation.

(c) This maps with geometric transformation 1(a) since $T(0,\ 0)$ is not equal to $(0,\ 0)$.

3. (a) $T(x,\ y) = (3x + 0y,\ 0x + 2y)$

(b) $T(x,\ y) = (-x + 2y,\ 4x + 7y)$

(c) $T(x,\ y) = (0x - 1y, -3x + 0y)$

4. This geometric transformation is not a linear transformation. Since point A is at the origin, if this is a linear transformation then A' will also be at the origin. However, then points A', B', and C' would not be collinear. This implies that lines don't map to lines, and thus this transformation is not linear.

5. Syd's reasoning is incorrect because after applying the distributive property and combining like terms, this transformation can be expressed as $T(x,\ y) = (2x + 6y,\ 3x - 4y)$, which has the form of a linear transformation.

6. Finley's reasoning is incorrect because in the linear transformation $T(x,\ y) = (ax + by,\ cx + dy)$, the values of a, b, c, and d can be 0. Thus, this linear transformation can be expressed as $T(x,\ y) = (3x + 0y,\ 2x + 0y)$, which is the form of a linear transformation.

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UNIT 4M

LESSON 4M.2

Expressing Linear Transformations with Matrix Multiplication

LEARNING OBJECTIVES

4M.1.2 Express a system of linear expressions using multiple representations.

4M.1.3 Multiply two matrices when the product is defined, both with and without technology.

LESSON OVERVIEW

CONTENT FOCUS

In this lesson, students are introduced to matrices and matrix notation. This lesson connects students' understanding of linear transformations as coordinate functions to matrices. This is the first of three lessons in which students work to define matrix multiplication and explore its applications. Students use matrix multiplication to determine the image of a single point in the coordinate plane. This requires students to represent linear transformations as matrix transformations by connecting the coefficients of a linear transformation expressed as a coordinate function to the entries of the linear transformation in matrix form. By the end of this lesson, students know how to multiply a 2×2 matrix by a 2×1 matrix to determine the image of a single point. This lesson prepares students to apply a linear transformation to multiple points in the coordinate plane simultaneously in Lesson 4M.3. In Lesson 4M.5, students multiply a 2×2 matrix by a 2×1 matrix to express a sequence of linear transformations as a single linear transformation.

AREA OF FOCUS

- Connections Among Multiple Representations

SUGGESTED TIMING

~120 minutes

HANDOUTS

Lesson

- 4M.2.A: Writing Linear Transformations with Matrix Notation
- 4M.2.B: Defining the Features of a Matrix
- 4M.2.C: Determining the Images of Points in the Plane

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LESSON DESCRIPTION

Part 1: Writing Linear Transformations with Matrix Notation

In this part of the lesson, students rewrite linear transformations expressed as coordinate functions using matrix notation. The goal of this part of the lesson is to get students to recognize the connection between the coefficients of a linear transformation written as a coordinate function and the entries in an equivalent matrix form known as a matrix transformation. This understanding allows students to write linear transformations as matrix transformations.

Part 2: Defining the Features of a Matrix

In this part of the lesson, students learn the definition of a matrix, observe its features, and identify special types of matrices.

Part 3: Matrix Multiplication to Perform Linear Transformations on Points in the Plane

In this part of the lesson, students use matrix multiplication to apply a linear transformation to a specific point. In this lesson, matrix multiplication is limited to 2×2 matrices multiplied by 2×1 matrices. At this introductory stage, matrix multiplication is performed without technology to evaluate a linear transformation for a given input.

FORMATIVE ASSESSMENT GOAL

This lesson prepares students to complete the following formative assessment activity.

Consider the following two geometric transformations:

1. $T(x, y) = (2y - 3x, -x(1 - y)$
2. $T(x, y) = (-2(y - 2x), 3(-x + y))$

Complete the following:

(a) Identify which geometric transformation is a linear transformation. Explain your reasoning.

(b) Rewrite the linear transformation as a matrix transformation.

(c) Find the image of the point $(-1, 2)$.

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Part 1: Writing Linear Transformations with Matrix Notation

~30 MIN.

In this part of the lesson, students rewrite linear transformations expressed as coordinate functions using matrix notation. The goal of this part of the lesson is to get students to recognize the connection between the coefficients of a linear transformation written as a coordinate function and the entries in an equivalent matrix form known as a matrix transformation. This understanding allows students to write linear transformations as matrix transformations.

Warm-Up

- Activate students' prior learning from Lesson 4M.1 by having them discuss the following questions with a partner:
  - What is the definition of a linear transformation?

    A linear transformation is a transformation that maps the origin to the origin and lines to lines.
  - Give some examples of linear transformations.

    Examples of linear transformations include rotations about the origin, reflections across lines through the origin, and dilations centered at the origin.
  - What does a linear transformation look like when it is expressed as a coordinate function?

    $T(x, y) = (ax + by, cx + dy)$
  - What is the mathematical term for *a*, *b*, *c*, and *d* when a linear transformation is expressed as a coordinate function?

    These are called coefficients.

Student Task

In this task, students are given five linear transformations explored in Lesson 4M.1 to write as coordinate functions and as matrix transformations. Students start by rewriting the linear transformations as coordinate functions before exploring matrices. Through this exploration, students make connections between the coefficients of a linear transformation written as a coordinate function and the linear transformation written as a matrix transformation. This task ends with students using matrices to express linear transformations.

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Facilitating the Task

- Begin by providing students with **Handout 4M.2.A: Writing Linear Transformations with Matrix Notation**. Allow students some time to closely observe the task and ask clarifying questions.
- Have students work individually to complete the "Written as a Coordinate Function" column of the student task. Then have them compare their answers with a partner.
- After students have had some time to complete this column of the task and share their responses, explain that the coefficients summarize the effect of the transformation. Explain that one way to organize the coefficients is in a form called a *matrix*. Matrices (plural of matrix) are useful tools for organizing information and are used in career fields as diverse as engineering, business, and statistics.
- Highlight for students that they will explore another way to express linear transformations in matrix form. Writing linear transformations in matrix form is often referred to as a *matrix transformation*.
- Before students start to write linear transformations as matrix transformations, show them a linear transformation written as coordinate function, $T(x, y) = (2x - 3y,\ -x + y)$ and highlight the coefficients $a = 2$, $b = -3$, $c = -1$, and $d = 1$.
- Next, show students the equivalent matrix form of the transformation as $T(x,\ y) = \begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix}$ and explain that matrix notation consists of large square brackets and includes numbers or mathematical expressions, called *entries*, organized in rows and columns. Explain that they will learn more about matrices over the rest of the unit.
- To support students' understanding of this new notation, the following color-coded picture may provide a helpful visualization:

$$T(x, y) = (\overbrace{2x - 3y}^{\text{row 1}}, \underbrace{-1x + 1y}_{\text{row 2}}) \Leftrightarrow T(x, y) = \begin{bmatrix} \overset{x \downarrow}{2} & \overset{y \downarrow}{-3} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Have students spend a few moments sharing what they notice and wonder about the matrix form representation in comparison to the coordinate function representation. Then ask the following questions:

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- What do you notice about the matrix $\begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix}$ and the coordinate function $T(x, y) = (2x\ -3y,\ -1x + 1y)$?

  It contains only the coefficients of the linear transformation, not the variables.

- What do you notice about the second matrix on the right?

  It contains only the variables x and y.

- Where are a, b, c, and d arranged in matrix transformation notation?

  They are all in the matrix on the left and are arranged from left to right and top to bottom in the order that they appear in the coordinate function representation of the linear transformation. This is expressed as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

- Let's focus on that matrix on the left. Where are the coefficients located that correspond to the x-coordinate of the linear transformation?

  The coefficients that correspond to the x-coordinate are located in the first column of the matrix.

- Where are the coefficients located that correspond to the y-coordinate of the linear transformation?

  The coefficients that correspond to the y-coordinate are located in the second column of the matrix.

- Now ask students to return to the task and work with a partner to complete the "Written as a Matrix Transformation" column.

Summarizing the Task

- Conclude the task by asking student volunteers to share their responses written in the "Written as a Matrix Transformation" column.
- As students share out, explain that the matrix on the left is called the *coefficient matrix* because of how its entries are assigned. In general, a linear transformation written as a coordinate function is in the form $T(x, y) = (ax + by,\ cx + dy)$. As a matrix transformation, this is expressed equivalently as $T(x, y) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.
- The key takeaway in this part of the lesson is for students to make connections between coordinate function and matrix transformation notations. Students should recognize that the coefficients of the transformation expressed as a coordinate function are the entries of the coefficient matrix of the corresponding matrix transformation.

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Part 2: Defining the Features of a Matrix

~30 MIN.

In this part of the lesson, students learn the definition of a matrix, observe its features, and identify special types of matrices.

Student Task

In this task, students record the meaning of the term *matrix*, its features, and the names of special types of matrices on the vocabulary graphic organizer presented on **Handout 4M.2.B: Defining the Features of a Matrix.** Students conclude the task by determining the dimensions of matrices and identifying the names of any special types of matrices. Students' understanding of dimensionality supports their understanding of matrix multiplication, which is introduced in the next part.

Facilitating the Task

- Begin by providing students with Handout 4M.2.B. Allow students some time to closely observe the task and ask clarifying questions.
- Remind students that the idea of using matrices to express linear transformations was introduced in Part 1. Explain that they will now discuss specific features they should know about matrices that are important as they continue to explore linear transformations expressed as matrix transformations.
- Next, use the matrix at the top of Handout 4M.2.B as a visual reference to define the term *matrix*, its features, and the names of special types of matrices. Ask students to write the definitions on their vocabulary graphic organizer as you explain them, referring to the matrix at the top half of the handout as needed. Answers to this handout can be found in the Assess and Reflect section of the lesson.
- After explaining the defining features of a matrix and the names of special types of matrices, give students some time to complete the task.

Summarizing the Task

- Ask students to share their responses for problems (a) though (f) of the handout.
- Before moving on, explain to students that matrices can have any number of rows and columns. In this lesson, they will continue to focus on 2×2 matrices—which have 2 rows and 2 columns—where the entries of the first column are the coefficients of the x-coordinate of a linear transformation, and the entries of the second column are the coefficients of the y-coordinate. Students will also work with 2×1 matrices, where the entry in the first row represents the x-coordinate of a point, and the entry in the second row represents the y-coordinate of that point.

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Part 3: Applying Linear Transformations to Points in the Plane

~60 MIN.

In this part of the lesson, students use matrix multiplication to apply a linear transformation to a specific point. In this lesson, matrix multiplication is limited to 2×2 matrices multiplied by 2×1 matrices. At this introductory stage, matrix multiplication is performed without technology to evaluate a linear transformation for a given input.

Student Task

The student task is provided on **Handout 4M.2.C: Determining the Images of Points in the Plane.** In this task, students start by evaluating a linear transformation written as a coordinate function to determine the image of a single point. Next, students are introduced to matrix multiplication, which is limited to multiplying the coefficient matrix of a two-variable linear transformation by a column matrix representing the coordinates of a point in the plane. Students conclude the task with an understanding of how to apply a linear transformation in matrix form to determine the image of a single point.

Facilitating the Task

- Begin by providing students with Handout 4M.2.C and arranging students in groups of 3 or 4. Allow students some time to closely observe the task and ask clarifying questions.
- Explain to students that they will explore how to determine the image of a point using two different but equivalent forms of linear transformations: the coordinate function representation and the matrix multiplication method.
- Starting with the coordinate function representation in Part 1 of the handout, walk students through the process given for problem A for using a coordinate function to find the image of a point. Repeat the process for problem B by substituting coordinate values into the coordinate function. Reinforce that students are evaluating the function for a given input, the coordinates of a point called the preimage, to determine the output, the coordinates of the image of that point.
- After completing problems A and B, give students some time to complete the rest of Part 1 of the student task.
- As you circulate around the room, be sure that students are correctly evaluating the coordinate function for the given inputs and determining the images of the points.
- Next, use problem A on the handout to illustrate to students how to determine the image of a point using *matrix multiplication.*

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- To represent the process of performing matrix multiplication, you can color code row 1 and one half of the column matrix in one color and row 2 and the other half of the column matrix in another color. After assigning colors, articulate the procedure for multiplying matrices represented by the two steps shown in the diagram.

$$T(3, 2) = \begin{bmatrix} 4 & -3 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \cdot 3 + (-3) \cdot 2 \\ \; \end{bmatrix} = \begin{bmatrix} 6 \\ \; \end{bmatrix}$$

$$T(3, 2) = \begin{bmatrix} 4 & -3 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \cdot 3 + (-3) \cdot 2 \\ -2 \cdot 3 + 5 \cdot 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

- Display problem A, explaining to students that when using matrix multiplication, the coefficient matrix on the left is multiplied by the 2×1 column matrix on the right. The column matrix on the right represents the coordinates of points in the plane. The coordinates of the point in the coordinate plane to which a transformation will be applied are the inputs of the coordinate function. In problem A, the input is $(3,2)$, the preimage point.
- Tell students that to multiply a 2×2 matrix by a 2×1 column matrix, they will begin by working with the entries of row 1 in the coefficient matrix and the entries of the 2×1 column matrix. They multiply the corresponding entries and add the resulting products. This sum is the entry in the top row of a new 2×1 column matrix. Repeat the same process with the entries of row 2 and the entries of the column matrix to find the entry in the bottom row of the new 2×1 column matrix.
- The column matrix that results from matrix multiplication represents the coordinates of the output, the image point. In problem A, the image is $(6, 4)$.
- Next, have each member of the group choose one coordinate pair from the handout (B, C, D, or E) and create a similar color-coded visual to demonstrate how the linear transformation is applied using matrix multiplication to determine the image of the point.
- After students complete their problem, have them share their work with another student who worked on the same problem. Encourage students to compare how they determined the image for their coordinate pair.
- Once students have demonstrated an understanding of how to multiply matrices, have them complete Parts 2 and 3 of the handout.

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Summarizing the Task

- Use the following questions to assess if students understand how to apply linear transformations to a point in the plane:
  - What are the images of the points after a linear transformation in Part 3 of the handout?

    Answers for the handout are provided in the Assess and Reflect section of the lesson.
  - How do your responses in Part 2 of the handout correspond to your responses in Part 3?

    The coordinates of the outputs in Part 3 of the handout correspond to the column matrix that is the product of matrix multiplication.
- Explain to students that matrices and matrix multiplication provide a different way to express linear transformations in an equivalent form as matrix transformations. It also provides a method for applying linear transformations to a point in the plane. This will be important later when students learn how to apply linear transformations simultaneously to multiple points in the coordinate plane.

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ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

Consider the following two geometric transformations:

1. $T(x, y) = (2y - 3x, -x(1 - y))$
2. $T(x, y) = (-2(y - 2x), 3(-x + y))$

Complete the following:

(a) Identify which of the geometric transformations is a linear transformation.

Transformation 2 is a linear transformation because it can be expressed in the form $T(x, y) = (ax + by, cx + dy)$ where $a = 4$, $b = -2$, $c = -3$, and $d = 3$.

(b) Rewrite the linear transformation as a matrix transformation.

$$T(x, y) = \begin{bmatrix} 4 & -2 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(c) Find the image of the point $(-1, 2)$.

$$T(x, y) = \begin{bmatrix} 4 & -2 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -8 \\ 9 \end{bmatrix}$$

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HANDOUTS

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 4M.2.A: Writing Linear Transformations with Matrix Notation

Linear Transformation	Written as a Coordinate Function $T(x, y) = (ax + by, cx + dy)$	Written as a Matrix Transformation
$T(x, y) = (3x + 2y, 4x)$	$T(x, y) = (3x + 2y, 4x + 0y)$	$T(x, y) = \begin{bmatrix} 3 & 2 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
$T(x, y) = (-y, 2x + y)$	$T(x, y) = (0x - 1y, 2x + 1y)$	$T(x, y) = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
$T(x, y) = (-(y + 1) + 1, -3x)$	$T(x, y) = (0x - 1y, -3x + 0y)$	$T(x, y) = \begin{bmatrix} 0 & -1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
$T(x, y) = (3x, 2x)$	$T(x, y) = (3x + 0y, 2x + 0y)$	$T(x, y) = \begin{bmatrix} 3 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$
$T(x, y) = (2(x + y) - 3x, 4(x + 2y) - y)$	$T(x, y) = (-1x + 2y, 4x + 7y)$	$T(x, y) = \begin{bmatrix} -1 & 2 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

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Handout 4M.2.B: Defining the Features of a Matrix

Matrices Graphic Organizer	
$$\begin{array}{cccc} \text{Column 1} & \text{Column 2} & \text{Column 3} & \text{Column 4} \\ \end{array}$$ $$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix} \begin{array}{l} \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \end{array}$$	
Matrix	A matrix is a rectangular array of numbers or expressions organized in rows and columns. The numbers or expressions in a matrix are called entries.
Column of a Matrix	A column of a matrix is a set of entries arranged vertically from top to bottom.
Row of a Matrix	A row of a matrix is a set of entries arranged horizontally left to right.
Dimension of a Matrix	The dimension of a matrix is expressed as the number of rows by the number of columns.
Square Matrix	A square matrix has the same number of rows as columns.
Row Matrix	A row matrix has dimension $n \times 1$.
Column Matrix	A column matrix has dimension $1 \times n$.
Things to Notice	Count the number of entries down a column of a matrix to determine its number of rows. Count the number of entries across a row of a matrix to determine its number of columns.

(a) 2×2; square matrix

(b) 3×2

(c) 3×3; square matrix

(d) 2×3

(e) 1×3; row matrix

(f) 3×1; column matrix

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Handout 4M.2.C: Determining the Images of Points in the Plane

Part 1

	(x, y)	$T(x, y) = (4x - 3y, -2x + 5y)$	$T(x, y)$
A	$(3, 2)$	$T(3, 2) = (4(3) - 3(2), -2(3) + 5(2)) = (6, 4)$ $\begin{bmatrix} 4 & -3 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \cdot 3 + (-3) \cdot 2 \\ -2 \cdot 3 + 5 \cdot 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$	$(6, 4)$
B	$(-1, 3)$	$T(-1, 3) = (4(-1) - 3(3), -2(-1) + 5(3)) = (-13, 17)$ $\begin{bmatrix} 4 & -3 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \cdot (-1) + (-3) \cdot 3 \\ -2 \cdot (-1) + 5 \cdot 3 \end{bmatrix} = \begin{bmatrix} -13 \\ 17 \end{bmatrix}$	$(-13, 17)$
C	$(-4, 5)$	$T(-4, 5) = (4(-4) - 3(5), -2(-4) + 5(5)) = (-31, 33)$ $\begin{bmatrix} 4 & -3 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} -4 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \cdot (-4) + (-3) \cdot 5 \\ -2 \cdot (-4) + 5 \cdot 5 \end{bmatrix} = \begin{bmatrix} -31 \\ 33 \end{bmatrix}$	$(-31, 33)$
D	$(0, -6)$	$T(0, -6) = (4(0) - 3(-6), -2(0) + 5(-6)) = (18, -30)$ $\begin{bmatrix} 4 & -3 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ -6 \end{bmatrix} = \begin{bmatrix} 4 \cdot 0 + (-3) \cdot (-6) \\ -2 \cdot 0 + 5 \cdot (-6) \end{bmatrix} = \begin{bmatrix} 18 \\ -30 \end{bmatrix}$	$(18, -30)$
E	$(2, -5)$	$T(2, -5) = (4(2) - 3(-5), -2(2) + 5(-5)) = (23, -29)$ $\begin{bmatrix} 4 & -3 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 4 \cdot 2 + (-3) \cdot (-5) \\ -2 \cdot 2 + 5 \cdot (-5) \end{bmatrix} = \begin{bmatrix} 23 \\ -29 \end{bmatrix}$	$(23, -29)$

Part 2

1. $\begin{bmatrix} 3 & -1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$

2. $\begin{bmatrix} -5 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \end{bmatrix} = \begin{bmatrix} -5 \\ 10 \end{bmatrix}$

3. $\begin{bmatrix} 1 & -4 \\ 12 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -14 \\ -15 \end{bmatrix}$

4. $\begin{bmatrix} 3 & 7 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} -41 \\ -9 \end{bmatrix}$

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Part 3

Input (x, y)	**Coordinate Function** $T(x,y) = (ax + by, cx + dy)$	**Output** (x, y)
$(2,\ 3)$	$T(x,\ y) = (3x - 1y,\ 4x + 0y)$	$(3,\ 8)$
$(1,\ 7)$	$T(x,\ y) = (-5x - 0y,\ 3x + 1y)$	$(-5,\ 10)$
$(-2,\ 3)$	$T(x,\ y) = (1x - 4y,\ 12x + 3y)$	$(-14,\ -15)$
$(-2,\ -5)$	$T(x,\ y) = (3x + 7y,\ 2x + 1y)$	$(-41,\ -9)$

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UNIT 4M

LESSON 4M.3

Determining Images of Multiple Points Simultaneously

LEARNING OBJECTIVES

4M.1.3 Multiply two matrices when the product is defined, both with and without technology.

4M.1.4 Determine the image of a set of points under a linear transformation or a sequence of linear transformations.

LESSON OVERVIEW

CONTENT FOCUS

In this lesson, students learn to use technology to compute the product of two matrices. Introducing the computational efficiency of technology helps students conceptualize and work with matrices whose dimensions are greater than 2×2. Students also explore how to represent the coordinates of n points simultaneously in a $2 \times n$ matrix. In this context, students first observe the noncommutativity of matrix multiplication. Finally, students formalize their thinking and observe that the dimensions of two matrix factors determine whether their product is defined.

LESSON DESCRIPTION

Part 1: Using Technology to Perform Matrix Multiplication

In this part of the lesson, students learn how to use technology to perform matrix multiplication. Through this exploration, students attend to the order in which matrices are multiplied, allowing them to observe the noncommutativity of matrix multiplication. This observation of noncommunity leads to the understanding that matrix multiplication is not defined when attempting to multiply a 2×1 matrix by a 2×2 matrix. Also, students learn how to use the Desmos matrix calculator to determine the image of a point in the coordinate plane.

AREAS OF FOCUS

- Connections Among Multiple Representations
- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~90 minutes

MATERIALS

- matrix calculator, such as **Desmos.com/matrix**

HANDOUT

Lesson

- 4M.3: Determining the Images of Multiple Points Simultaneously

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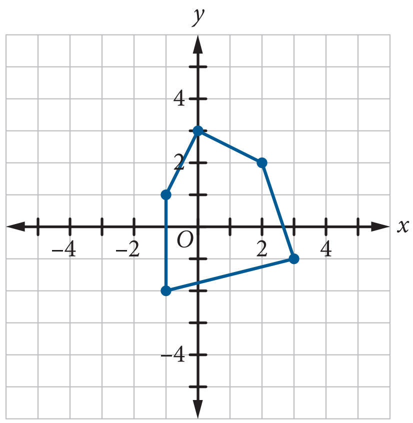
Part 2: Determining the Image of Multiple Points
In this part of the lesson, students learn that a $2 \times n$ matrix can be used to represent the coordinates of n points. Students discover that representing multiple coordinate pairs in a single matrix allows them to efficiently determine the image of n points simultaneously.

FORMATIVE ASSESSMENT GOAL

This lesson prepares students to complete the following formative assessment activity.

A polygon is shown in the following image:



The matrix transformation given by $T(x, y) = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ is applied to all points in the coordinate plane. What are the vertices of the image of this polygon?

UNIT 4M

Part 1: Using Technology to Perform Matrix Multiplication

~45 MIN.

In this part of the lesson, students learn how to use technology to perform matrix multiplication. Through this exploration, students attend to the order in which matrices are multiplied, allowing them to observe the noncommutativity of matrix multiplication. This observation of noncommunity leads to the understanding that matrix multiplication is not defined when attempting to multiply a 2×1 matrix by a 2×2 matrix. Also, students learn how to use the Desmos matrix calculator to determine the image of a point in the coordinate plane.

Warm-Up

- You may find it helpful to provide the following warm-up to assess each student's progress toward mastery of the content discussed in Lesson 4M.2. Students should work individually to complete the warm-up.
- Present students with the linear transformation $T(x, y) = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Have students determine the images of the points $(0, 0)$, $(4, 2)$, $(7, 8)$, and $(3, 6)$ under this linear transformation using matrix multiplication.
- Limit this warm-up to five minutes as it is also designed to motivate the need for a more efficient way to determine the images of numerous points.
- Before moving on, ensure that students understand how the coordinates of the preimage (the input of the linear transformation) and the coordinates of the image (the output of the linear transformation) are represented in the matrix representation.
  - How do we represent the input and output of the linear transformation using matrix representation?

    We represent the input and output of a linear transformation as a 2×1 matrix (a column matrix) with the x-coordinate of the preimage (or image) in the first row and the y-coordinate of the preimage (or image) in the second row.

Student Task

In this task, students explore how to use the Desmos matrix calculator to determine the product of two matrices. Students explore dimensionality by developing the idea of the noncommutativity of matrix multiplication. Students make this observation when using the matrix calculator to find the products AB and BA, where A is a 2×2 matrix and B is a 2×1 matrix. Students conclude the task knowing how to use the matrix calculator to determine the image of a single point under a linear transformation.

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Use the following linear transformation, $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, and input, $B = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$, to respond to the following questions:

1. What is the dimension of matrix *A*?
2. What is the dimension of matrix *B*?
3. Use Desmos to verify if matrix multiplication is defined for AB, BA, or both. Record your findings.
4. For instances in which matrix multiplication is not defined, explain why you think this is happening.

Facilitating the Task

- Begin by displaying the student task and allowing students some time to closely observe the task and ask clarifying questions. Students should identify A as a 2×2 matrix and B as a 2×1 matrix.
- Have students continue to work independently to ensure that each understands how to use technology to determine the image of a point under a specified linear transformation.
- Remind students that they have learned how to determine the image of a point given the coordinate function representation or given a matrix representation of the linear transformation.
- Explain to students that in this part of the lesson they will learn how to use technology to determine the image of a specified point.
- Instruct students to use the Desmos matrix calculator to complete problem 3. Navigate to the Desmos matrix calculator and show students how to enter matrices A and B in Desmos. You can use this as an opportunity to reinforce that the dimensions of a matrix are defined as the number of rows by the number of columns. Thus, matrix A has dimension 2×2 and matrix B has dimension 2×1.
- After students enter both matrices in Desmos, show them how to perform matrix multiplication.
- Have each student make and record their prediction about whether the image of point $(7, 8)$ under the linear transformation described by matrix A is determined by the product AB, the product BA, or both. Many students will think that $AB = BA$ since in their prior experience with multiplication has been commutative. At this point in the lesson,

> **Classroom Ideas**
> This part of the lesson can be done on a graphing calculator such as a TI-83 or T-84 if students do not have access to the Desmos matrix calculator.

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you do not need to correct students' thinking. Students will discover that matrix multiplication is not commutative when they attempt to compute AB and BA. Students should find that $AB = \begin{bmatrix} 38 \\ 39 \end{bmatrix}$. Desmos will return an error that explains the dimensions are incompatible when trying to compute BA.

- Engage students in a class conversation about the relationship between the product $AB = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 7 \\ 8 \end{bmatrix}$ and the product $BA = \begin{bmatrix} 7 \\ 8 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$. Encourage students to use their understanding of matrix multiplication to explain why the product BA is not defined. Students can record their thinking in problem 4.

Guiding Student Thinking

Students further explore the noncommutativity of matrix multiplication in Lesson 4M.5 where they consider sequences of linear transformations.

Summarizing the Task

- To summarize the task, elicit responses from students by asking the following questions:
  - Explain why the product of AB does not equal BA. Justify your reasoning.

    The product of AB does not equal BA because multiplication is only defined for one of the pairs of factors. AB is defined because the number of entries in each row of matrix A corresponds to the number of entries in each column of matrix B. Using Desmos, the product of AB is $\begin{bmatrix} 38 \\ 39 \end{bmatrix}$. BA is not defined because the number of entries in each row of matrix B does not correspond to the number of entries in each column of matrix A. Also, Desmos outputs an error message when trying to perform matrix multiplication.

- Remind students that the first entry in the product of matrix multiplication is defined as the sum of the products of each entry in the first row of the first matrix and the corresponding entry in the first column of the second matrix. This means that the number of columns in the first matrix needs to be equivalent to the number of rows in the second matrix. This reasoning should help students understand why the product BA is not defined. Since AB is defined and BA is not defined, these matrices are not equivalent. In other words, AB is not equal to BA. Ensure that students observe that the order in which matrices are multiplied matters, so matrix multiplication is noncommutative.

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- Now that students understand why matrix multiplication is not defined for BA, elicit responses from students about what they noted for problems 1 and 2. Highlight for students that matrix multiplication is defined for two matrices when the number of entries in each row of the left matrix is equal to the number of entries in each column of the right matrix. The dimension of AB, produced by multiplying matrices of dimension 2×2 and 2×1, is 2×1. The dimension of BA would be determined by the dimensions of the matrix factors, 2×1 and 2×2, but that operation is not defined for these matrices in that order.
- The key takeaway for this part of the lesson is for students to understand when matrix multiplication is defined and know how to use technology to determine the product. This will be important when students learn how to apply a linear transformation to multiple points simultaneously in the next part of the lesson.

Part 2: Determining the Image of Multiple Points

~45 MIN.

In this part of the lesson, students learn that a $2 \times n$ matrix can be used to represent the coordinates of n points. Students discover that representing multiple coordinate pairs in a single matrix allows them to efficiently determine the image of n points simultaneously.

Student Task

The student task is provided on **Handout 4M.3: Determining the Images of Multiple Points Simultaneously**. Students learn how to represent the coordinates of multiple points using a single matrix, and then use the Desmos matrix calculator to find the images of all those points simultaneously. Within this task, students make an argument for why using matrix multiplication is more efficient than using a coordinate function representation to determine the images of multiple points.

Facilitating the Task

- Begin by distributing Handout 4M.3 and assigning students to work in small groups of 3 or 4.
- Give students a couple of minutes to come up with a plan for how to approach problem 1 on the handout. Some students may think about how to determine the image of the figure all at once instead of identifying the image of the figure one point at a time using matrix multiplication. The goal of this part of the task is to help students see that they can determine the image of a polygon by determining the images of all its vertices simultaneously using a single matrix that includes all the vertices' coordinates and matrix multiplication.

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Instructional Rationale

While the construction of a polygon is not the main point of this lesson, this task is designed to reinforce ideas of linear transformations from Lesson 4M.1 and to prepare students for Lesson 4M.4, in which students interpret the determinant of a matrix as the scale factor that relates the area of the preimage and the area of the image.

- Select groups to share their strategies for how to approach the task. If no student suggests it, ask students if it is sufficient to determine the images of the four vertices. Have students justify their answer.
  - When determining the image of the parallelogram, why is it sufficient to just determine the images of its four vertices?

    Since lines map to lines under a linear transformation, finding the images of the endpoints of each line segment (the vertices) is sufficient to determine the image of the line segment.

  - Is the process that you followed for determining the image of the parallelogram tedious or efficient?

    Answers will vary. Most students will say this is a tedious process.
- Explain to students that they will now learn a more efficient way to use the matrix representation of a linear transformation to determine the images of all vertices at once.
- Remind students that matrices are structures used to organize information in ways that make arithmetic easier to perform. Highlight for students that they have recorded the coordinates of a point in a 2×1 column matrix, but matrices can have any number of rows and columns that are needed. Ask students to use what they know so far to make a prediction about how to record the coordinates of two points in a single matrix. Have them share their ideas.
- Students will likely create either a 2×2 matrix or a 4×1 matrix by stacking the coordinates. Remind students that the process of matrix multiplication involves multiplying the entries of each row by the corresponding entries in the column of the second matrix, with the resulting products summed. This means that the matrix encoding the coordinates of these two points should have two rows and two columns where each column corresponds to one point, with the first row containing the x-coordinates and the second row containing the y-coordinates.

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Instructional Rationale

This is students' introduction to multiplication of a 2×2 matrix by a 2×2 matrix. Students will get practice performing multiplication with matrices of these dimensions by hand later in the key concept when they work with sequences of linear transformations. However, students do not need to carry out this computation by hand in this lesson.

- For problem 2, students explore four options that may help them determine a more efficient way to find the image of a parallelogram. The goal of this part of the lesson is to have students leverage what they know about multiplying matrices to evaluate which option is best.
- As you circulate around the room, be sure that students are using Desmos to test all options presented on the student task.

Summarizing the Task

- Summarize the task by eliciting responses from groups of students for problem 2. Students should argue that it is more efficient to include all points in a single matrix before applying the linear transformation to it than to apply the linear transformation to four column matrices separately. If some students struggle to see this, have them reflect on the process they took in their warm-up to find the images of four points separately using matrix multiplication. They should recognize that the process of applying a linear transformation to one point at a time is more tedious.
- Highlight students' reasons for the options they selected by asking the following questions:
  - What option did you choose as an efficient way to determine the image of the parallelogram? Explain your reasoning.

    Options C and D are efficient ways to determine the image of the parallelogram because I can find the images of all vertices of the parallelogram at the same time.
  - What are the similarities and differences between the options you chose to be the most efficient for determining the image of the parallelogram?

    The columns that represent each input are ordered differently in each option, but the resulting image is the same for each of the vertices.
- Elicit responses from students for problem 4 by asking the following question:
  - What did you notice about the image of triangle ABC after applying the linear transformation $T(x, y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$?

    Linear transformation T mapped the vertices of triangle ABC to themselves.

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- Explain to students that the linear transformation in problem 4 is called the *identity transformation*. The matrix representing this transformation is called the identity matrix. The *identity matrix* is a square matrix with entries of ones along the main diagonal and all other entries of zero.
- The key takeaway for students in this part of the lesson is understanding that there are more efficient ways to use matrices when applying a linear transformation to multiple points. Students also need to understand how dimensionality indicates whether matrix multiplication is defined. Lastly, be sure that students can represent the coordinates of n points using a $2 \times n$ matrix and that they can use technology to determine the images of these n points using a $2 \times n$ matrix and matrix multiplication. Follow this lesson up with additional practice problems in which students identify the dimensions of matrices and determine whether a product of two matrices is defined.

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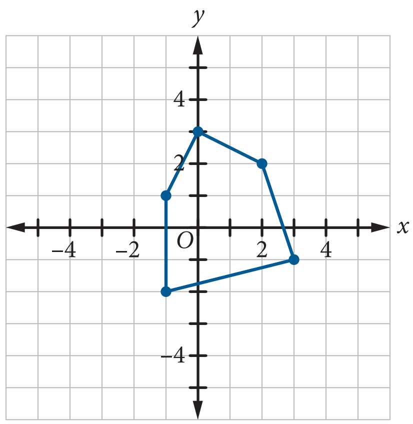
UNIT 4M

Assess and Reflect on the Lesson

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

A polygon is shown in the following image:



The matrix transformation given by $T(x, y) = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ is applied to all points in the coordinate plane. What are the vertices of the image of this polygon?

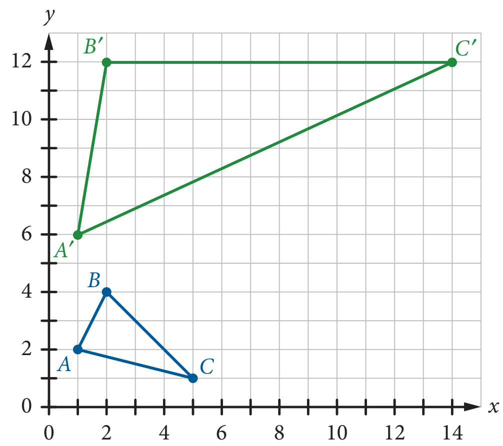
$(1, -4), (6, 0), (6, 3), (1, 2), (-5, -1)$

HANDOUT

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 4M.3: Determining the Images of Multiple Points Simultaneously

1. Student answers will vary. See the Facilitating the Task section for additional guidance.
2. Options C and D provide a more efficient way to determine the image of the parallelogram using a single matrix.
3. Only option C is correct. The image is shown in the figure.
4. The image of the vertices of triangle ABC is represented by the matrix $\begin{bmatrix} -1 & 2 & 4 \\ -1 & -4 & 3 \end{bmatrix}$.



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UNIT 4M

LESSON 4M.4

Area and the Determinant of a Matrix

LEARNING OBJECTIVE

4M.1.5 Interpret the determinant of a 2×2 matrix geometrically.

LESSON OVERVIEW

CONTENT FOCUS

In this lesson, students observe that a unit square under a linear transformation is not necessarily a square but instead another type of parallelogram. Students explore the relationship between the area of the image of a unit square and the determinant of the matrix that represents the corresponding linear transformation. Students extend their understanding of this relationship in other contexts and recognize the determinant as the scale factor that relates the area of the image of a polygon to the area of its preimage.

AREA OF FOCUS

- Connections Among Multiple Representations

SUGGESTED TIMING

~115 minutes

MATERIALS

- matrix calculator, such as **Desmos.com/matrix**

HANDOUTS

Lesson

- 4M.4.A: Determining the Area of the Image of a Unit Square

Practice

- 4M.4.B: Determinants of Matrices and Area

LESSON DESCRIPTION

Part 1: Determining the Area of the Image of a Unit Square

In this part of the lesson, students apply a given linear transformation to a unit square and determine its image. Students observe that the image of a unit square is a different type of parallelogram and apply skills developed in geometry to explore ways to find its area. This exploration leads to students applying the general form of a linear transformation to the coordinates of a unit square, and then finding the area of the image parallelogram to determine the formula for the determinant. The determinant in this context represents the area of the image of a unit square. Students conclude this part of the lesson by using the determinant as a tool to find the area of an image parallelogram.

Part 2: Interpreting Negative Determinants

In this part of the lesson, students continue to apply linear transformations to a unit square and determine the area of the image. Students explore linear

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transformations for which the value of the determinant of its matrix representation is negative. Students compare the graphs of the images of two polygons to observe the sign of the determinant corresponds to the orientation of the polygon. Understanding that area is a nonnegative quantity, students recognize that the area of the image is the absolute value of the determinant.

Part 3: Relating Areas of Rectangles to the Determinant
In this part of the lesson, students investigate the relationship between the determinant of a matrix and the area of the image of a polygon under a linear transformation represented by that matrix. Students explore the area of the image of a linear transformation in which the preimage is a rectangle. Students observe the relationship between determinants and the areas of the images of polygons other than a unit square.

FORMATIVE ASSESSMENT GOAL

This lesson prepares students to complete the following formative assessment activity.

A triangle with an area of 7 square units is a preimage of the transformation $T(x, y) = \begin{bmatrix} -3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. What is the area of the image of the triangle? Explain how you determined the area of the image.

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UNIT 4M

Part 1: Determining the Area of the Image of a Unit Square

~45 MIN.

In this part of the lesson, students apply a given linear transformation to a unit square and determine its image. Students observe that the image of a unit square is a different type of parallelogram and apply skills developed in geometry to explore ways to find its area. This exploration leads to students applying the general form of a linear transformation to the coordinates of a unit square, and then finding the area of the image parallelogram to determine the formula for the determinant. The determinant in this context represents the area of the image of a unit square. Students conclude this part of the lesson by using the determinant as a tool to find the area of an image parallelogram.

Student Task

The student task is provided on **Handout 4M.4.A: Determining the Area of the Image of a Unit Square.** In Part 1 of the handout, students explore how to find the area of the image of a unit square under a transformation using the general form of a linear transformation. This exploration leads students to develop the formula for the value of the determinant of a 2×2 matrix. Students conclude the task with the understanding that the value of the determinant of a matrix represents the area of the image of a unit square under a transformation represented by that matrix.

Facilitating the Task

- Begin by providing students with Handout 4M.4.A and allowing them some time to closely observe the task and ask clarifying questions. Students should work in pairs to complete this task.
- As you circulate around the room, you may need to provide students with some guidance on drawing a unit square whose vertices are $(0,\ 0)$, $(0,\ 1)$, $(1,\ 0)$, and $(1,\ 1)$ for problem 1. To help students conceptualize this unit square, consider asking questions such as the following:
    - What do you think the area of a unit square will be?

      The area of a unit square is 1 square unit.
    - If we want to investigate a unit square in the first quadrant that has one of its vertices at the origin, what would be the coordinates of the three other vertices that define this unit square?

      $(1,\ 0)$, $(1,\ 1)$, and $(0,\ 1)$.

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Guiding Student Thinking

For the purposes of this lesson, the unit square of interest is defined to have the vertices $(0,\ 0)$, $(0,\ 1)$, $(1,\ 0)$, and $(1,\ 1)$. If students ask about other unit squares that exist in the plane, reinforce that any square with a side length of 1 unit can be considered a unit square. Using the vertices provided in this lesson ensures that the conditions of a linear transformation are met.

- Next, have each pair of students count off from 1 to 4. The pair's group number determines which of the four linear transformations provided on the handout they will explore throughout the task.

Instructional Rationale

Each of the four matrices on the handout has a positive determinant. This supports a focused class discussion in this part of the lesson. Students will explore how to interpret the meaning of a negative determinant in Part 2 of the lesson.

- After students draw the unit square on their handouts, have them determine the image of the unit square under their assigned linear transformation, as represented by a 2×2 matrix. This task provides an opportunity for students to review how to determine the image of a set of points under a linear transformation from Lesson 4M.3 and construct a 2×4 matrix to represent the coordinates of all four points simultaneously.
- Ask students to determine the type of quadrilateral formed by this linear transformation of the unit square. Students can verify that the slopes of opposite sides are equal either algebraically or by visual inspection. Students might be surprised that the image of a square is not a square, and not even a rectangle, but instead a nonrectangular parallelogram.
- Explain to students that in this lesson they will focus on the relationship between the areas of the preimage and image of a figure under a linear transformation. They will start by exploring how the area of the unit square representing the preimage compares to the area of the image, a parallelogram.
- Remind students that they determined that the area of the unit square is 1 square unit. For problem 2(c), they will determine the area of the parallelogram they constructed in problem 2(b).
- Give students time to work with their partners to develop a plan for how they will determine the area of this parallelogram.

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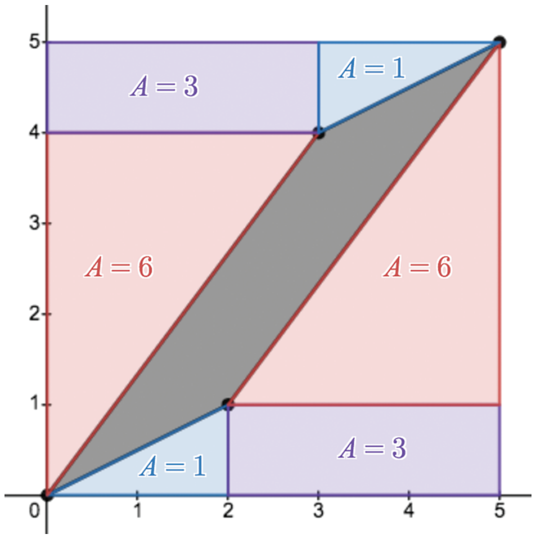
Guiding Student Thinking

Many students will try to use the formula for the area of a parallelogram to determine the area of the image of the unit square. While this is a mathematically valid approach, it is not an efficient strategy. Students would need to use the distance formula to determine the length of the base, determine the slope of the base, algebraically determine the line defining the altitude, determine the intersection of the altitude with the base, and then determine the length of the altitude. It is acceptable—and beneficial—to have students spend a few minutes exploring this approach, as it will help them understand the difference between mathematically valid and mathematically efficient.

- Have a few student groups share their approach to determining the area of the polygon that is the image of the unit square. If no group suggests it, present students with an alternative approach in which you draw a large square around a parallelogram using the origin and the vertex farthest away from the origin. This approach is illustrated in the figure below for the transformation $T(x, y) = (3x + 2y,\ 4x + y)$. Here, the area of the inner parallelogram is $(5 \times 5) - (1 + 3 + 6 + 1 + 3 + 6) = 5$. Therefore, the area of the parallelogram is 5 square units.



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Instructional Rationale

While students might come up with different ways to partition the square into triangles and rectangles or partition the parallelogram into triangles and rectangles, having all students work with the same partitions will support a class conversation and the development of the formula $ad - bc$ in the next part of the lesson.

- Give students time to use this method of decomposing the square into triangles and rectangles to determine the area of the interior parallelogram. Before moving on, make sure each group has correctly determined the area of their parallelogram.
- Have each pair select another transformation from the table and determine the resulting image of the unit square and the area of the image following the transformation.
- After a couple of minutes, ask students if they want to use the same method again or if they want to try to develop a formula that would be more efficient. This will help motivate the next task in which students develop a formula for the image of the unit square under a linear transformation.
- To set students up for problem 3, explain that they will determine the image of the unit square under the linear transformation $T(x, y) = (ax + by, cx + dy)$. This will help them determine a formula for finding the area of a parallelogram in quadrant I with a vertex at the origin. To help students develop this formula, consider asking questions such as the following:
  - What are the coordinates of the image of the unit square under this transformation?

    $T(0, 0) = (0, 0)$, $T(1, 0) = (a, c)$, $T(1, 1) = (a + b, c + d)$, and $T(0, 1) = (b, d)$.
- Have students construct the parallelogram in problem 3 and use their figures to determine the areas of the four triangles and two rectangles needed to complete the larger square. As you circulate around the room, be sure that students are plotting the points of the image accurately before determining the areas of the six figures.

Meeting Learners' Needs

For students who would benefit from additional support with plotting the points of the image in the coordinate plane, rewrite the image of the unit square as $\begin{bmatrix} 0 & a & a+b & b \\ 0 & c & c+d & d \end{bmatrix}$. This form allows students to connect their interpretation of the image with vertices that have numeric coordinates to the image of this unit square where the entries are variables.

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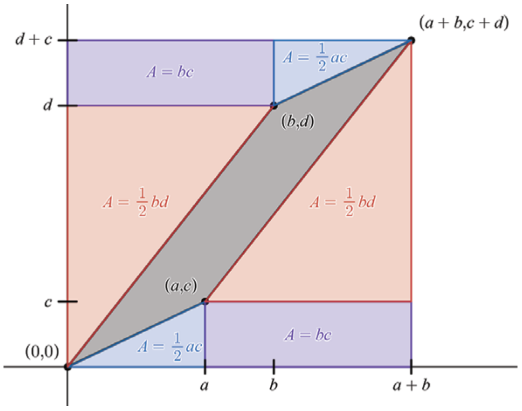
UNIT 4M

- Finally, have students determine the area of the parallelogram by subtracting the areas of the six smaller shapes from the area of the largest rectangle. This gives the computation:

$$\begin{aligned}\text{Area of a Parallelogram} &= (a+b)(c+d)-\left(bc+\frac{1}{2}ac+\frac{1}{2}bd+bc+\frac{1}{2}ac+\frac{1}{2}bd\right)\\ &= (ac+ad+bc+bd)-(2bc+ac+bd)\\ &= ac+ad+bc+bd-2bc-ac-bd\\ &= ad-bc\end{aligned}$$

Summarizing the Task

- After students have had time to determine the image of the unit square under the linear transformation $T(x,y)=(ax+by,cx+dy)$, have pairs of students share their responses for problem 2. In particular, have them discuss their responses for problem 2(c), in which they discuss the area they determined for the image of the unit square.
- If you find that students struggled to set up or simplify the equation resulting from the formula, display the following image and walk students through how the areas of the six figures were determined before computing the area of the parallelogram.



- Once students have an understanding of how to find the area of the unit square under the transformation $T(x,y)=(ax+by,cx+dy)$ tell them that the formula they just developed, $ad-bc$, gives the value of the determinant. In the context of a linear transformation, the *determinant* is the area of the image of the unit square.

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- Have students use the formula for the determinant to find the area of the image of the unit square under each of the three other linear transformations in the table. Students can compare their answers with the group that first studied each linear transformation.
- One key takeaway is for students to be able to identify the formula representing the value of the determinant by applying the general form of a linear transformation, $T(x, y) = (ax + by, cx + dy)$, to the unit square. Before moving on, students should understand that this formula can be used to find the determinant, which is equivalent to the area of the parallelogram representing the image of the unit square under any linear transformation.
- Another key takeaway is for students to develop the formula for the area of an image parallelogram following a linear transformation of the unit square and connect it to a new concept called the determinant. This formula can be used to find the value of the determinant of a 2×2 matrix.

Part 2: Interpreting Negative Determinants

~45 MIN.

In this part of the lesson, students continue to apply linear transformations to a unit square and determine the area of the image. Students explore linear transformations for which the value of the determinant of its matrix representation is negative. Students compare the graphs of the images of two polygons to observe that the sign of the determinant corresponds to the orientation of the polygon. Understanding that area is a nonnegative quantity, students recognize that the area of the image is the absolute value of the determinant.

Student Task

Students complete Part 2 of Handout 4M.4.A, which focuses on understanding when the value of the determinant is negative.

Facilitating the Task

- Students can continue to work with their partner to complete Part 2 of Handout 4M.4.A.
- As you circulate around the room, remind students that they can use what they learned in Part 1 of the lesson to compare and contrast the images of the unit squares under the two given linear transformations and the values of their determinants.

Classroom Ideas

By this point, students have had a lot of practice determining the images of the unit square under linear transformations. You can split up the class into two groups and have each group determine the image of one of the transformations, and then share their resulting parallelograms with the class.

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Summarizing the Task

- After students have had some time to complete Part 2 of the handout, have pairs of student volunteers share what they noticed in their comparisons of the images of the two linear transformations and of their determinants. If students are struggling to develop meaningful insights into their observations, you could ask questions like the following:
  - What differences do you notice between the two parallelograms?

    One possible answer is that vertices B' and D' are switched.
  - What similarities do you notice between the two parallelograms?

    Possible answers include that they have the same vertices and they have the same areas.
  - What do you notice about the determinants of the 2×2 matrices defining each transformation?

    The determinant associated with transformation T is positive 5 whereas the determinant associated with transformation L is negative 5.
  - How do you think we might interpret a negative determinant?

    A negative determinant indicates the oriention of the image differs from that of the preimage.

Guiding Student Thinking

If students struggle to identify the differences between the two parallelograms, encourage them to label the vertices of the unit square *ABCD* and then label the corresponding vertices of the image of the parallelogram $A'B'C'D'$.

- Explain to students that a negative determinant indicates information about a polygon's orientation in the plane, while the absolute value of the determinant gives the area of the polygon under the linear transformation defined by that matrix.
- The key takeaway for this part of the lesson is for students to recognize that there are instances in which the value of a determinant is negative. Since the determinant represents the area of the parallelogram in this context, and area is defined to be nonnegative, the absolute value of the determinant must be calculated to accurately quantify the area. Students should also understand that switching the entries in the transformation matrix changes the orientation of image points. The orientation of the image points, while different, still result in a parallelogram in the coordinate plane that appears to be the same under both linear transformations.

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Part 3: Relating Areas of Rectangles to the Determinant

~25 MIN.

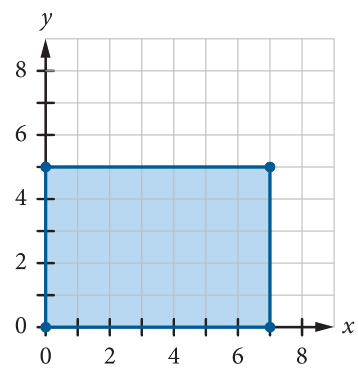
In this part of the lesson, students investigate the relationship between the determinant of a matrix and the area of the image of a polygon under a linear transformation represented by that matrix. Students explore the area of the image of a linear transformation in which the preimage is a rectangle. Students observe the relationship between determinants and the areas of the images of polygons other than a unit square.

Student Task

Students complete Part 3 of Handout 4M.4.A in which they investigate if the same relationship between the areas of the images and preimages of polygons exists for figures other than the unit square.

Facilitating the Task

- Students can continue to work with their partner to complete Part 3 of Handout 4M.4.A.
- Tell students that the polygons to which we would like to apply transformations are usually more complex than the unit square. Explain that in this part of the lesson they will determine whether the determinant relates the areas of the images and preimages of other polygons, or if its interpretation is limited to the areas of the image and preimage of the unit square.
- As you circulate around the room, be sure that students have selected numbers between 2 and 10 to represent the dimensions of their rectangles. Once students have determined their dimensions, they need to calculate the areas of the rectangles and then graph them in the coordinate plane, as shown in the following graph.



- Have students complete problem 4 on the handout and determine the image of the rectangle under the linear transformation $T(x, y) = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

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- Give students time to determine the area of the image, which is a parallelogram, using the method from the first part of the lesson.
- For problem 4, you can remind students to use the method they learned in Part 1 to determine the image of the rectangle under the linear transformation $T(x,y)=\begin{bmatrix}2 & 3\\1 & 4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$.

Meeting Learners' Needs

You might need to remind students that the formula $ad-bc$ gives the area of the image of the unit square, and not the area of the image of any other polygon under the linear transformation.

Guiding Student Thinking

You can help students visualize that an area of 10 units means that the area of the figure is 10 unit squares, each of which has an area of 1 square unit. If, under the linear transformation, the area of the unit square is now the value of the determinant, then the area of the image of the rectangle will be 10 copies of this new parallelogram with area $ad-bc$ square units.

Summarizing the Task

- Have each student pair record the area of the preimage and the area of the image on a class display. You can then ask questions such as the following:
  - What do you notice about the relationships between the area of the preimage and the area of the image?

    The area of the image is always five times as large as the area of the preimage.
  - How is this scale factor of 5 related to this linear transformation?

    Five is the determinant of the matrix that defines the linear transformation.
  - If you know that a rectangle has an area of 10 square units, what will be the area of the image of the rectangle under the linear transformation $T(x,y)=\begin{bmatrix}2 & 3\\1 & 4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$?

    The area of the image of the rectangle will be 50 square units, or 5 times as large as the area of the preimage.
- Before moving on, make sure students understand that the absolute value of the determinant of a matrix representing a linear transformation gives the scale factor relating the area of the image under that transformation to the area of the preimage.

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- For additional practice, have students complete the problems on **Handout 4M.4.B: Determinants of Matrices and Area**. These problems are meant to further develop students' understanding of the relationship between the determinant of a matrix representing a linear transformation and the scale factor relating the area of the image under that transformation to the area of the preimage.

Guiding Student Thinking

In problem 3 on Handout 4M.4.B, students investigate how the area of the preimage of a triangle is related to the area of its image under a linear transformation. Help students see that when a linear transformation is applied to any type of polygon, the area of the image is equal to the product of the area of the preimage and the absolute value of the determinant of the transformation matrix.

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UNIT 4M

Assess and Reflect on the Lesson

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

A triangle with an area of 7 square units is a preimage of the transformation $T(x, y) = \begin{bmatrix} -3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. What is the area of the image of the triangle? Explain how you determined the area of the image.

The determinant of the matrix in the given linear transformation is $(-3)(2) - (5)(1) = -6 - 5 = -11$. Because the absolute value of the determinant of a matrix in a linear transformation is the scale factor between the areas of a polygon's preimage and image, the area of the image of the triangle is $7|-11| = 77$ square units.

HANDOUTS

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 4M.4.A: Determining the Area of the Image of a Unit Square

Part 1

1. Graphs may vary depending on which linear transformation the student was assigned. All images are shown in problem 2(b).

2. (a)

	Linear Transformation	Image of the Unit Square
1.	$\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$	$\begin{bmatrix} 0 & 2 & 1 & 3 \\ 0 & 4 & 3 & 7 \end{bmatrix}$
2.	$\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$	$\begin{bmatrix} 0 & 4 & 1 & 5 \\ 0 & 2 & 3 & 5 \end{bmatrix}$
3.	$\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 4 & 1 & 5 \\ 0 & 3 & 2 & 5 \end{bmatrix}$
4.	$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$	$\begin{bmatrix} 0 & 3 & 2 & 5 \\ 0 & 1 & 4 & 5 \end{bmatrix}$

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(b)

Image 1

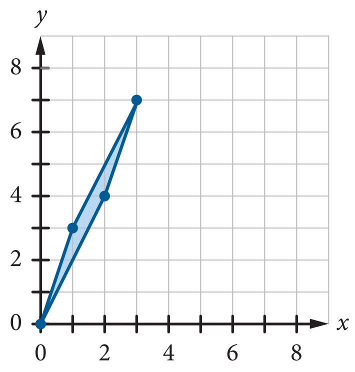


Image 2

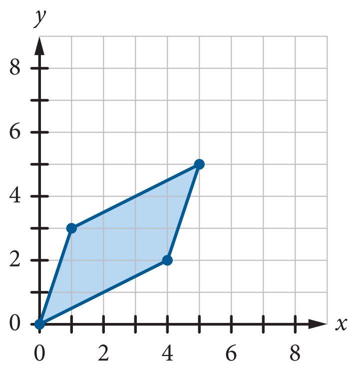


Image 3

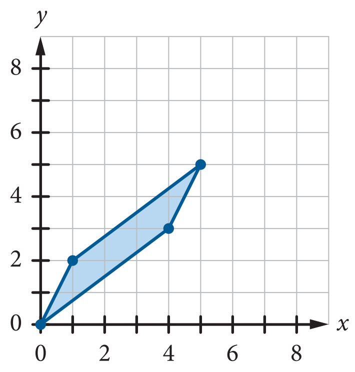
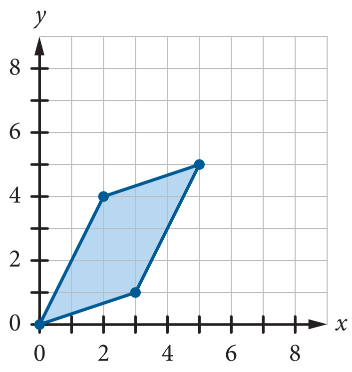


Image 4



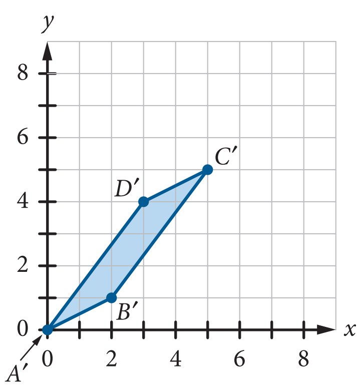
(c) Area of image rectangle 1: 2 square units; area of image rectangle 2: 10 square units; area of image rectangle 3: 5 square units; area of image rectangle 4: 10 square units.

3. The area of the parallelogram is $ad - bc$. See the Summarizing the Task section in Part 1 of the lesson to reference the image.

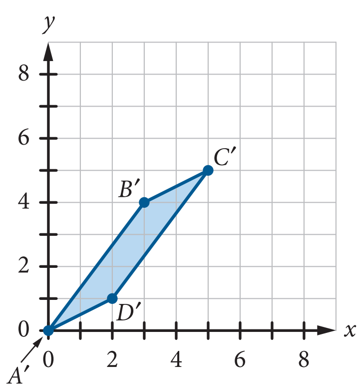
Part 2

1. (a) The vertices B' and D' are switched in the coordinate plane.

$T(x, y) = (2x + 3y, 1x + 4y)$



$L(x, y) = (3x + 2y, 4x + 1y)$



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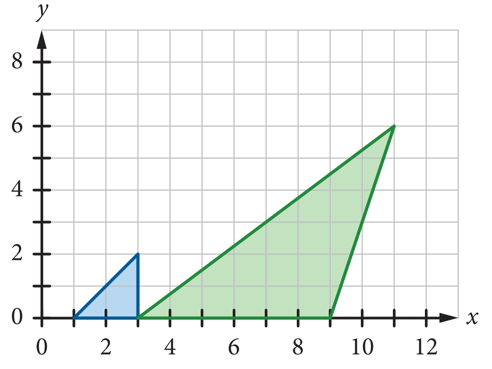
(b) The determinants have the same absolute value. The area of the image after the linear transformation $T(x, y) = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ is applied to the unit square is 5 square units.

Part 3

The answers for problems 1–4 will vary. See the Facilitating the Task section in Part 3 of the lesson for additional guidance.

Handout 4M.4.B: Determinants of Matrices and Area

1. (a) 2 square units
   (b) 14 square units
   (c) 2 square units
2. 70 square units
3. (a) 2 square units
   (b) On the following graph, the preimage is shown in blue and the image is shown in green.



   (c) 18 square units
   (d) The area of the image is equal to the product of the area of the preimage and the determinant, 9. In other words, the determinant is the scale factor that we can multiply the area of the preimage by to get the area of the image.
4. The area of triangle *ABC* is 5 square units.
5. Two possible values are $k = \frac{-9}{4}$ or $k = \frac{-3}{4}$.

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LESSON 4M.5

Sequences of Linear Transformations

LEARNING OBJECTIVE

4M.1.6 Express a sequence of linear transformations as a single linear transformation.

LESSON OVERVIEW

CONTENT FOCUS

In this lesson, students connect their understanding of sequences of transformations (formally, computing compositions of transformations) to matrix multiplication. Students recognize that applying a sequence of two transformations is equivalent to applying a transformation that is the product of the matrix representations of those transformations. Students leverage their understanding of matrix multiplication from Lesson 4M.3, in which they multiplied 2×2 matrices with $2 \times n$ matrices, to multiplying two 2×2 matrices to represent a sequence of two transformations.

LESSON DESCRIPTION

Part 1: Exploring Commutativity

In this part of the lesson, students explore the commutativity of several mathematical operations on functions. This exploration leads them to assess whether matrix multiplication is commutative. To support their investigation of whether matrix multiplication is commutative, they test three different cases. By the end of this part of the lesson, students conclude that matrix multiplication is not commutative, even when the dimensions of the matrices allow for them to be multiplied in either order.

AREA OF FOCUS

- Connections Among Multiple Representations

SUGGESTED TIMING

~75 minutes

MATERIALS

- matrix calculator, such as **Desmos.com/matrix**

HANDOUTS

Lesson

- 4M.5.A: Exploring Commutativity
- 4M.5.B: Composing Linear Transformations

Practice

- 4M.5.C: Applying Sequences of Linear Transformations

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Part 2: Composing Linear Transformations
In this part of the lesson, students use their prior knowledge about the composition of functions to observe that composing linear transformations corresponds to the product of the matrices that define those linear transformations. Students also apply their knowledge of matrix multiplication to recognize that the order in which matrices are multiplied matters when expressing a sequence of linear transformations as a single linear transformation. Students conclude this part of the lesson with an understanding of how to apply a sequence of linear transformations using a single matrix to points in the coordinate plane.

FORMATIVE ASSESSMENT GOAL

This lesson prepares students to complete the following formative assessment activity.

The transformation $T(x, y) = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ maps point A to point B and the transformation $S(x, y) = \begin{bmatrix} 1 & -1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ maps point B to point C. Using the transformations T and S, describe a transformation R that maps point A to point C. Explain the steps you took to determine transformation R.

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Part 1: Exploring Commutativity

~30 MIN.

In this part of the lesson, students explore the commutativity of several mathematical operations on functions. This exploration leads them to assess whether matrix multiplication is commutative. To support their investigation of whether matrix multiplication is commutative, they test three different cases. By the end of this part of the lesson, students conclude that matrix multiplication is not commutative, even when the dimensions of the matrices allow for them to be multiplied in either order.

Warm-Up

- Begin by asking students the following questions:
  - What are the four basic mathematical operations?

    The four basic mathematical operations are addition, subtraction, multiplication, and division.
  - What does it mean for a mathematical operation to be commutative? Share an example of an operation that is commutative and an example of an operation that is not commutative.

    A mathematical operation is commutative when the order of the values on which the operation is performed does not matter. For example, $1+2$ is equivalent to $2+1$. Subtraction is a nonexample of commutativity. For example, $1-2$ is not equivalent to $2-1$.

Meeting Learners' Needs

If students have difficulty getting started, consider reminding them of the definition of commutativity and/or providing specific examples for students to assess, such as those in the following table.

- Display the following table and ask students to record in column 3 if the example provided is commutative.

Operation	Example	Is It Commutative?
Adding two functions	$f(x)+g(x)$ $g(x)+f(x)$ for $f(x)=3x$ and $g(x)=4x^2-5$	Yes
Multiplying two functions	$f(x)\cdot g(x)$ $g(x)\cdot f(x)$ for $f(x)=3x$ and $g(x)=4x^2-5$	Yes

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UNIT 4M

Operation	Example	Is It Commutative?
Composing two functions	$f(g(x))$ $g(f(x))$ for $f(x)=3x$ and $g(x)=4x^2-5$	No
Performing two transformations	Select a point in quadrant I, reflect over the y-axis, and then rotate 90° clockwise. Rotate 90° clockwise and then reflect over the y-axis.	No

- Is working through one example enough to determine if an operation is commutative?

  One counterexample is sufficient to show that an operation is not commutative. If one example appears to show that an operation is commutative, further exploration might be required.

- Can you think of actions in your everyday life that are commutative?

  Putting on a shirt and pants is the same as putting on pants and then a shirt. The result is the same.

Student Task

The student task can be found on **Handout 4M.5.A: Exploring Commutativity**. In this task, students explore three distinct cases to assess whether matrix multiplication is or is not commutative. Students conclude the task with the understanding that matrix multiplication in general is not commutative. Answers to the handout can be found in the Assess and Reflect section of the lesson.

Facilitating the Task

- Begin by providing each student with a copy of Handout 4M.5.A and access to a matrix calculator application, such as Desmos. Students should work in pairs to complete the handout. Give students some time to closely observe the task and ask clarifying questions.
- As you circulate around the room, you may observe that some students need guidance to start the task. You can ask some questions like the following:
  - Do you think matrix multiplication will be commutative?

    No, because dimensions matter, and you can't multiply a 2×3 matrix by a 2×2 matrix but you can multiply a 2×2 matrix by a 2×3 matrix.

  - If you can perform both operations, do you think the results will be the same?

    It is acceptable for students to have different conjectures at this point. The task should help them reach a conclusion.

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- For Sets 1 and 2, students should recognize that both sets of matrices require them to multiply a 2×2 matrix by a 2×2 matrix. Set 1 is consistent with commutativity because matrix A is the identity matrix, which was introduced in Lesson 4M.3. You can remind students that they learned previously that the identity matrix defines the identity transformation, which maps all points in the plane onto themselves. But Set 2, which also requires students to multiply a 2×2 matrix by a 2×2 matrix, is not commutative.
- Students will continue to explore Sets 3 and 4 and eventually assess that matrix multiplication is not commutative.

Summarizing the Task

- After students have time to work through the task and record what they noticed about each set, ask questions like the following:
  - What did you notice about Set 1?

    The product of AB is $\begin{bmatrix} 2 & -3 \\ -4 & 0 \end{bmatrix}$, which is the same answer as BA. The product is the same regardless of the order in which the matrices are multiplied.
  - Would you conjecture that matrix multiplication is commutative?

    Based on this example, multiplicative commutativity appears to be true for 2×2 matrices.
  - For Set 2, what did you notice about the products AB and BA?

    The product of AB is $\begin{bmatrix} 0 & -6 \\ 28 & 12 \end{bmatrix}$. The product of BA is $\begin{bmatrix} -8 & 17 \\ -8 & -4 \end{bmatrix}$. We can carry out the matrix multiplication, but we don't get the same product.
  - What did you notice about both Sets 3 and 4?

    For Set 3, $AB = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \end{bmatrix}$. The product of BA is undefined. For Set 4, AB is $\begin{bmatrix} 1 & -13 & 22 & -24 \\ 7 & -11 & 24 & -38 \end{bmatrix}$ and BA is not defined.
  - In Sets 3 and 4, you multiplied a 2×2 matrix by a $2 \times n$ matrix for $n = 3$ and $n = 4$. Does $AB = BA$? Why or why not?

    In each set, $AB \neq BA$ because the number of entries in the rows is not equal to the number of entries in the columns, so matrix multiplication is not defined for the product expression BA.
- Before moving on, discuss the following with students to summarize the key takeaways of the student task:
  - You can multiply a 2×2 matrix by a 2×2 matrix and the product will be another 2×2 matrix.

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- When you multiply two 2×2 matrices, the products are often different if you change the order in which they are multiplied.
- When you multiply a 2×2 matrix by a $2 \times n$ where $n \geq 3$, AB is defined but BA is not defined. Therefore $AB \neq BA$.
- The fact that the order in which we multiply matrices matters means that matrix multiplication is *noncommutative*. Note that there are some special cases when these products are equal and explain that those will be explored more in the next lesson.

- Another key takeaway for students is to understand that matrix multiplication in general is not commutative. There are instances where matrix multiplication is defined for matrices A and B, but AB doesn't always equal BA. In the next part, students work toward expressing a sequence of linear transformations as a single linear transformation. Students understand that this involves matrix multiplication, where the order of the linear transformations matters.

Part 2: Composing Linear Transformations

~45 MIN.

In this part of the lesson, students use their prior knowledge about the composition of functions to observe that composing linear transformations corresponds to the product of the matrices that define those linear transformations. Students also apply their knowledge of matrix multiplication to recognize that the order in which matrices are multiplied matters when expressing a sequence of linear transformations as a single linear transformation. Students conclude this part of the lesson with an understanding of how to apply a sequence of linear transformations using a single matrix to points in the coordinate plane.

Student Task

The student task can be found on **Handout 4M.5.B: Composing Linear Transformations**. In this task, students use their understanding of function composition and matrix multiplication to apply a sequence of linear transformations S and T to points in the coordinate plane. By the end of this task, students understand that the composition of two linear transformations can be expressed as a single matrix and that this matrix can be used to find the output, $T(S(x, y))$.

Facilitating the Task

- Explain to students that they will explore how to express a sequence of linear transformations using function composition and matrix multiplication. Remind students that in Part 1, they learned that matrix multiplication is not commutative,

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so they should keep this in mind when they express a sequence of linear transformations as a single transformation.

- Have students recall their work with function composition in Unit 2 before they attempt to learn how to apply a sequence of linear transformations to points in the coordinate plane.
- Explain to students that just as they can compose functions, they can also compose multiple linear transformations. Also, earlier in the warm-up, we reviewed examples of some operations that are commutative and some that aren't. One operation reviewed was the composition of functions. Recall that function composition is not commutative and, like the composition of functions, expressing a sequence of linear transformations as a single linear transformation is not commutative since the composition of linear transformations involves matrix multiplication.
- Distribute Handout 4M.5.B to each student. Students can continue to work with their partners to complete this task. Give students some time to closely observe the task and ask clarifying questions.
- Once students are done asking clarifying questions, review the example in the first row of the table with students before having them complete the remainder of the task with their partner.

Guiding Student Thinking

It may be helpful to talk about the meaning of *composition* in the context of linear transformations. Ask students to explain what $S(x, y)$ means and again emphasize the idea of the input and the output. Then ask what $T(S(x, y))$ means. Be sure to make the connection that the output of $S(x, y)$ now becomes the input for $T(x, y)$.

- Next, tell students that they may find it tedious to use the coordinate function representation to perform a composition of linear transformations. Point out that they found in prior lessons that they can write linear transformations as matrices. They can consider using matrices as tools to compose the linear transformations instead. Show students the following steps, which use matrices:
  - Write the matrix representation of the linear transformation S.

$$S(x, y) = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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- Use this representation to compute $S(3, 5)$ and show the resulting 2×1 matrix. You can use technology for this computation if you want.

$$S(3,\ 5) = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 14 \end{bmatrix}$$

- Write the matrix representation of the linear transformation T.

$$T(x,\ y) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Use this representation to compute $T\left(S(3, 5)\right)$. Note that the output from $S(3, 5)$ is the input for T, just as with regular function composition.

$$T(S(3,\ 5)) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 14 \end{bmatrix} = \begin{bmatrix} 29 \\ 59 \end{bmatrix}$$

- After walking students through the process of using matrices to compose linear transformations, ask the following questions:
  - What matrices are being multiplied when applying linear transformation S then T in sequence to the point $(3,\ 5)$?

    You multiply the matrix form of the linear transformations S, $\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$ and T, $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ by $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$. This can be represented as $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.
  - Does the order in which the matrices of the linear transformations are multiplied matter before finding the image of the point?

    Yes. You get different image points depending on the order in which you multiply the matrices.

- Give students an opportunity to verify that the order in which matrices representing the linear transformations S and T are multiplied matter. Have them apply each order to the point $(3,\ 5)$. They should observe that outputs will differ.
- Remind students that the goal of this part of the lesson is to express a sequence of linear transformations as a single linear transformation. For the example we are doing now, we know that the two matrices can be multiplied before applying the product to an input. This would be $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$. Expressed as a single transformation, this product is $\begin{bmatrix} 8 & 1 \\ 18 & 1 \end{bmatrix}$.
- Now that students have seen how to express a sequence of linear transformations as a single linear transformation, have them complete the rest of the table by applying $\begin{bmatrix} 8 & 1 \\ 18 & 1 \end{bmatrix}$ to the other given inputs.

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Summarizing the Task

- A key takeaway from this part of the lesson is that the composition of two linear transformations results in a new linear transformation that is determined by multiplying the matrices that define the original linear transformations. The order in which these transformations are applied (and thus the order in which they are multiplied) matters.
- For additional practice, have students complete the practice problems on **Handout 4M.5.C: Applying Sequences of Linear Transformations**. These problems are meant to further develop students' understanding that applying a sequence of two transformations is equivalent to applying a single transformation represented by the product of the matrix representations of those transformations.

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UNIT 4M

Assess and Reflect on the Lesson

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

The transformation $T(x, y) = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ maps point A to point B and the transformation $S(x, y) = \begin{bmatrix} 1 & -1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ maps point B to point C. Using the transformations T and S, describe a transformation R that maps point A to point C. Explain the steps you took to determine transformation R.

Since T maps point A to point B and S maps point B to point C, the composition of the transformations $S(T(x, y))$ maps point A to point C. Because the composition of two linear transformations is equivalent to the transformation matrix that is the product of the two linear transformations, the corresponding transformation matrix C is $\begin{bmatrix} 1 & -1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 16 & 27 \end{bmatrix}$. Thus, the transformation $R(x, y) = \begin{bmatrix} -1 & 4 \\ 16 & 27 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ maps point A to point C.

HANDOUTS

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 4M.5.A: Exploring Commutativity

Set 1: $AB = \begin{bmatrix} 2 & -3 \\ -4 & 0 \end{bmatrix}$ and $BA = \begin{bmatrix} 2 & -3 \\ -4 & 0 \end{bmatrix}$.

Set 2: $AB = \begin{bmatrix} 0 & -6 \\ 28 & -12 \end{bmatrix}$ and $BA = \begin{bmatrix} -8 & 17 \\ -8 & -4 \end{bmatrix}$.

Set 3: $AB = \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 1 \end{bmatrix}$ and BA is not defined.

Set 4: $AB = \begin{bmatrix} 1 & -13 & 22 & -24 \\ 7 & -11 & 24 & -38 \end{bmatrix}$ and BA is not defined.

Handout 4M.5.B: Composing Linear Transformations

Students use the coordinate function representation to compose linear transformations in the example to help motivative the need for using matrix multiplication. Students are expected to use matrix multiplication for the problems in the table to determine

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the output. See the Facilitating the Task section in Part 2 of the lesson for additional guidance.

Input	$S(x, y) = (2x - y, 3x + y)$	$T(S(x, y))$	Output
Example: $(3, 5)$	$S(3, 5) = (2(3) - 5, 3(3) + 5) = (1, 14)$	$T(S(3, 5)) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 14 \end{bmatrix} = \begin{bmatrix} 29 \\ 59 \end{bmatrix}$	$(29, 59)$
$(1, 0)$	$S(1, 0) = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$	$T(S(1, 0)) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 18 \end{bmatrix}$	$(8, 18)$
$(-2, 4)$	$S(-2, 4) = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -8 \\ -2 \end{bmatrix}$	$T(S(-2, 4)) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -12 \\ 32 \end{bmatrix}$	$(-12, 32)$
$(-3, -2)$	$S(-3, -2) = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ -11 \end{bmatrix}$	$T(S(-3, -2)) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -4 \\ -11 \end{bmatrix} = \begin{bmatrix} -26 \\ -56 \end{bmatrix}$	$(-26, -56)$

Handout 4M.5.C: Applying Sequences of Linear Transformations

1. (a) $A' = (4, 6), B' = (7, 13), C' = \left(\frac{-5}{2}, 0\right)$
   (b) $S(T(x, y))$
   (c) $\begin{bmatrix} 1/2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$
2. (a) First apply T, and then apply S. The resulting vertices are: $A' = (0, 4)$, $B' = (0, 12)$, $C' = (3, 10)$, and $D' = (6, 12)$.
   (b) $B(x, y) = (x - y, 2x + 2y)$, or equivalently $B(x, y) = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.
3. (a) $(-2.33, 5.96)$. Students' answers may vary slightly due to rounding error.
   (b) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. If students use the Desmos Matrix Calculator to determine this matrix, they should understand that -1.1102×10^{16} is essentially 0 and is displayed in that form due to rounding algorithms in Desmos calculations.
   (c) Students either can go back to their answer from problem 3(a) and apply the given transformation a third time, or they can apply the transformation described in problem 3(b) directly to the point $(4, 5)$.
4. (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
   (b) $(4, 5)$. Students can apply the 120° rotation 3 times, they can use the matrix they constructed in problem 4(a), or they can reason that a 360° rotation about the origin will map any point onto itself.

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UNIT 4M

LESSON 4M.6

Undoing Transformations and Finding Preimages

LEARNING OBJECTIVES

4M.1.7 Determine the inverse of a square matrix, including matrices describing linear transformations, both with and without technology.

4M.1.8 Determine the preimage of a specified point under a given linear transformation.

LESSON OVERVIEW

CONTENT FOCUS

In this lesson, students use their knowledge of the equivalence of a sequence of matrix transformations and the product of the matrices that define those transformations to uncover properties of the inverse of a matrix. Students investigate the relationship between the entries of a matrix and the entries of the inverse of that matrix. Next, students examine the relationship between the determinants of matrices that are inverses of each other. This exploration leads to students generalizing a formula for finding the inverse of any 2×2 matrix. Finally, students apply the inverse property $AA^{-1} = I = A^{-1}A$ to determine if matrices are inverses of each other.

AREA OF FOCUS

- Connections Among Multiple Representations

SUGGESTED TIMING

~135 minutes

MATERIALS

- matrix calculator, such as **Desmos.com/matrix**

HANDOUTS

Lesson

- 4M.6.A: Inverse Transformations
- 4M.6.B: Defining the Inverse of a Matrix

Practice

- 4M.6.C: Undoing Transformations and Finding Preimages

LESSON DESCRIPTION

Part 1: Observing Inverse Relationships in Linear Transformations

The first part of this lesson introduces the idea that inverses of linear transformations may exist. Given a transformation T, an inverse, T^{-1}, may exist such that the preimages and images of T correspond to the images and preimages, respectively, of T^{-1}. Students begin to develop this understanding by computing the images of four linear transformations, and then comparing the images across these four transformations to observe if an inverse relationship exists.

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Part 2: Determining the Inverse of a Transformation Matrix
In this part of the lesson, students use their understanding of inverse transformations they developed in Part 1 to discover a formula for the inverse of a 2×2 matrix. Students first observe the relationship between the entries of a 2×2 matrix and the entries of its inverse. By the end of this part of the lesson, students are able to use this formula to find the inverse of any 2×2 matrix.

Part 3: Applying Properties of Matrix Inverses
In this part of the lesson, students identify the inverse of a given matrix and verify that the matrices are inverses of each other using the property that the product of a matrix and its inverse is the identity matrix. Using mathematical notation, this relationship is expressed as $AA^{-1} = I = A^{-1}A$. Students also use inverse matrices and inverse transformations to determine the preimages of various given images.

FORMATIVE ASSESSMENT GOAL

This lesson prepares students to complete the following formative assessment activity.

Consider the transformation $T(x,y)=\begin{bmatrix}2 & 5\\ 3 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$. Find the inverse of this transformation, $T^{-1}(x,y)$. Explain how you would verify that T^{-1} is the inverse of T.

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Part 1: Observing Inverse Relationships in Linear Transformations

~45 MIN.

The first part of this lesson introduces the idea that inverses of linear transformations may exist. Given a transformation T, an inverse, T^{-1}, may exist such that the preimages and images of T correspond to the images and preimages, respectively, of T^{-1}. Students begin to develop this understanding by computing the images of four linear transformations, and then comparing the images across these four transformations to observe if an inverse relationship exists.

Student Task

The student task for this part of the lesson is provided on **Handout 4M.6.A: Inverse Transformations.** Students begin the task by computing the images under their assigned linear transformation. Next, students compare the images under their linear transformation to those of their group members. The goal is for students to identify which member of their group has a linear transformation that is the inverse of theirs.

Facilitating the Task

- Provide each student with a copy of Handout 4M.6.A. Give students time to closely observe the task and ask clarifying questions. Students should work in groups of 4 to complete the task.
- In each group, have students count off from 1 to 4. The number that a student is assigned corresponds to the number of the transformation on the handout that they should work with.

Instructional Rationale

The four transformations that students will work with in this part of the lesson were chosen so that two of the transformations are inverses of the other two transformations. The matrices that represent transformations were chosen so that they have unique entries and a prime determinant. This will support students in Part 2 of the lesson when they determine the general formula for the inverse of a 2×2 matrix.

- Each transformation in the table is accompanied by a set of preimage points. Students will compute the images of the points using their assigned transformation. Students are strongly encouraged to find these images using technology.

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Guiding Student Thinking

When computing the images of the five points given for their assigned transformation, students may be tempted to work with one point at a time. It may help to remind students that they can compute the images of all four preimage points simultaneously using a single preimage matrix, just as they did in Lesson 4M.3. In this case, students can construct a 2×5 matrix whose columns consist of the coordinates of all five image points. They can then find the product of the 2×2 matrix that defines their transformation and this 2×5 matrix. The columns of the resulting 2×5 matrix are the coordinates of the images of the five corresponding preimage points.

- Tell students to leave the two rows at the bottom of their coordinate list blank for now so they can add points there later in the lesson.

Summarizing the Task

- Begin summarizing the task by having students compare the preimage and image points of their assigned transformations with those of their group members. Support students by asking questions like the following:
  - What do you notice about your preimage and image points in comparison to those of your group members?

    The preimage points for my assigned transformation are the image points that one of my group members found with their transformation, and the image points of my transformation are the preimage points used by that same group member.
  - Where in this course have you previously seen this relationship between tables of values of functions in which the input and output values are swapped?

    Sets of input and output values are swapped when two functions are inverses of each other.
- Highlight for students that two linear transformations for which their sets of input and output values are swapped—as observed with their table and one of their groupmate's tables—indicates that the linear transformations are inverses of one another.
- Let students know that linear transformations 1 and 4 are inverses of each other, and linear transformations 2 and 3 are inverses of each other.
- As a class, decide on a random point $(a,\ b)$, such as $(2,\ 1)$ or $(-5,\ 2)$. Each student should add this point to the first blank row of the table of their assigned transformation and compute the image of this point under their assigned transformation.

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- Within each group, have students 1 and 4 and students 2 and 3 pair up as partners. Each student should add their partner's preimage and image pairs in the appropriate columns of the last row of their table, accounting for the inverse relationship of their transformations without performing further computations. Then elicit responses from students by asking the following question:
  - If your partner wanted to include your preimage and image coordinate pairs in their table, how would it appear?

    My preimage coordinate pair will be a coordinate pair in my partner's image column and my image coordinate pair will appear in the same row of my partner's preimage column.
- The key takeaway for students in this part of the lesson is to recognize that for a linear transformation that maps a set of inputs to a set of outputs, another linear transformation may exist that maps those outputs back to the original inputs. That is, some linear transformations can be undone using another linear transformation.

Meeting Learners' Needs

Some students may benefit from a review of inverse functions. Try to have students recall that:

- A function associates an input with its output, and its inverse function (if it exists) associates the output with its input.
- The inputs and outputs of a function are switched in an inverse function: if the point $(a,\ b)$ is on the graph of $y = f(x)$, then the point $(b,\ a)$ is on the graph of $y = f^{-1}(x)$. Therefore, the domain and range of f^{-1} are the range and domain of f, respectively.
- Not all functions are invertible. A function is invertible if and only if each input is mapped to a unique output.

Part 2: Determining the Inverse of a Transformation Matrix

~45 MIN.

In this part of the lesson, students use their understanding of inverse transformations they developed in Part 1 to discover a formula for the inverse of a 2×2 matrix. Students first observe the relationship between the entries of a 2×2 matrix and the entries of its inverse. By the end of this part of the lesson, students are able to use this formula to find the inverse of any 2×2 matrix.

Student Task

The student task for this part of the lesson is provided on **Handout 4M.6.B: Defining the Inverse of a Matrix**. In this task, students generalize the relationship between the

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entries of matrices that are inverses of each other. Students start by comparing the entries of the matrices representing linear transformations $A(x, y)$ and $D(x, y)$ from Handout 4M.6.A. The observations made from comparing these entries help students define the inverse of any 2×2 matrix.

Facilitating the Task

- Provide each student with a copy of Handout 4M.6.B. Give students time to closely observe the task and ask clarifying questions. Students should work in groups of 4 to complete the task.

Instructional Rationale

In this lesson, all inverse computations are expected to be completed by hand. If you would like your students to use technology throughout the lesson, you can introduce the inverse operation in the Desmos matrix calculator or another online matrix calculator.

- Have students complete problem 1 by identifying the matrices in the transformations $A(x, y)$ and $D(x, y)$ from Handout 4M.6.A. Note to students that they can use the term Matrix A to describe the transformation $A(x, y)$ and Matrix D to describe the transformation $D(x, y)$.
  - What is the relationship between this pair of transformations? How do you know?

    These transformations are inverses of each other because they switch input and output pairs.
- Point out to students that Matrix D has a fraction for each of its entries, while Matrix A does not. Have them observe that the numerators of these fractions seem to be similar to the entries of Matrix A. Then proceed to elicit observations from students regarding these similarities.

Guiding Student Thinking

Students may not be able to observe the relationship between the matrix entries and the fractional entries of the inverse matrix. If necessary, students can write each entry of the inverse so that it is clearly separated into two factors (e.g., $\frac{1}{\det(A)}\cdot(entry)$). When expressed in this way, students may be able to focus solely on the *entry* factor.

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- There are key observations students should make regarding the relationship between the entries of a matrix and its inverse. These observations are:
  - The values in row 1, column 1 and in row 2, column 2 switch places from $A(x, y)$ to $D(x, y)$, and vice versa.
  - The signs of the values in row 2, column 1 and in row 1, column 2 change from $A(x, y)$ to $D(x, y)$, and vice versa.

Meeting Learners' Needs
Some students may need additional guidance when working through this part of the task. You may want to highlight or draw boxes around the entries in each matrix when these observations are discussed so students can keep track of the relationship between the entries of a matrix and its inverse.

- As you circulate around the room, you may find that students need additional support to complete problem 3. Ask students a question like the following:
  - What do you remember about the determinant of a matrix?

    The determinant of a matrix is the scale factor between the areas of the preimage and image polygons under a linear transformation. The determinant of the 2×2 matrix is equal to $ad - bc$.

- For problem 4, once students observe how the entries are related between a matrix and its inverse, refocus their attention on the denominator of each fraction by asking the following question:
  - How are the values of the denominators in the inverse matrix related to the original matrix?

    The denominator of each entry of the inverse is equal to the determinant of Matrix A.

- For problem 5, students should leverage the observations they made earlier in the task to predict the inverse of the transformation matrix.

Summarizing the Task

- Begin summarizing the task by eliciting responses from groups of students for problem 2. Record all observations made by students on a whiteboard or poster paper. There are several key observations that students should highlight.
- Once all the listed observations have been made, have students use their observations to determine whether these observations hold for the matrices that represent $B(x, y)$ and $C(x, y)$ from Part 1 of the lesson. Elicit responses from the class by asking the following question:

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- Do the observations we made earlier also hold for the other pair of inverse matrices on Handout 4M.6.A, $B(x, y)$ and $C(x, y)$?

  Yes. The positions of the first entry (row 1, column 1) and last entry (row 2, column 2) are switched, the signs of the second entry (row 1, column 2) and third entry (row 2, column 1) are switched, and then each entry of the inverse is divided by the determinant of the other matrix.

- Elicit responses from students to verify that they used their observations to predict the inverse of the matrix $\begin{bmatrix} -2 & 1 \\ 5 & -4 \end{bmatrix}$ in problem 5. This understanding will help them generalize a method for finding the inverse of any 2×2 matrix.
- Finally, have students use their observations to predict the inverse of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and have them describe in words the relationship between entries of A and its inverse. Encourage students to use $\det(A)$ or $ad - bc$ to express the fractions that are the entries of the inverse matrix.
- After students have had a chance to find the inverse of Matrix A above, ask the following question:
  - Does every matrix have an inverse?

    No. The determinant of the original matrix is required to find the inverse matrix. The determinant is defined only for square matrices. Since every matrix entry in the inverse is divided by the determinant of the original matrix, whenever the determinant is 0, entries for an inverse matrix cannot exist because division by zero is not defined. Therefore, when the inverse of a matrix that represents that transformation doesn't exist (that is, when the determinant of that matrix is 0), the inverse of a transformation does not exist.

- Before moving on to the final part of the lesson, have students formalize their observations by recording the defining properties of the inverse of a matrix transformation. These properties are as follows:

  Given a standard transformation $T(x, y) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, the inverse transformation will exist if $ad - bc \neq 0$ and be represented by a matrix, $\frac{1}{\det(A)} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ such that:

  1. The positions of entries a and d are switched.
  2. The signs of entries b and c are switched.
  3. Each entry is divided by $ad - bc$, the determinant of the matrix in the original transformation $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

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Part 3: Applying Properties of Matrix Inverses

~45 MIN.

In this part of the lesson, students identify the inverse of a given matrix and verify that the matrices are inverses of each other using the property that the product of a matrix and its inverse is the identity matrix. Using mathematical notation, this relationship is expressed as $AA^{-1} = I = A^{-1}A$. Students also use inverse matrices and inverse transformations to determine the preimages of various given images.

Student Task

1. Use the observations made in Part 2 to compute the inverse of each of the following matrices, should they exist:

   (a) $\begin{bmatrix} 5 & -4 \\ -1 & 3 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

   (c) $\begin{bmatrix} 5 & 2 \\ -7 & -3 \end{bmatrix}$

   (d) $\begin{bmatrix} -3 & 1 \\ 5 & -2 \end{bmatrix}$

2. Verify that the matrices you generated are the inverses of each of the given matrices. Provide numerical, geometric, and algebraic explanations of how to verify inverse matrices. You may use an online matrix calculator.

Facilitating the Task

- Begin by displaying the student task in a location that provides access to all students. Give students time to closely observe the task and ask clarifying questions. Students can work in pairs to complete the task.
- Students will compute the inverses of each of the given transformations using the inverse formula they developed at the end of Part 2.
- Have students compare their computations with those of their partner to ensure they have the same answers. As you circulate around the room, ask students how they might be able to verify that the inverses they found are the inverses for the corresponding matrices in problem 1. Give them time to share their responses with the class.
- Below are possible methods that may align to what students share with the class. If the following methods are not provided by students, a discussion of methods 1 and 3 may be beneficial.
  - Method 1: In Part 1, we identified inverse transformations by comparing the sets of inputs and outputs of the transformations. If one transformation has a specific set of inputs and outputs (or equivalently, preimages and images), then we can expect the inverse transformation to have the set of outputs as inputs and the set of corresponding inputs as outputs.

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- Method 2: If the transformation is represented geometrically, the inverse of that transformation undoes the original transformation. For example, if a transformation rotates all points 90° clockwise about the origin, then the inverse should rotate all points 90° counterclockwise about the origin. This can be verified by substituting sample points into these transformations in sequence to be sure the points are mapped from their original locations and then mapped back to their original locations.
- Method 3: Since an inverse transformation should undo the original transformation, performing the first transformation followed by the second transformation as we did in Lesson 4M.5 should result in the identity transformation. This means we can compose the two transformations and determine if it results in the identity transformation.

- Method 3 is the preferred method for verifying whether two transformations or matrices are inverses of each other. Have students discuss how they might use this method to algebraically determine whether two matrices are inverses of each other.

Guiding Student Thinking

Students might benefit from revisiting what it means algebraically to apply the transformation T followed by the transformation T^{-1}. They might remember that in Lesson 4M.5, the composition of two transformations can be interpreted as a new transformation whose matrix representation is given by the product of the matrices in T and T^{-1}.

- Before helping students understand why method 3 is the desired method for verifying whether two transformations or matrices are inverses, ask the following questions:
  - What happens when we compose two linear transformations?

    We determine a single transformation represented by the product of the two matrices that represents the original transformations performed in sequence.
  - To verify that two transformations are inverses, we find their product. If the product results in the identity transformation, what matrix representation should we expect?

    The identity transformation is represented by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Therefore, the product of a matrix and its inverse (or the composition of a transformation and its inverse) should result in the identity matrix.

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UNIT 4M

- Formally define the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as the 2×2 *identity matrix*. Have students use this preferred method of verification to check that the inverses they computed are in fact inverses of each other.

Summarizing the Task

- Begin summarizing the task by eliciting responses from students to verify that the inverse matrices found in problem 1 of the Student Task are accurate. The answers to problem 1 follow below:

  1(a) The determinant of the original matrix is 11. Therefore, the inverse is $\begin{bmatrix} \frac{3}{11} & \frac{4}{11} \\ \frac{1}{11} & \frac{5}{11} \end{bmatrix}$.

  1(b) The determinant of the original matrix is –2. Therefore, the inverse is $\begin{bmatrix} -2 & -1 \\ -\frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.

  1(c) The determinant of the original matrix is –1. Therefore, the inverse is $\begin{bmatrix} 3 & -2 \\ 7 & -5 \end{bmatrix}$.

  1(d) The determinant of the original matrix is 1. Therefore, the inverse is $\begin{bmatrix} -2 & -1 \\ -5 & -3 \end{bmatrix}$.

- Next, students should now be able to use the matrix property $AA^{-1} = I = A^{-1}A$. Have them complete problem 2 of the task using this property as well as Desmos to compute the product of the two matrices to verify algebraically whether two matrices are inverses of each other. Students should recognize that if the product results in the identity matrix, the matrices are inverses.

- A problem set is also provided for students to engage in more practice with inverse transformations and images and preimages under various transformations. Students can complete the practice problems on **Handout 4M.6.C: Undoing Transformations and Finding Preimages** to further develop their understanding of finding the inverse of a matrix and using the property $AA^{-1} = I = A^{-1}A$ to verify if matrices are inverses of each other.

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Assess and Reflect on the Lesson

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

Consider the transformation $T(x,y)=\begin{bmatrix}2 & 5\\3 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$. Find the inverse of this transformation, $T^{-1}(x,y)$. Explain how you would verify that T^{-1} is the inverse of T.

$T^{-1}(x,y)=\begin{bmatrix}-\frac{1}{13} & \frac{5}{13}\\ \frac{3}{13} & -\frac{2}{13}\end{bmatrix}$. I would verify that these matrices are inverses by showing that $TT^{-1}=I=T^{-1}T$, where I is the identity matrix.

HANDOUTS

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 4M.6.A: Inverse Transformations

1. $A(x, y)=\begin{bmatrix}3 & 4\\2 & 7\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$		2. $B(x, y)=\begin{bmatrix}4 & 5\\1 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$	
(x, y)	$A(x, y)$	(x, y)	$B(x, y)$
(1, 2)	(11, 16)	(2, 0)	(8, 2)
(–3, 4)	(7, 22)	(–1, 3)	(11, 8)
(5, –2)	(7, –4)	(–2, 5)	(17, 13)
(–1, 6)	(21, 40)	(9, –7)	(1, –12)
(7, 4)	(37, 42)	(6, –3)	(9, –3)
(,)	(,)	(,)	(,)
(,)	(,)	(,)	(,)

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3. $C(x, y) = \begin{bmatrix} \frac{3}{7} & -\frac{5}{7} \\ -\frac{1}{7} & \frac{4}{7} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$		4. $D(x, y) = \begin{bmatrix} \frac{7}{13} & -\frac{4}{13} \\ -\frac{2}{13} & \frac{3}{13} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$	
(x, y)	$C(x, y)$	(x, y)	$D(x, y)$
(8, 2)	(2, 0)	(11, 16)	(1, 2)
(11, 8)	(−1, 3)	(7, 22)	(−3, 4)
(17, 13)	(−2, 5)	(7, −4)	(5, −2)
(1, −12)	(9, −7)	(21, 40)	(−1, 6)
(9, −3)	(6, −3)	(37, 42)	(7, 4)
(,)	(,)	(,)	(,)
(,)	(,)	(,)	(,)

Handout 4M.6.B: Defining the Inverse of a Matrix

1.

Matrix from Transformation $A(x, y)$	Matrix from the Inverse Transformation $D(x, y)$
$\begin{bmatrix} 3 & 4 \\ 2 & 7 \end{bmatrix}$	$\begin{bmatrix} \frac{7}{13} & -\frac{4}{13} \\ -\frac{2}{13} & \frac{3}{13} \end{bmatrix}$

2. Students should make the following observations when comparing linear transformation $A(x, y)$ to linear transformation $D(x, y)$:
   - The values in row 1, column 1 and in row 2, column 2 switch places from $A(x, y)$ to $D(x, y)$, and vice versa.
   - The signs of the values in row 2, column 1 and in row 1, column 2 change from $A(x, y)$ to $D(x, y)$, and vice versa.

3. The value of $\det(A) = 3 \cdot 7 - 4 \cdot 2 = 21 - 8 = 13$. The denominator of each fraction in Matrix D is 13.

4. All observations hold true.

5. $\begin{bmatrix} -\frac{4}{3} & -\frac{1}{3} \\ -\frac{5}{3} & -\frac{2}{3} \end{bmatrix}$

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Handout 4M.6.C: Undoing Transformation and Finding Preimages

1. The inverse transformation $T^{-1}(x, y) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ rotates every point in the plane 90° clockwise about the origin.

2. (a) $T^{-1}(x, y) = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

   (b) The determinant of the inverse matrix is $\frac{1}{5}$.

   (c) After applying the linear transformation, T, to the unit square, the image is $\begin{bmatrix} 2 & 3 & 0 & 5 \\ 1 & 4 & 0 & 5 \end{bmatrix}$.

   (d) Determine the inverse of the linear transformation, $T(x, y) = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. The transformation matrix required to map the image found in part (c) to its preimage is $T^{-1}(x, y) = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$.

   (e) The determinants of a matrix and its inverse matrix are reciprocals of each other. This makes sense because, while the transformation T maps the unit square to a parallelogram with an area of 5 units, its inverse transformation maps the parallelogram with an area of 5 units back to the unit square.

3. Mikaela could use matrix multiplication to determine which matrix, B or C, describes the inverse of T by determining which product, BT or CT, is equal to the identity matrix. Because $TB = BT = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, Matrix B defines the inverse transformation.

4. The determinant tells us that k is either 1 or –4. This gives two possible inverses, one for each value of k.

   $T^{-1}(x, y) = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{3}{5} & \frac{2}{5} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ or $T^{-1}(x, y) = \begin{bmatrix} -\frac{4}{5} & -\frac{1}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

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UNIT 4M

PRACTICE PERFORMANCE TASK

Using Matrices to Construct Polynomial Functions

LEARNING OBJECTIVES

4M.2.1 Express a system of linear equations using multiple representations.

4M.2.2 Solve a system of linear equations using inverse matrices.

4M.2.3 Construct the algebraic representation of a polynomial function that passes through a set of given points using technology.

PRACTICE PERFORMANCE TASK DESCRIPTION

In this practice performance task, students explore an application of matrices for solving systems of equations. Given a set of n points, it is possible to determine a polynomial equation with a degree of at most $n-1$ that passes through the given points. Students use what they know about solving systems of equations with inverse matrices to determine the parameters of the polynomial equation.

AREAS OF FOCUS

- Engagement in Mathematical Argumentation
- Connections Among Multiple Representations

SUGGESTED TIMING

~45 minutes

MATERIALS

- online matrix calculator, such as **Desmos.com**

HANDOUT

- Unit 4M Practice Performance Task: Using Matrices to Construct Polynomial Functions

AP Connections

This performance task supports AP preparation through alignment to the following AP Calculus Course Skills:

- **1.E** Apply appropriate mathematical rules or procedures, with and without technology.
- **2.A** Identify common underlying structures in problems involving different contextual situations.
- **3.G** Confirm that solutions are accurate and appropriate.

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ELICITING PRIOR KNOWLEDGE

The goal of this task is for students to demonstrate their understanding of how to use matrices to solve a system of equations. This practice performance task expects students to set up a system of equations, express the system as an equivalent matrix equation, and use inverse matrices to solve the equation. In this application of matrices, to solve systems of equations, students determine the equation of a polynomial that passes through a given set of points.

- To begin, introduce students to the practice performance task. The student handout for the task is shown before the scoring guidelines for reference.
- To prepare students to engage in the task, you could ask questions like the following:
  - Suppose you had the system of equations $\begin{cases} 3x+4y=8 \\ 7x-9y=10 \end{cases}$. How can you express this system as a matrix equation?

    The matrix equation form of this system is $\begin{bmatrix} 3 & 4 \\ 7 & -9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 8 \\ 10 \end{bmatrix}$.
  - Suppose you had the system of equations $\begin{cases} 2x+4y+z=3 \\ 6x-8y-3z=-5 \\ -x+7y-4z=11 \end{cases}$. How can you express this system as a matrix equation?

    The matrix equation form of this system is $\begin{bmatrix} 2 & 4 & 1 \\ 6 & -8 & -3 \\ -1 & 7 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 3 \\ -5 \\ 11 \end{bmatrix}$.
  - How can you solve a system of equations expressed as a matrix equation?

    One way to solve a system of equations expressed as a matrix equation is to multiply the left side of both sides of the matrix equation by the inverse of the coefficient matrix.
- If students struggle with the warm-up problems, it could indicate that they are not yet fully prepared to engage in the practice performance task. You may find it beneficial to provide a just-in-time review of the concepts critical for success with this task: expressing a system of equations as a matrix equation and solving a system of equations with inverse matrices.

SUPPORTING STUDENTS

This practice performance task is a little different from others in this course because it includes some instruction for students for setting up systems of equations. The following are a few possible implementation strategies you can use to help students engage with the task. You should rely on your knowledge of your students and your

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professional expertise to determine how to provide appropriate scaffolds while maintaining the cognitive demand of the task.

- **Gradual Implementation:** This practice performance task includes instructions for students for setting up the systems of equations. You might find it helpful to work through part (a) with the entire class before having students complete the rest of the task.
- **Collaboration:** This practice performance task is an excellent opportunity for students to work with a partner. Since this application of matrices and systems of equations will likely be new for many students, they may benefit from engaging in an academic conversation with a partner about how to complete the task.
- **Using Technology:** Encourage students to use a matrix calculator, such as Desmos, to find inverse matrices and perform the matrix multiplication in parts (d) and (e). If students have not solved a system of more than two equations in more than two variables, they may need more significant support to complete the practice performance task successfully.

SCORING STUDENT WORK

Because this is a practice performance task, you can decide to have students score their own solutions, have students score their classmates' solutions, or score the solutions yourself. Be sure to use the results of the practice performance task to identify patterns and trends that can inform further instruction.

Using Matrices to Construct Polynomial Functions

You have learned that matrices are incredibly useful mathematical tools for representing geometric transformations and for solving systems of equations. Another application of matrices is determining the equation of a polynomial that passes through a given set of points by solving a system of equations. Here's how it works:

Suppose you want to know the equation of a line that passes through the points $(2, -5)$ and $(-4, 1)$. Since any two distinct points lie along a unique line, $(2, -5)$ and $(-4, 1)$ must be solutions to an equation $mx + b = y$. You can substitute the values you know into the equation, which creates two equations: $2m + b = -5$ and $-4m + b = 1$. These equations make up a system of two equations where the unknowns are m and b, rather than the more commonly used variables x and y:

$$\begin{cases} 2m + b = -5 \\ -4m + b = 1 \end{cases}$$

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There are 19 possible points for this practice performance task.

Student Stimulus and Part (a)

(a) Represent the system of equations as a matrix equation and then solve the matrix equation using an inverse matrix. Show the work that leads to your solution.

Sample Solution

The system of linear equations can be expressed as

$$\begin{bmatrix} 2 & 1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \end{bmatrix}.$$

To solve the system of equations, we multiply the left side of both sides of the equation by the inverse of the coefficient matrix:

$$\begin{bmatrix} \frac{1}{6} & -\frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{6} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} -5 \\ 1 \end{bmatrix}$$

This yields $\begin{bmatrix} m \\ b \end{bmatrix}\begin{bmatrix} -1 \\ -3 \end{bmatrix}$.

Scoring note: If the student incorrectly writes the matrix equation for the first point but uses a correct procedure to solve the matrix equation, then they can receive the third point.

Points Possible

3 points maximum

1 point for correctly expressing the system of equations as a matrix equation

1 point for a correct explanation of how to solve the system of equations using an inverse matrix

Scoring note: The student does not have to calculate the inverse of the coefficient matrix to receive the second point, but they must explain how to solve the system of equations.

1 point for correctly solving the system of equations using an inverse matrix

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Student Stimulus and Part (b)

(b) The final step after solving the system of equations is to substitute the values of m and b you found into the equation $mx + b = y$ to determine the equation of the line that passes through the points $(2, -5)$ and $(-4, 1)$. You can use a graphing utility to verify that the line whose equation you found passes through both points.

Sample Solution	Points Possible
Based on the solution of the system of equations, the equation of the line that passes through the points $(2, -5)$ and $(-4, 1)$ is $y = -x - 3$.	**1 point maximum** 1 point for correctly using the student's solution from part (a) as the parameters of the linear equation

Student Stimulus and Part (c)

(c) Suppose you wanted to find the equation of a parabola that passes through the points $(-1, -5)$, $(2, 4)$, and $(3, -1)$. Because these points are on a parabola, they must be solutions to a quadratic equation, which has the form $ax^2 + bx + c = y$. Write a system of equations where the variables are a, b, and c, and then express the system in matrix form.

Sample Solution	Points Possible
The system of equations for this set of points is $\begin{cases} 1a - 1b + c = -5 \\ 4a + 2b + c = 4 \\ 9a + 3b + c = -1 \end{cases}$. This system can be expressed as a matrix equation as: $\begin{bmatrix} 1 & -1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \\ -1 \end{bmatrix}$. *Scoring note*: If the student makes an error writing the system of equations but correctly expresses their system in matrix equation form, then the student should receive the fourth point.	**4 points maximum** 1 point for writing a correct equation using the point $(-1, -5)$ 1 point for writing a correct equation using the point $(2, 4)$ 1 point for writing a correct equation using the point $(3, -1)$ 1 point for correctly expressing the system of equations as a matrix equation

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Student Stimulus and Part (d)

(d) Solve the system of equations that you wrote in part (c) and use the solutions to determine the equation of the parabola that passes through all three points. Explain how you determined your solution. You can use a graphing utility to verify that the parabola whose equation you found passes through all three points.

Sample Solution	Points Possible
To solve the system of equations, I multiply the left side of the matrix equation by the inverse of the coefficient matrix. This yields the solution: $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 2 \end{bmatrix}$ Based on the solution of the system of equations, the equation of the parabola that passes through the points $(-1, -5)$, $(2, 4)$, and $(3, -1)$ is $y = -2x^2 + 5x + 2$. *Scoring note*: If the student incorrectly writes the algebraic equations or the matrix equation in part (c) but uses a correct procedure to solve the matrix equation and correctly interprets their solution, then they can receive points in this part.	**3 points maximum** 1 point for providing a correct explanation of how to solve the system of equations using an inverse matrix 1 point for finding the correct solution based on the student's system of equations from part (c) 1 point for correctly using the student's solution to write the equation of the parabola through the given points

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Student Stimulus and Part (e)

(e) Suppose you wanted to find the equation of a third-degree polynomial that passes through the points $(-4, -11)$, $(-2, 11)$, $(1, -1)$, and $(3, 31)$. Write a system of equations that you can use to find the coefficients of the equation, express it as a matrix equation, solve the equation using an inverse matrix, and write the equation of the polynomial.

Sample Solution	Points Possible
The general form of a third-degree polynomial is $ax^3 + bx^2 + cx + d = y$. So, the system of equations that can be used to find the third-degree polynomial that passes through the points $(-4, -11)$, $(-2, 11)$, $(1, -1)$, and $(3, 31)$ is $\begin{cases} -64a + 16b - 4c + d = -11 \\ -8a + 4b - 2c + d = 11 \\ a + b + c + d = -1 \\ 27a + 9b + 3c + d = 31 \end{cases}$. This system can be expressed as a matrix equation as: $\begin{bmatrix} -64 & 16 & -4 & 1 \\ -8 & 4 & -2 & 1 \\ 1 & 1 & 1 & 1 \\ 27 & 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -11 \\ 11 \\ -1 \\ 31 \end{bmatrix}$ To solve this system, I multiply the left side of both sides of the matrix equation by the inverse of the coefficient matrix. This yields the solution: $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -5 \\ 1 \end{bmatrix}$ Therefore, the equation of the third-degree polynomial that passes through the given points is $y = x^3 + 2x^2 - 5x + 1$.	**8 points maximum** 1 point for writing a correct equation using the point $(-4, -11)$ 1 point for writing a correct equation using the point $(-2, 11)$ 1 point for writing a correct equation using the point $(1, -1)$ 1 point for writing a correct equation using the point $(3, 31)$ 1 point for expressing the system of equations as a matrix equation *Scoring note*: If the student makes an error in writing the system of equations but correctly expresses their system in matrix equation form, then the student should receive the fifth point. 1 point for providing a correct explanation of how to solve the system of equations using an inverse matrix 1 point for finding the solution based on the student's system of equations from part (c) 1 point for using the solution to write the equation of the polynomial through the given points *Scoring note*: If the student incorrectly writes the matrix equation but uses a correct procedure to solve the matrix equation and correctly interprets their solution, then they can receive sixth, seventh, and eighth points.

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PROVIDING FEEDBACK ON STUDENT WORK

Because this is a practice performance task, you could choose to share the scoring guidelines with students before you score their work. This would give students an opportunity to learn what a complete response looks like and allow them to self-assess the completeness and correctness of their answer. Be sure to identify trends in students' responses to inform further instruction. These trends should include topics that students consistently displayed mastery of, as well as conceptual errors that students commonly made. Possible trends and suggested guidance for each part of the task follow, although the patterns you observe in your classroom may differ.

(a) If students are unsure about how to express a system of linear equations in matrix form, they may require additional practice in writing the coefficients of the linear equations in matrix form. Students may also benefit from more thorough instruction about why the matrix equation has a form in which the product of the coefficient matrix and the variable matrix is equal to the constant matrix.

Teacher Notes and Reflections

(b) Because students are used to solving systems of equations for variables rather than for parameters, they may struggle to interpret their solution and substitute the values as parameters into the equation $y = mx + b$. It may help to call their attention to the matrix form of the solution, $\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$, so they can more readily observe how the entries of this matrix correspond to the parameters.

Teacher Notes and Reflections

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(c) Some students may be unsure of how the values the coefficients of the a terms in the equations of the system are determined. It may help to call students' attention to the general form of a second-degree polynomial, $ax^2 + bx + c = y$, and discuss that when the values for x in each of the coordinates are substituted into the general form, they are squared in the first term.

Teacher Notes and Reflections

(d) If students have not previously solved a system of three equations in three variables, they may attempt to develop a manual technique for determining the inverse of the 3×3 coefficient matrix. Encourage students to use technology to find the inverse of the 3×3 matrix and then perform the required multiplication rather than attempting to invert the 3×3 matrix by hand, which is beyond the scope of this course.

Teacher Notes and Reflections

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(e) Part (e) is intended to be a culmination of the skills students have learned through the practice performance task as well as an extension of the task to a higher degree polynomial than they have previously encountered. The most challenging part of the task for students may be recognizing that the general form of a third-degree polynomial is $ax^3 + bx^2 + cx + d = y$ and that the system of equations will have four equations and four variables. You can encourage students to use all the available information they have and apply a similar approach to what they used when working with a second-degree polynomial.

Teacher Notes and Reflections

Try to assure students that converting their score into a percentage does not provide an accurate measure of how well they performed on the task. You can use the following suggested score interpretations with students to discuss their performance.

Points Received	How Students Should Interpret Their Score
17 to 19 points	"I know all of these algebraic concepts really well. This is top-level work. (A)"
12 to 16 points	"I know all of these algebraic concepts well, but I made a few mistakes. This is above-average work. (B)"
7 to 11 points	"I know some of these algebraic concepts well, but not all of them. This is average-level work. (C)"
3 to 6 points	"I know only a little bit about these algebraic concepts. This is below-average work. (D)"
0 to 2 points	"I don't know much about these algebraic concepts at all. This is not passing work. (F)"

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UNIT 4M

LESSON 4M.7

Introduction to Recursive Processes

LEARNING OBJECTIVES

4M.3.1 Express a recursive process that models the relationship between two or three related subgroups of a population using multiple representations.

4M.3.2 Determine the future size of subgroups of a population given the current size of each subgroup.

4M.3.4 Determine the previous sizes of subgroups of a population given the current size if each subgroup.

LESSON OVERVIEW

CONTENT FOCUS

In this lesson, students explore a recursive process that determines how members of a population move between subgroups over a unit of time. To engage in this exploration, students need to understand how to represent recursive processes using a system of equations, transition diagrams, and an equation involving matrices. Understanding these various representations leads students to recognize that exponentiating a transition matrix is a more efficient technique for determining the future or previous sizes of subgroups within a population.

LESSON DESCRIPTION

Part 1: Representing Recursive Processes Algebraically

In this part of the lesson, students learn how to represent recursive processes using a transition diagram and algebraic expressions. Students use the verbal representation of the scenario to determine the size of each subgroup of a population after one unit of time. This understanding leads students to represent the number of members of a population within each subgroup using an algebraic expression.

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Connections Among Multiple Representations

SUGGESTED TIMING

~110 minutes

LESSON SEQUENCE

- This lesson is part of a lesson sequence (~220 minutes total) with Lesson 4M.8.

MATERIALS

- matrix calculator, such as **Desmos.com/matrix**

HANDOUTS

Lesson

- 4M.7.A: Determining the Future and Previous Sizes of Subgroups of a Population

Practice

- 4M.7.B: Exploring the Movement Between Subgroups of a Population

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Students' understanding of how to represent the sizes of subgroups of a population with algebraic expressions supports the second part of the lesson in which they develop a matrix equation to help determine the size of each subgroup.

Part 2: Using Transition Matrices to Describe the Movement between Subgroups
In this part of the lesson, students augment the diagrams they created in Part 1 to describe the movement between subgroups. Students add numerical details to their diagrams that describe how the members of the population move between its subgroups. Next, students interpret the transition diagram to construct the matrix equation needed to determine the size of each subgroup over time. Students end this part of the lesson with an understanding of how to interpret the entries of a transformation matrix to describe the movement of members of subgroups of a population at any point in time.

Part 3: Determining Previous Sizes of Subgroups of a Population
In this part of the lesson, students apply their understanding of matrix equations and the inverse of a transition matrix to determine the previous sizes of subgroups of a population. Students can accomplish this in two different ways. First, they can find the inverse of the transition matrix and then multiply it by the matrix representing existing sizes of the subgroups. Alternatively, students can set up and then solve the matrix equation to determine the sizes of the subgroups. Students conclude this part of the lesson knowing how to determine previous sizes of subgroups by going back one unit of time.

FORMATIVE ASSESSMENT GOAL

This lesson prepares students to complete the following formative assessment activity.

An invasive species of beetle made its way to a remote island in the middle of the Pacific. The beetles feast on the fast-growing palm trees on the island. Each month, the beetles consume 40% of the island's healthy palm trees, and only 25% of the unhealthy palm trees recover and become healthy trees again. Initially, there are 260 healthy palm trees and zero unhealthy palm trees on the island.

(a) Generate a transition diagram to represent the scenario.

(b) Write a matrix equation that could be used to determine the numbers of heathy and unhealthy palm trees.

(c) Predict how many healthy palm trees and how many unhealthy palm trees are present on the island after four weeks. Round your answers to the nearest whole number.

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Part 1: Representing Recursive Processes Algebraically

~45 MIN.

In this part of the lesson, students learn how to represent recursive processes using a transition diagram and algebraic expressions. Students use the verbal representation of the scenario to determine the size of each subgroup of a population after one unit of time. This understanding leads students to represent the number of members of a population within each subgroup using an algebraic expression. Students' understanding of how to represent the sizes of subgroups of a population with algebraic expressions supports the second part of the lesson in which they develop a matrix equation to help determine the size of each subgroup.

Student Task

The student task is provided on Part 1 of **Handout 4M.7.A: Determining the Future and Previous Sizes of Subgroups of a Population.** In this part of the handout, students are introduced to a scenario in which they explore how frogs move between a pond habitat and a stream habitat. To do this, students draw a diagram that represents the movement between the population and use algebraic expressions to determine the size of each subgroup within the population.

Facilitating the Task

- Begin by providing students with Handout 4M.7.A and allowing them some time to closely observe the task and ask clarifying questions. Students should work in small groups of 3 or 4 to complete this task.
- Explain to students that in this lesson, they will explore contexts that are likely different from those they encountered in their prior mathematics courses.
- As you circulate around the room, you may need to provide students with some guidance on problem 1. To help prepare students for this context, consider asking questions such as:
  - What two different subgroups are represented in the problem?
    The two subgroups are frogs in the pond habitat and frogs in the stream habitat.
  - Each week, what percentage of frogs in the pond habitat stay in the pond habitat?
    Sixty percent of the frogs in the pond habitat stay in the pond habitat.

Meeting Learners' Needs

Before students begin working on the task, it may be helpful to read the questions in Part 1 of the handout together and unpack the key details. Students may struggle to generate a diagram that represents the situation without a clear understanding of the two subgroups and how members of the population move between habitats.

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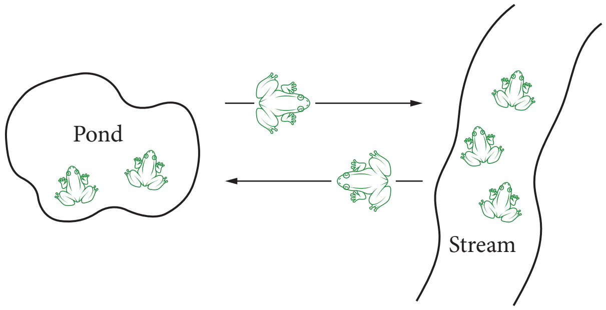
- Each week, what percentage of frogs in the stream habitat stay in the stream habitat?

  Eighty percent of the frogs in the stream habitat stay in the stream habitat.

- Student diagrams for problem 1 of the handout will vary in detail. At this point in the lesson, students should not be expected to produce a transition diagram, and it is acceptable for students to focus on the movement of frogs between habitats without including numerical details about how they move between them. Transition diagrams will be discussed later in the lesson. A sample student diagram is shown below.



- Give students about five minutes to work on problem 2. Students may use a variety of methods. Some students may draw a diagram, make a chart, or use algebraic tools.
- For problem 3, encourage students to use at least two different methods to determine the number of frogs in each habitat after 1 week. This will help them generalize their thinking in the next part of the lesson.
- For problem 4, students generalize their two worked examples from problem 3 to develop algebraic expressions for the number of frogs in each habitat after 1 week in terms of the initial number of frogs in each habitat. Some students might describe the process in terms of a diagram. Encourage them to represent their thinking algebraically.

Meeting Learners' Needs
If students finish problem 2 early, you can encourage them to determine the number of frogs in each habitat using a different method. For example, if the student used a diagram to answer the question, you can encourage them to represent the situation and solve the problem algebraically.

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- For students who need support to develop the algebraic formula, you can encourage students to start by developing the basic structure of the equation using words:

the number of frogs who stayed in the pond	+	the number of frogs who moved to the pond	=	the number of frogs who are in the pond after 1 week

- Next, help students write an expression that represents each quantity in this equation. Students can use their work from problems 2 and 3 to help them generalize the quantities for each part of the formula.

Guiding Student Thinking

There is a nuance in how variables and labels are used throughout this lesson. We use *P* and *S* in the transition diagram as labels to represent the two habitats: the pond habitat and the stream habitat. We use variables *p* and *s* to represent the *number* of frogs in each habitat. We can add subscripts to the variables to indicate the week number at which the number of frogs in each habitat is counted.

Summarizing the Task

- Begin summarizing this part of the lesson by sharing the sample diagram in the Facilitating the Task section so students can compare it with their own diagrams. Be sure to explain that while everyone's diagrams may look different, every diagram should show two different habitats and indicate movement between them. Let students know that the sample diagram you display is only one way to represent this context.
- Next, have student groups volunteer to report to the class their process for answering problem 2. Remind students that this is new material, so it is fine if they are unsure whether their answer or approach is correct. As students share their approaches, be sure to highlight at least two different ways they went about determining the number of frogs in each habitat after one week.
- As student groups present their work, encourage the rest of the class to identify any similarities they observe in the thought processes of the different approaches. This will help students identify the algebraic structures that allow matrices to be used to represent this context.

Classroom Ideas

If student groups all used similar methods, you can present alternative solution strategies that other students could have used.

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- Next, have student groups share with the class their responses to problem 3 and how they used these responses to generate their algebraic expressions in problem 4.
- Encourage students to look for similarities in the algebraic expressions they used to determine the number of frogs in each habitat. Consider asking questions such as:
  - What aspects of your process stayed the same?

    The values representing the percentages of frogs that stayed in each habitat and moved from each habitat remained constant.
  - What aspects of the process were different in problem 3?

    The values representing the initial number of frogs in each habitat were different.
- Before moving on, be sure that students are able to interpret each term in the expressions as the number of frogs staying in the pond habitat, $0.6p$, the number of frogs moving from the stream habitat to the pond habitat, $0.2s$, the number of frogs moving from the pond habitat to the stream habitat, $0.4p$, and the number of frogs staying in the stream habitat, $0.8s$.

Part 2: Using Transition Matrices to Describe the Movement between Subgroups

~45 MIN.

In this part of the lesson, students augment the diagrams they created in Part 1 to describe the movement between subgroups. Students add numerical details to their diagrams that describe how the members of the population move between its subgroups. Next, students interpret the transition diagram to construct the matrix equation needed to determine the size of each subgroup over time. Students end this part of the lesson with an understanding of how to interpret the entries of a transformation matrix to describe the movement of members of subgroups of a population at any point in time.

Student Task

Students should continue to Part 2 of Handout 4M.7.A, in which they focus on constructing a transition diagram that represents the context they explored in Part 1. Students conclude this part of the task by interpreting the transition matrix that describes how members of a population move between subgroups.

Facilitating the Task

- By the end of Part 1, students developed the understanding that $0.6p + 0.2s$ is an algebraic expression that represents the number of frogs in the pond habitat after 1 week, and $0.4p + 0.8s$ is an algebraic expression that represents the number of frogs in the stream habitat after 1 week.

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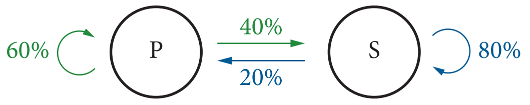
- Explain to students that they will now learn how to construct a *transition diagram* that illustrates the relationships described in this situation. A transition diagram provides more detail than the diagram students created in problem 1 because it shows that the percentages of frogs that stayed in each habitat and moved from each habitat each week is constant. Add the phrase *transition diagram* to your classroom's word wall, if you have one.
- Display the following transition diagram and make connections to the verbal representation and algebraic expressions discussed in Part 1 of the lesson. Show how this diagram is constructed and labeled to describe the movement between the frogs in the pond habitat and the frogs in the stream habitat.

One Week Transition Diagram



- Next, introduce the idea of using subscripts to keep track of the week number that is represented by each variable. For this situation, p_0 represents the initial number of frogs in the pond habitat (week 0), p_1 represents the number of frogs in the pond habitat after 1 week, and p_2 represents the number of frogs in the pond habitat after 2 weeks.
- To help students practice interpreting this subscript notation, consider asking questions such as:
  - What does the notation s_0 represent?

    s_0 represents the number of frogs initially in the stream at week 0.
  - What does the notation p_7 represent?

    p_7 represents the number of frogs in the pond after 7 weeks.
- Help students see that the expressions they wrote in problem 4 can be expressed in terms of the equations below:

$$0.6p_0 + 0.2s_0 = p_1$$
$$0.4p_0 + 0.8s_0 = s_1$$

Guiding Student Thinking

Students likely worked with subscripts in Pre-AP Algebra 1 when learning about arithmetic and geometric sequences. You can help students make the connection between this type of recursive process and the formulas they wrote in Pre-AP Algebra 1.

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- Call attention to the structure of these equations, in which there are two equations, each with two unknowns. Remind students that they can represent these types of equations using matrices as:

$$\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}\begin{bmatrix} p_0 \\ s_0 \end{bmatrix}=\begin{bmatrix} p_1 \\ s_1 \end{bmatrix}$$

- Help students see that the first column of the matrix represents the behavior of the frogs initially in the pond habitat, while the second column represents the behavior of the frogs initially in the stream habitat. The first row represents the percentage of frogs from each habitat that end up in the pond habitat, and the second row represents the percentage of frogs from each habitat that end up in the stream habitat. This matrix is also known as the *transition matrix* because it describes how members of the population move between subgroups.

Meeting Learners' Needs

An alternative way to think about the assignment of the entries in the transition matrix is that each row describes where the frogs are going, while each column describes where the frogs are coming from. Students often find this way of thinking about the columns and rows helpful when organizing recursive processes with more than two categories. Some problems that involve more than two categories are included in the practice problems at the end of the lesson.

Guiding Student Thinking

Now that students have connected this context to matrices, have them verify either by hand or using technology that this equation produces the same answers that they found for problems 2 and 3.

Summarizing the Task

- Before moving on, be sure that students can interpret each entry in the transition matrix. If students are still struggling with interpreting the transition matrix, they may benefit from seeing the transition matrix labeled as follows:

$$\begin{array}{cc} & \begin{array}{cc} \textbf{P} & \textbf{S} \end{array} \\ \begin{array}{c} \textbf{P} \\ \textbf{S} \end{array} & \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix} \end{array}$$

This translates to the following entries in the matrix $\begin{bmatrix} \text{PP} & \text{SP} \\ \text{PS} & \text{SS} \end{bmatrix}$.

- Using this labeling, each entry in the matrix can be represented by a pair of letters read from column to row. So, the first entry, PP, or $P \rightarrow P$, should be interpreted

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as 60% of the frogs in the pond remain in the pond. SP, or $S \to P$, should be interpreted as 20% of the frogs in the stream habitat move to the pond habitat. PS, or $P \to S$, means 40% of the frogs in the pond habitat move to the stream habitat. Lastly, SS, or $S \to S$, means that 80% of the frogs in the stream habitat remain in the stream habitat.

- The key takeaway for students in this part of the lesson is to understand that there are many situations in which subgroups of a population move between categories in predictable ways. Such scenarios can be represented with transition diagrams and represented as equations that include matrix multiplication. It is important that students conclude this part of the lesson with the ability to use all three representations—the transition diagram, the system of equations, and the matrix equation—to represent the scenario. Students also need to be able to interpret the coefficient matrix as a representation of the number of members of the population within each subgroup at a given unit of time.

Part 3: Determining Previous Sizes of Subgroups of a Population

~20 MIN.

In this part of the lesson, students apply their understanding of matrix equations and the inverse of a transition matrix to determine the previous sizes of subgroups of a population. Students can accomplish this in two different ways. First, they can find the inverse of the transition matrix and then multiply it by the matrix representing existing sizes of the subgroups. Alternatively, students can set up and then solve the matrix equation to determine the sizes of the subgroups. Students conclude this part of the lesson knowing how to determine previous sizes of subgroups by going back one unit of time.

Student Task

Students advance to Part 3 of Handout 4M.7.A. This part of the handout focuses on students' understanding of the inverse of a transition matrix and matrix equations to find the previous sizes of subgroups of a population. Students conclude the task knowing how to determine the sizes of subgroups going back one time-unit.

Facilitating the Task

- Begin by asking students to advance to Part 3 of the handout to complete problem 6. Students should continue to work in their small groups.
- To get students started on Part 3, remind them of what steps they took to find the preimage of a point using matrix multiplication to solve a matrix equation. Probe students by asking the following questions:

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- If given a linear transformation T and an image of a point, how would you determine the preimage?

  I would determine the inverse of the transformation matrix and then multiply it by the matrix that represents the image.

- Explain the procedure for solving the matrix equation $AX = B$. Why does this procedure work?

  Assuming the products are defined, I multiply A^{-1} on the left of both sides of the equation getting $A^{-1}AX = A^{-1}B$. I know this works because $A^{-1}A = I$ and $IX = X$, so $IX = X = A^{-1}B$.

- For problem 6, as you circulate around the room, encourage students to write a matrix equation to represent the scenario. Be sure that students set up the matrix equation $\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}\begin{bmatrix} p_2 \\ s_2 \end{bmatrix} = \begin{bmatrix} 128 \\ 272 \end{bmatrix}$. Highlight for students that this equation has the same structure as those used in previous lessons to determine the preimages of points and solve systems of equations. Help students develop a method for finding the sizes of the subgroups at week 0 by considering the method for finding the sizes of the subgroups one time-step back. Multiplying the current sizes of the subgroups once by the inverse transformation matrix yields the sizes of the subgroups 1 time-step back (week 2).

Guiding Student Thinking

It may be beneficial for students to consider how the previous sizes of subgroups are related to multiplying the existing sizes of the subgroups by the inverse of the transition matrix. In other words, students can think of finding the previous sizes of subgroups as applying the inverse operation to the transition matrix, and then multiplying the result by the matrix representing the sizes of the subgroups, as represented by the expression $\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}^{-1}\begin{bmatrix} 100 \\ 300 \end{bmatrix}$. This is how students determined the preimage of a point in Lesson 4M.6.

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Guiding Student Thinking

As students think about the inverse transition matrix, they might be tempted to negate each entry in the transition matrix. Instead, remind students that the inverse of the transition matrix will be another 2×2 matrix such that the product of the transition matrix and the inverse transition matrix is the 2×2 identity matrix. It is outside the scope of this course to interpret the entries of the inverse of the transition matrix.

Summarizing the Task

- Summarize the task by allowing groups of students to share out how they determined the sizes of the subgroups in problem 6. Students will likely use one of two main methods to determine the sizes of the subgroups. Some students may have set up the matrix equation representing the scenario, while others may have multiplied $\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}^{-1}$ directly to the matrix representing the sizes of the subgroup. Highlight any student methods that included exponentiating the inverse transformation matrix to find the previous sizes of subgroups over multiple time-steps.
- Before moving on, make sure students understand that determining the size of each subgroup one week prior is mathematically equivalent to determining the preimage or solving a system of equations.

Guiding Student Thinking

As students think about the inverse transition matrix, they might be tempted to negate each entry in the transition matrix. Instead, remind them that the inverse of the transition matrix will be another 2×2 matrix such that the product of the transition matrix and the inverse transition matrix is the 2×2 identity matrix. It is outside the scope of this course to interpret the entries of the inverse of the transition matrix.

- For additional practice, have students complete the practice problems on **Handout 4M.7.B: Exploring the Movement Between Subgroups of a Population.** These problems are meant to further develop students' understanding of how to model members of a population moving between subgroups with transition diagrams. Students also practice determining future sizes of the subgroups over a *k*-unit period of time, interpreting the entries of a transition matrix in context, and determining previous sizes of subgroups within a population.

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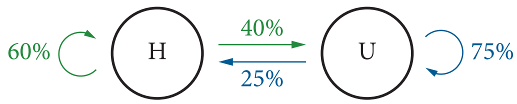
Assess and Reflect on the Lesson

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

An invasive species of beetle made its way to a remote island in the middle of the Pacific. The beetles feast on the fast-growing palm trees on the island. Each month, the beetles consume 40% of the island's healthy palm trees, and only 25% of the unhealthy palm trees recover and become healthy trees again. Initially, there are 260 healthy palm trees and zero unhealthy palm trees on the island.

(a) Generate a transition diagram to represent the scenario.



(b) Write a matrix equation that could be used to determine the numbers of heathy and unhealthy palm trees.

The matrix equation that models the palm tree population on the island is given by $\begin{bmatrix} 0.6 & 0.25 \\ 0.4 & 0.75 \end{bmatrix}\begin{bmatrix} h_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} h_1 \\ u_1 \end{bmatrix}$, which is equivalent to $\begin{cases} 0.6h_0 + 0.25u_0 = h_1 \\ 0.4h_0 + 0.75u_0 = u_1 \end{cases}$ where h_0 represents the initial healthy palm tree population and u_0 represents the initial unhealthy palm trees.

(c) Predict how many healthy palm trees and how many unhealthy palm trees are present on the island after four weeks. Round your answers to the nearest whole number.

To determine the week 4 population, we use the model $\begin{bmatrix} 0.6 & 0.25 \\ 0.4 & 0.75 \end{bmatrix}^4\begin{bmatrix} h_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} h_4 \\ u_4 \end{bmatrix}$, with $h_0 = 260$ and $u_0 = 0$. Substituting these values into the model and multiplying, we get that $\begin{bmatrix} h_4 \\ u_4 \end{bmatrix} = \begin{bmatrix} 102 \\ 158 \end{bmatrix}$, or $h_4 = 102$ and $u_4 = 158$. We conclude that after 4 weeks, there will be about 102 healthy palms on the island and about 158 unhealthy palms on the island.

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UNIT 4M

HANDOUTS

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 4M.7.A: Determining the Future and Previous Sizes of Subgroups of a Population

PART 1

1. Student answers will vary. A sample response is provided in the Facilitating the Task section in Part 1 of the lesson.
2. (a) There are 44 frogs in the pond habitat.

   (b) There are 76 frogs in the stream habitat.
3. There are 62 frogs in the pond habitat and 108 frogs in the stream habitat.
4. The expression $0.6p + 0.2s$ represents the number of frogs in the pond habitat after 1 week and the expression $0.4p + 0.8s$ represents the number of frogs in the stream habitat after 1 week.

PART 2

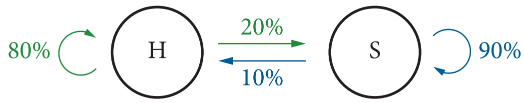
5. See the Facilitating the Task section in Part 2 of the lesson for guidance.

PART 3

6. At week 3, there are 128 frogs in the pond habitat and 272 frogs in the stream habitat. The sizes of each subgroup at week 2 are the product of the inverse of the transformation matrix and the matrix representing the sizes of the subgroups at week 3, or $\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}^{-1} \begin{bmatrix} 128 \\ 272 \end{bmatrix} = \begin{bmatrix} 120 \\ 280 \end{bmatrix}$. So, at 2 weeks, there are 120 frogs in the pond habitat and 280 frogs in the stream habitat.

Handout 4M.7.B: Exploring the Movement Between Subgroups of a Population

1. (a) **One Week Transition Diagram**



   (b) $\begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix} \begin{bmatrix} h_0 \\ s_0 \end{bmatrix} = \begin{bmatrix} h_1 \\ s_1 \end{bmatrix}$. Equivalently, we can interchange the rows and columns of the matrices for an equivalent equation: $\begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} s_0 \\ h_0 \end{bmatrix} = \begin{bmatrix} s_1 \\ h_1 \end{bmatrix}$.

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Instructional Rationale

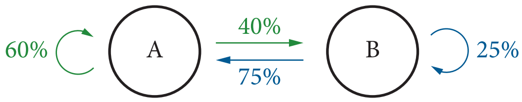
The organization of the entries in the transition matrix is not unique. Interchanging the rows and columns of all matrices in the equation produces an equivalent algebraic representation. Both equations will yield the same solutions for the numbers of healthy and infected plants.

(c) There are 1,603 healthy plants and 427 infected plants.

(d) Solving the equation $\begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}\begin{bmatrix} h_0 \\ s_0 \end{bmatrix} = \begin{bmatrix} 320 \\ 80 \end{bmatrix}$ yields

$\begin{bmatrix} h_0 \\ s_0 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{bmatrix}^{-1}\begin{bmatrix} 320 \\ 80 \end{bmatrix} = \begin{bmatrix} 400 \\ 0 \end{bmatrix}$. So, there were 400 healthy plants and 0 infected plants the previous week.

2. (a) **One Week Transition Diagram**



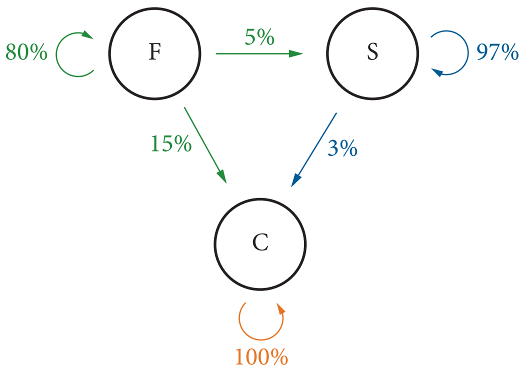
(b) $\begin{bmatrix} 0.6 & 0.75 \\ 0.4 & 0.25 \end{bmatrix}\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$ or equivalently $\begin{bmatrix} 0.25 & 0.4 \\ 0.75 & 0.6 \end{bmatrix}\begin{bmatrix} b_0 \\ a_0 \end{bmatrix} = \begin{bmatrix} b_1 \\ a_1 \end{bmatrix}$.

(c) There are 255 shoppers at Store A and 125 shoppers at Store B.

(d) Solving the equation $\begin{bmatrix} 0.6 & 0.75 \\ 0.4 & 0.25 \end{bmatrix}\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} 420 \\ 240 \end{bmatrix}$ yields

$\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.75 \\ 0.4 & 0.25 \end{bmatrix}^{-1}\begin{bmatrix} 420 \\ 240 \end{bmatrix} = \begin{bmatrix} 500 \\ 160 \end{bmatrix}$. So there were 500 shoppers at Store A and 160 shoppers at Store B the week before.

3. (a) **One Day Transition Diagram**



(b) There are multiple ways to set up an equation with matrices depending on how students organize their rows and columns. One possible equation is

$$\begin{bmatrix} 0.8 & 0 & 0 \\ 0.15 & 1 & 0.03 \\ 0.05 & 0 & 0.97 \end{bmatrix}\begin{bmatrix} f_0 \\ c_0 \\ s_0 \end{bmatrix} = \begin{bmatrix} f_1 \\ c_1 \\ s_1 \end{bmatrix}.$$

(c) There are 36 kg of fresh food and 7.1 kg of spoiled food in the refrigerator, and 6.9 kg of food has been eaten.

Guiding Student Thinking

To help students organize the entries of the 3×3 matrix describing this system, you can have students first organize the information in a chart such as the one below.

	starts fresh	starts eaten	starts spoiled
ends up fresh	0.8	0	0
ends up eaten	0.15	1	0.03
ends up spoiled	0.05	0	0.97

This yields the equation $\begin{bmatrix} 0.8 & 0 & 0 \\ 0.15 & 1 & 0.03 \\ 0.05 & 0 & 0.97 \end{bmatrix}\begin{bmatrix} f_0 \\ c_0 \\ s_0 \end{bmatrix} = \begin{bmatrix} f_1 \\ c_1 \\ s_1 \end{bmatrix}$. Help students see that the order of subgroups in the rows of the transition matrix match the order of the groups in the columns.

(d) Solving the equation $\begin{bmatrix} 0.8 & 0 & 0 \\ 0.15 & 1 & 0.03 \\ 0.05 & 0 & 0.97 \end{bmatrix}\begin{bmatrix} f_0 \\ c_0 \\ s_0 \end{bmatrix} = \begin{bmatrix} 16 \\ 3.12 \\ 4.88 \end{bmatrix}$ yields

$\begin{bmatrix} f_0 \\ c_0 \\ s_0 \end{bmatrix} = \begin{bmatrix} 0.8 & 0 & 0 \\ 0.15 & 1 & 0.03 \\ 0.05 & 0 & 0.97 \end{bmatrix}^{-1}\begin{bmatrix} 16 \\ 3.12 \\ 4.88 \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \\ 4 \end{bmatrix}$. So there were 20 kg of fresh food and 4 kg of spoiled food at the beginning of the day.

4. (a) **One Minute Transition Diagram**

(b) $\begin{bmatrix} 0.85 & 0.55 \\ 0.15 & 0.45 \end{bmatrix}\begin{bmatrix} a_0 \\ s_0 \end{bmatrix} = \begin{bmatrix} a_1 \\ s_1 \end{bmatrix}$

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(c) There are 316 fish that are awake and 84 fish that are asleep.

(d) Solving the equation yields $\begin{bmatrix} a_0 \\ s_0 \end{bmatrix} = \begin{bmatrix} 0.85 & 0.55 \\ 0.15 & 0.45 \end{bmatrix}^{-1} \begin{bmatrix} 320 \\ 210 \end{bmatrix} = \begin{bmatrix} 95 \\ 435 \end{bmatrix}$. So there were 95 fish that were awake and 435 fish that were asleep at 12:00 p.m.

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UNIT 4M

LESSON 4M.8

Stabilized Recursive Processes

LEARNING OBJECTIVES

4M.3.2 Determine the future size of subgroups of a population given the current size of each subgroup.

4M.3.3 Identify how the sizes of related subgroups of a population stabilize over time.

LESSON OVERVIEW

CONTENT FOCUS

In this lesson, students continue to explore how the members of a population move between its subgroups. Students use the same scenarios presented in Lesson 4M.7 to observe the efficiency of exponentiating a transition matrix to determine the future sizes of subgroups within a population over multiple time-steps, whereas the previous lesson introduced students with a way of finding the sizes of subgroups of a population only one time-unit away. The lesson concludes with students investigating how the sizes of each subgroup of a population tend to stabilize over time.

LESSON DESCRIPTION

Part 1: Determining Future Sizes of Subgroups Over a *k*-unit Period of Time

In this part of the lesson, students build on their understanding of transition matrices to investigate how the members of a population move between its subgroups. They work toward understanding that the operation of exponentiation on the transition matrix is a more efficient method for determining the sizes of subgroups of a population after k units of time than repeated matrix multiplication.

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Connections Among Multiple Representations

SUGGESTED TIMING

~110 minutes

LESSON SEQUENCE

- This lesson is part of a lesson sequence (~220 minutes total) with Lesson 4M.7.

MATERIALS

- matrix calculator, such as **Desmos.com/matrix**

HANDOUTS

Lesson

- 4M.8.A: Determining the Sizes of Subgroups of a Population

Practice

- 4M.8.B: Exploring How Sizes of Subgroups of a Population Stabilize Over Time

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Part 2: Examining How Populations Change Over a Fixed Time Interval
In this part of the lesson, students apply the technique of exponentiating a transition matrix to determine the future sizes of subgroups of a population. Students also continue to interpret the entries of the transition matrix to understand how the members of a population move between subgroups after *k* units of time.

Part 3: Stabilizing Systems with Two Subgroups
In this part of the lesson, students continue to analyze a scenario in which the frogs move between subgroups within a population. Students focus their exploration on how the sizes of the two subgroups of the population stabilize over time. Students observe that as the numbers of frogs moving between habitats change over time, the number of frogs in each habitat eventually stabilizes.

FORMATIVE ASSESSMENT GOAL

This lesson prepares students to complete the following formative assessment activity.

An invasive species of beetle made its way to a remote island in the middle of the Pacific. The beetles feast on the fast-growing palm trees on the island. Each month, the beetles consume 40% of the island's healthy palm trees, and only 25% of the unhealthy palm trees recover and become healthy trees again.

After any three-week period, what percentage of palm trees that started the three-week period healthy are now unhealthy?

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Part 1: Determining Future Sizes of Subgroups Over a *k*-unit Period of Time

~30 MIN.

In this part of the lesson, students build on their understanding of transition matrices to investigate how the members of a population move between its subgroups. They work toward understanding that the operation of exponentiation on the transition matrix is a more efficient method for determining the sizes of subgroups of a population after *k* units of time than repeated matrix multiplication.

Student Task

The student task is provided on Part 1 of **Handout 4M.8.A: Determining the Sizes of Subgroups of a Population**. In this part of the task, students focus on developing a more efficient way to determine the matrix equation that can be used to determine the number of frogs in each habit after *k* weeks.

Instructional Rationale

The scenario presented on Handout 4M.8.A is the same scenario presented in Lesson 4M.7. This was done to reduce the cognitive load students would experience while learning the new concepts presented in this lesson. In this part of the lesson, students expand their understanding of exponentiation to matrices to find future sizes of subgroups in multiple time-steps, as opposed to finding sizes of subgroups one time-unit away as was introduced in Lesson 4M.7.

Facilitating the Task

- Begin by providing students with Handout 4M.8.A and allowing them time to closely observe the task and ask clarifying questions. Students should notice that the context in this lesson is the same as the context in the previous lesson. Let them know that the same context is used to continue learning how members of a population move between its subgroups and to reduce the cognitive load associated with exploring new concepts. Students should work in small groups of 3 or 4 to complete this task.
- As you circulate around the room, you may find that some students need help determining a strategy for completing problem 1(b). Help students understand that they can treat the numbers of frogs in each habitat after 1 week (35 in the pond habitat and 90 in the stream habitat) as a new set of initial conditions to be used with the transition matrix again. Ensure that students can properly use subscripts to distinguish between the values of each quantity at different moments of time.

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- Consider asking questions such as the following to help motivate the need to develop an efficient method for determining the number of frogs in each habitat for time-steps further into the future.
  - Can you explain how you determined the number of frogs in each habitat after 2 weeks?

    I determined the number of frogs in each habitat after 1 week and then used those values as the input in column matrix form and then multiplied them by the transition matrix as if they were new initial values.

  - Could you use a similar process to determine the number of frogs in each habitat after 3 weeks?

    Yes, I can use the values of $p_2 = 39$ and $s_2 = 86$ as the inputs in the equation $\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}\begin{bmatrix} p_2 \\ s_2 \end{bmatrix} = \begin{bmatrix} p_3 \\ s_3 \end{bmatrix}$.

  - Could you use a similar process to determine the number of frogs in each habitat after 1 year (52 weeks)? How many calculations would you need to perform?

    Yes, you would need to perform 52 calculations using the transition matrix $\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}$ multiplied by the number of frogs in each habitat from each previous week, starting with the initial input values. Alternatively, you could use the values of p_{51} and s_{51}, if they were known, as inputs in the matrix equation $\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}\begin{bmatrix} p_{51} \\ s_{51} \end{bmatrix} = \begin{bmatrix} p_{52} \\ s_{52} \end{bmatrix}$.

- As students discuss their solution methods for determining the number of frogs in each habitat after 1 year, they should understand that the process of iterating this computation 52 times is mathematically valid but extremely tedious, and that each additional computation increases the possibility of introducing an error. Students likely will be discouraged at the thought of iterating this calculation 52 times to determine the number of frogs in each habitat after 1 year. This should help motivate students to develop a new and more efficient process for calculating these values.

- Review with students the following sequence of calculations to help introduce the operation of raising a matrix to an exponent:
  - Recall that to determine the number of frogs in each habitat after 1 week, we computed:

$$\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}\begin{bmatrix} p_0 \\ s_0 \end{bmatrix} = \begin{bmatrix} p_1 \\ s_1 \end{bmatrix}$$

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- To determine the number of frogs in each habitat after 2 weeks, we computed:

$$\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}\begin{bmatrix} p_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} p_2 \\ s_2 \end{bmatrix}$$

- Using substitution for $\begin{bmatrix} p_1 \\ s_1 \end{bmatrix}$, we get:

$$\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}\begin{bmatrix} p_0 \\ s_0 \end{bmatrix} = \begin{bmatrix} p_2 \\ s_2 \end{bmatrix}$$

$$\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}^2\begin{bmatrix} p_0 \\ s_0 \end{bmatrix} = \begin{bmatrix} p_2 \\ s_2 \end{bmatrix}$$

- For problem 2, encourage students to use technology to carry out the calculations for problem 1 again using exponentiation. Students should observe that they can efficiently determine the number of frogs in each habitat after 1 year (52 weeks) with the calculation: $\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}^{52}\begin{bmatrix} 25 \\ 100 \end{bmatrix} \approx \begin{bmatrix} 42 \\ 83 \end{bmatrix}$. As students interpret their answer, help them see that when they round the numbers of frogs to whole numbers, they should do so in such a way that ensures the total number of frogs stays the same (125).

Meeting Learners' Needs

If students are struggling to understand that the exponent on the transition matrix corresponds to the time-step of interest, you can work through more calculations and substitutions like the one illustrated above. Students also may benefit from seeing a matrix raised to an exponent expressed in an equivalent form that uses repeated multiplication.

Summarizing the Task

- To summarize the task, elicit responses from students about how they would go about finding the number of frogs in each habitat at 18 months, 24 months, and k months, assuming 1 month is 4 weeks. It is not necessary for students to carry out the calculations for the number of frogs in each habitat at those times, but look for students to share that they would represent each scenario by a matrix equation with the coefficient matrix raised to the proper exponent: 72 for 18 months, 96 for 24 months, and $4k$ for k months.
- The key takeaway for this part of the lesson is for students to understand how the operation of exponentiation applies to matrices and that the transition matrix to the *kth* power represents the changes in subgroups over a k-unit period of time.

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Part 2: Examining How Populations Change Over a Fixed Time Interval

~30 MIN.

In this part of the lesson, students apply the technique of exponentiating a transition matrix to determine the future sizes of subgroups of a population. Students also continue to interpret the entries of the transition matrix to understand how the members of a population move between subgroups after k units of time.

Student Task

Students should advance to Part 2 of Handout 4M.8.A. This part of the task reinforces for students the relative efficiencies of different methods for determining the sizes of the subgroups of a population over multiple time-steps. Students also interpret the entries of a transition matrix to respond to questions on the student task.

Facilitating the Task

- Students continue to Part 2 of the handout. Give them time to work with their groups to complete the table. While students work, make sure they record their computations in the last column. Encourage them to use different, though equivalent, methods. Students will likely use one of three approaches:
  - Students iteratively multiply by the transition matrix to determine the number of frogs in each habitat each week and record the values only every other week. Students who use this approach are utilizing what they learned in Part 1 of this lesson. They would benefit from additional support in recognizing that a different method will more efficiently allow them to determine the number of frogs in each habitat more than one week into the future.
  - Students use the formula $\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}^k \begin{bmatrix} 125 \\ 0 \end{bmatrix}$ where k represents the week number. Students who use this approach are utilizing what they learned in Part 1 of this lesson. Be mindful of students who might be using the formula without really understanding what it represents.
  - Students multiply each output pair by $\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}^2$. This approach is represented in the solutions provided in the Assess and Reflect section to support students in understanding this approach through the rest of this part of the lesson.
- For problem 5, students leverage their understanding of the entries of the transition matrix. If some students struggle to understand the relationship between the entries of the transition matrix over multi-week periods of time, you can ask some questions like the following:

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- What percentage of frogs in the stream habitat can we expect to end up in the stream habitat after 3 weeks?

  68.8%

- What percentage of frogs in the stream habitat would we expect to move to the pond habitat after 3 weeks?

  31.2%

- If there are 80 frogs in the stream habitat after 12 weeks, how many of these frogs move to the pond habitat between week 12 and week 15?

  $(0.312)(80) \approx 25$

Guiding Student Thinking

It is important for students to recognize that the calculations for the percentage of frogs that would be expected to stay or end up in the stream habitat are based on the behavior described by the transition matrix over an n-unit period of time. Similarly, the calculations for the percentage of frogs that would be expected to move out of the stream habitat to the pond habitat are based on the behavior described by the transition matrix over an n-unit period of time.

Summarizing the Task

- Begin to summarize the task by having groups of students describe their methods for completing problem 4. If all three of the methods previously described were used by students in the class, highlight the approaches from the simplest to most nuanced. Then compare the efficiencies of these methods.
- After students share their approaches to computing the number of frogs in each habitat, be sure that they can interpret the entries of the transition matrix. Ask students questions like the following:
  - What expression can you use to determine the number of frogs in each habitat after 4 weeks?

    $\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}^2 \begin{bmatrix} 55 \\ 70 \end{bmatrix}$
- If no student suggests it, propose the expression $\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}^2 \begin{bmatrix} 55 \\ 70 \end{bmatrix}$ to determine the number of frogs in each habitat after 4 weeks. Have students determine whether this expression makes sense before having them evaluate it using technology.

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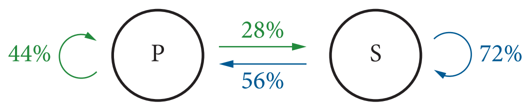
- Have students evaluate the transition matrix raised to the second power as $\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}^2 = \begin{bmatrix} 0.44 & 0.28 \\ 0.56 & 0.72 \end{bmatrix}$. Give students a couple of minutes to interpret the entries of the product matrix.

- Help students interpret the meanings of the entries in this matrix as the movement of frogs between the two habitats over a 2-week period of time. Specifically, 44% of frogs in the pond habitat stay or end up in the pond habitat, 28% of frogs in the stream habitat move to the pond habitat, 56% of frogs in the pond habitat move to the stream habitat, and 72% of frogs in the stream habitat stay in the stream habitat. You can highlight for students that the entries in each column sum to 1, meaning that movement of 100% of the frogs in each habitat is accounted for.

Two Week Transition Diagram



- Before moving on, make sure that students can interpret the entries of a transition matrix to describe how the members of a population move between subgroups over a multiweek period of time.

- The key takeaway from this part of the lesson is for students to understand how the skills they learned in Part 1 relate to exponentiating a transition matrix when computing the quantities in each subgroup within a population over multiple time-steps. Students should also be able to interpret the entries of a transition matrix that describes how members of a population move between subgroups over multiple time-steps.

Meeting Learners' Needs

To help students interpret these values, you can draw a transition diagram like the one shown here that represents how the frogs move over a 2-week period of time.

Students may also benefit from seeing this information expressed as a transition matrix labeled as follows:

$$\begin{array}{c} \\ \mathbf{P} \\ \mathbf{S} \end{array} \begin{array}{c} \begin{array}{cc} \mathbf{P} & \mathbf{S} \end{array} \\ \begin{bmatrix} 0.44 & 0.28 \\ 0.56 & 0.72 \end{bmatrix} \end{array}$$

These row and column labels can be used to identify the entries in the matrix, from column to row, as $\begin{bmatrix} PP & SP \\ PS & SS \end{bmatrix}$.

So, the first entry, PP can be interpreted as 44% of the frogs in the pond remain or end up in the pond. SP can be interpreted as 28% of the frogs in the stream habitat move to the pond habitat. PS means 56% of the frogs in the pond habitat move to the the stream habitat. Lastly, 72% of the frogs in the stream habitat remain in the stream habitat.

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UNIT 4M

Part 3: Stabilizing Systems with Two Subgroups

~50 MIN.

In this part of the lesson, students continue to analyze a scenario in which the frogs move between subgroups within a population. Students focus their exploration on how the sizes of the two subgroups of the population stabilize over time. Students observe that as the numbers of frogs moving between habitats change over time, the number of frogs in each habitat eventually stabilizes.

Student Task

Students advance to Part 3 of Handout 4M.8.A. In this task, students calculate the numbers of frogs in the pond and the stream habitats over a 30-week period. By the end of the task, students observe that the subgroups stabilize at week 15.

Facilitating the Task

- Have students continue to Part 3 of the handout. Give them time to observe the task closely and ask clarifying questions.
- Assign students to work in groups of 4, and then have them count off from 1 to 4 within their groups.
- Next, students should complete problem 7 using their assigned number to determine which set of zoo data they will work with. Make sure students attend to the correct week number, as the week numbers in the first column do not change by equal intervals.

Instructional Rationale

Each student will be working with initial population data from a different zoo to ensure that there are four examples for each group to generalize from.

- After students complete their tables for problem 7, have them complete a notice and wonder chart about the trends they observe in their tables.
- Remind students that each member of their group started with a different number of frogs in each habitat. Have students use their notice and wonder chart to predict what the tables of other students in their group might look like.
- Next, have students compare their notice and wonder charts within their groups. Ask students to examine their group members' tables to check their prediction(s).
- Have each group share out what they discovered by comparing their tables. Students should recognize that the number of frogs in the pond habitat and number of frogs in the stream habitat eventually stabilize at certain sizes and do not change again as the number of weeks increases. Some students may analyze the rates of change of

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the number of frogs in the pond habitat and number of frogs in the habitat stream over time and reason that the change in the size of each subgroup decreases over time. Finally, some students might observe that the number of frogs in the pond habitat is half the number of frogs in the stream habitat.

Instructional Rationale

The subgroup sizes in this part of the handout were selected so that the total number of frogs in each zoo is a multiple of 3. This ensures that the ratio of frogs in the stream habitat to frogs in the pond habitat after each number stabilizes is exactly 2. If the total number of frogs were not a multiple of 3, this ratio would approach, but not equal, 2. It is outside the scope of this course for students to understand the limits of sequences.

Summarizing the Task

- To help students understand this phenomenon, have each group share the number of frogs in each habitat once the number of frogs in each habitat stabilizes for each zoo. Consider asking the following questions:
  - When the number of frogs in each habitat eventually stops changing after a certain time-step, the values are said to *stabilize*. Did every group see the number of frogs in the pond habitat and the number of frogs in the stream habitat stabilize? If so, at what week did the number of frogs stabilize in the pond and stream habitats?

    Yes, the number of frogs in each habitat stabilized at week 15.
  - Do you notice any patterns in all four zoos in the relationship between the number of frogs in the pond habitat and the number of frogs in the stream habitat once the number of frogs in each habitat stabilizes?

    The number of frogs in the pond is half the number of frogs in the stream.
  - Does the initial number of frogs in each habitat influence the number of frogs in each habitat after their values stabilize?

    No, the total number of frogs in each zoo matters, not how these frogs are initially distributed among the habitats. We see this by noticing the data sets for both Student 1 and Student 2 reach the same number of frogs in each habitat.
  - Since the number of frogs in each habitat stabilizes, does this mean that no frogs are moving between the pond and the stream habitats?

    No, the transition matrix still represents the percentage of frogs that move between the pond and the stream habitats.

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- Remind students that the transition diagram represents how the frogs move between the pond habitat and the stream habitat. Saying the population distribution stabilizes means that the size of each subgroup approaches (gets closer and closer to) a constant value over time. We have explored when this happens for populations with two or three subgroups that stabilize over time. As the sizes of subgroups stabilize, the ratios of the sizes of pairs of subgroups in that population also approach constant values. These ratios are independent of size of the population.
- Highlight for students that when each subgroup within the population stabilized, the ratio of the number of frogs in the pond habitat to the number of frogs in the stream habitat was $\frac{1}{2} = 0.5$. This means that for every frog in the pond habitat, there are two frogs in the stream habitat. Give students a chance to compare their ratio to those of other members of their group and analyze the sizes of each subgroup at every zoo. This will show students that the ratio is independent of the size of the population.
- Have students return to problem 7 on the handout. Ask them to determine how many frogs in the pond habitat move to the stream habitat and how many frogs in the stream habitat move to the pond habitat after each time-step of the stabilized system. For example, Student 1 will see that 68 frogs move from the pond habitat to the stream habitat (40% of 170) and 68 frogs move from the stream habitat to the pond habitat (20% of 340). This demonstrates the stability of the system: although there are still frogs moving between the habitats, the numbers moving between habitats are the same, so the total number of frogs in each habitat remains the same.
- For additional practice, have students complete the practice problems on **Handout 4M.8.B: Exploring How Sizes of Subgroups of a Population Stabilize Over Time**. These problems are meant to further develop students' understanding of how to model

Meeting Learners' Needs

For students who are ready for a challenge, have them demonstrate algebraically that the number of frogs that move from the pond habitat to the stream habitat is precisely the same as the number of frogs that move from the stream habitat to the pond habitat. In other words, they show that when the number of frogs in each habitat has stabilized, 20% of the number of frogs in the steam habitat is equivalent to 40% of the number of frogs in the pond habitat. The zoo data for Student 1 can generate the following equations: $0.2s_k = 0.4p_k$ and $p_k = 510 - s_k$. Using substitution to solve for one variable, this can be expressed as $0.2s_k = 0.4(500 - s_k)$. Solving for s_k yields $s_k = 340$, which means there are 340 frogs in the stream habitat when the number of frogs in each habitat stabilizes.

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the movement of members between subgroups of a population using transition diagrams. Students also practice determining the future sizes of subgroups of a population over k-unit time-steps and continue to interpret the entries of a transition matrix. Students also demonstrate their understanding of how subgroups of a population stabilize over time.

- The key takeaway for students for this part of the lesson is understanding how to use a transition matrix to describe when the movement of members of subgroups of a population stabilizes over time. After students complete more practice problems, help them generalize that when a population is split into subgroups, the size of each subgroup will stabilize over time. When there are two subgroups, the ratio of the size of each subgroup approaches a constant value.

Instructional Rationale

The transition matrices in this course are all designed so that the systems stabilize. These transition matrices have columns whose entries sum to 1.

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Assess and Reflect on the Lesson

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

An invasive species of beetle made its way to a remote island in the middle of the Pacific. The beetles feast on the fast-growing palm trees on the island. Each month, the beetles consume 40% of the island's healthy palm trees, and only 25% of the unhealthy palm trees recover and become healthy trees again.

After any 3-week period, what percentage of palm trees that started the 3-week period healthy are now unhealthy?

To determine what percentage of healthy palm trees become unhealthy each week, we must determine the matrix that describes the percentages of healthy and unhealthy palm trees whose states remain the same (healthy trees that stay healthy, and unhealthy trees that stay unhealthy) or change (from healthy to unhealthy, or from unhealthy to healthy). The transition matrix is shown below.

To observe these trends over 3 weeks (3 time-units), compute the transition matrix $\begin{bmatrix} 0.6 & 0.25 \\ 0.4 & 0.75 \end{bmatrix}$ to the third power:

$$\begin{bmatrix} 0.6 & 0.25 \\ 0.4 & 0.75 \end{bmatrix}^3 \approx \begin{bmatrix} 0.411 & 0.368125 \\ 0.589 & 0.631875 \end{bmatrix}.$$

Each entry in this matrix describes how the trees change (or do not change) between being healthy and unhealthy. The entry in the second row and first column gives the percentage of healthy trees that become unhealthy. Thus 58.9% of plants that start the 3-week period healthy will end up unhealthy.

HANDOUTS

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 4M.8.A: Determining the Sizes of Subgroups of a Population

Part 1

1. (a) After 1 week, there are 35 frogs in the pond habitat and 90 frogs in the stream habitat.

   (b) After 2 weeks, there are 39 frogs in the pond habitat and 86 frogs in the stream habitat.

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(c) After 3 weeks, there are about 41 frogs in the pond and about 84 frogs in the stream habitat.

(d) Students are likely to get frustrated before determining an answer. This is intended.

2. (a) After 1 week, there are 35 frogs in the pond habitat and 90 frogs in the stream habitat.

(b) After 2 weeks, there are 39 frogs in the pond habitat and 86 frogs in the stream habitat.

(c) After 3 weeks, there are about 41 frogs in the pond and about 84 frogs in the stream habitat.

(d) After 52 weeks, there are about 42 frogs in the pond and about 83 frogs in the stream habitat.

3. $\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}^k \begin{bmatrix} p_0 \\ s_0 \end{bmatrix} = \begin{bmatrix} p_k \\ s_k \end{bmatrix}$

Part 2

4.

Week Number	Number of Frogs in the Pond Habitat	Number of Frogs in the Stream Habitat	Computation
0	125	0	These values are given.
2	55	70	$\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}^2 \begin{bmatrix} 125 \\ 0 \end{bmatrix}$
4	about 44	about 81	$\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}^4 \begin{bmatrix} 125 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}^2 \begin{bmatrix} 55 \\ 70 \end{bmatrix}$
6	about 42	about 83	$\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}^6 \begin{bmatrix} 125 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}^2 \begin{bmatrix} 44 \\ 81 \end{bmatrix}$

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5. Over each 2-week period, 44% of the frogs remain or end up in the pond habitat. This is found by exponentiating the matrix $\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}$ by 2 giving $\begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{bmatrix}^2 = \begin{bmatrix} 0.44 & 0.28 \\ 0.56 & 0.72 \end{bmatrix}$. Because $\begin{bmatrix} 0.44 & 0.28 \\ 0.56 & 0.72 \end{bmatrix}\begin{bmatrix} 300 \\ 0 \end{bmatrix} = \begin{bmatrix} 132 \\ 168 \end{bmatrix}$, 132 of these frogs are in the pond habitat.

6. This will never happen because the number of frogs in each habitat stabilizes after 16 weeks.

Part 3

7. The solutions for Student 1 are shown in the following table.

Week Number	Number of Frogs in the Pond	Number of Frogs in the Stream
0	320	190
1	230	280
5	171.536	338.464
10	170.016	339.984
15	170	340
20	170	340
30	170	340

The solutions for Student 2 are shown in the following table.

Week Number	Number of Frogs in the Pond	Number of Frogs in the Stream
0	400	200
1	280	320
5	202.048	397.952
10	200.021	399.979
15	200	400
20	200	400
30	200	400

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The solutions for Student 3 are shown in the following table.

Week Number	Number of Frogs in the Pond	Number of Frogs in the Stream
0	110	100
1	86	124
5	70.4096	139.59
10	70.0042	139.996
15	70	140
20	70	140
30	70	140

The solutions for Student 4 are shown in the following table.

Week Number	Number of Frogs in the Pond	Number of Frogs in the Stream
0	190	320
1	178	332
5	170.205	339.795
10	170.002	339.998
15	170	340
20	170	340
30	170	340

Handout 4M.8.B: Exploring How Sizes of Subgroups of a Population Stabilize Over Time

1. (a) Approximately 248 shoppers at Store A and 132 shoppers at Store B. One way of determining this answer is to compute $\begin{bmatrix} 0.6 & 0.75 \\ 0.4 & 0.25 \end{bmatrix}^4 \begin{bmatrix} 200 \\ 180 \end{bmatrix} \approx \begin{bmatrix} 248 \\ 132 \end{bmatrix}$. An equivalent method consists of the following steps: First, compute the product of the transition matrix and the initial number of shoppers at each store to determine the number of shoppers at each store after 1 week. Next, compute the product of the transition matrix and the numbers of shoppers at each store after 1 week to determine the number of shoppers at each store after 2 weeks. Then continue this process twice more to determine the numbers of shoppers at each store after 3 weeks and then 4 weeks.

   (b) There are 248 shoppers are at Store A and 132 shoppers at Store B.

   (c) The ratio of the shoppers at Store A to shoppers at Store B is $\frac{248}{132} \approx 1.88$.

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2. (a) A–D are correct.
   (b) When the system stabilizes, 314 fish are awake and 86 are asleep.
   (c) The ratio of awake fish to sleeping fish after the size of each subgroup stabilizes is about 3.64.
   (d) No, if there are 320 sleeping fish then there are 380 awake fish, giving a ratio of $380/320 = 1.1875$, which is not equal to 3.65.
3. (a) After 9 miles, the number of walkers stabilize to be approximately 167 walkers. This is observed in the matrix equation $\begin{bmatrix} 0.8 & 1 \\ 0.2 & 0 \end{bmatrix}^9 \begin{bmatrix} 1000 \\ 0 \end{bmatrix} \approx \begin{bmatrix} 833.333 \\ 166.667 \end{bmatrix}$.
   (b) The ratio of the number of runners to the number of walkers when the size of each group of participants stabilizes is $\frac{833}{167} \approx 5$.
   (c) The ratio of runners to walkers after the size of each subgroup stabilizes is approximately 5 runners to 1 walker. This gives $\frac{200}{x} \approx 5$, which means there are approximately 40 walkers and about 240 total participants.
4. Yes, the system stabilizes once all the food is eaten. Students might need support to understand that this counts as a stable system even when two rows have entries that are all zero (i.e., entries of 1 remain in only one category).

Guiding Student Thinking

To support students' understanding that the system stabilizes once the food is all eaten, present the transition matrix on a whiteboard and label the columns and rows for their current and future states over one time-step. In the matrix that follows, S represents food that is spoiled, E represents food that is eaten, and N represents nonspoiled food. Students need to exponentiate the transition matrix to increasing powers to observe if the system stabilizes. This means that the entry once the spoiled food is eaten is 1. Also, the entry once nonspoiled food is eaten is 1. After about 160 days, the 3% of the food that was initially spoiled is now eaten and the 15% of the food that was nonspoiled is now eaten.

$$\begin{array}{c} \\ \mathbf{S} \\ \mathbf{E} \\ \mathbf{N} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{S} & \mathbf{E} & \mathbf{N} \end{array} \\ \begin{bmatrix} 0.97 & 0 & 0.05 \\ 0.03 & 1 & 0.15 \\ 0 & 0 & 0.80 \end{bmatrix} \end{array}$$

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Unit 4M

Performance Task

PERFORMANCE TASK

Migrating Populations

LEARNING OBJECTIVES

4M.3.1 Express a recursive process that models the relationship between two or three related subgroups of a population using multiple representations.

4M.3.2 Determine the future sizes of subgroups of a population given the current size of each subgroup.

4M.3.3 Identify how the sizes of related subgroups of a population stabilize over time.

4M.3.4 Determine the previous sizes of subgroups of a population given the current size of each subgroup.

PERFORMANCE TASK DESCRIPTION

In this performance task, students explore a context in which they analyze how members of subgroups move within a population. Students are given a transition diagram and the initial size of each subgroup, which can be used to set up a matrix equation. Students use what they know about transition matrices and matrix multiplication to determine future and previous sizes of the subgroups of a population. Students also use technology to determine when the subgroups within the population stabilize.

AREA OF FOCUS

- Connections Among Multiple Representations

SUGGESTED TIMING

~60 minutes

MATERIALS

- matrix calculator, such as **Desmos.com/matrix**

HANDOUT

- Unit 4M Performance Task: Migrating Populations

AP Connections

This performance task supports AP preparation through alignment to the following AP Calculus Course Skills:

- **1.E** Apply appropriate mathematical rules or procedures, with and without technology.
- **2.E** Identify common underlying structures in problems involving different contextual situations.
- **3.F** Explain the meaning of mathematical solutions in context.

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UNIT 4M

ELICITING PRIOR KNOWLEDGE

The goal of this performance task is for students to demonstrate their understanding of the use of transition matrices to solve application problems. This performance task expects students to model recursive processes using a matrix, perform matrix multiplication using technology, interpret the transition matrix at any time-step, and use the inverse of the transformation matrix to identify previous sizes of subgroups within a population. Students are also expected to use mathematical notation appropriately to convey the solutions.

- To begin, introduce students to the performance task. The student handout for the task is shown before the scoring guidelines for reference.
- To prepare students to engage in the task, you could ask questions like the following:
  - The following matrix describes the observed trends in how customers choose to shop at either Store 1, Store 2, or Store 3 each week. Assuming that past trends continue, predict the percentage of people who currently shop at Store 3 that will decide to shop in Store 2 next week.

	Store 1	**Store 2**	**Store 3**
Store 1	0.90	0.06	0.02
Store 2	0.10	0.91	0.05
Store 3	0	0.04	0.93

    Based on past trends, 5% of the customers who normally shop in Store 3 will decide to shop in Store 2 next week.

  - Suppose you want to find the number of customers shopping in each store 10 weeks from now. If the numbers of customers shopping at Store 1, Store 2, and Store 3 are represented by x_0, y_0, and z_0, respectively, set up a matrix equation that you could use to determine the number of customers in each store that week.

$$\begin{bmatrix} 0.90 & 0.06 & 0.02 \\ 0.10 & 0.91 & 0.05 \\ 0 & 0.04 & 0.93 \end{bmatrix}^{10} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} x_{10} \\ y_{10} \\ z_{10} \end{bmatrix}$$

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- If students struggle with the warm-up problems, it could indicate that they are not yet fully prepared to engage in the performance task. You may find it beneficial to provide a just-in-time review of the concepts critical for success with this task: expressing a transition diagram as a matrix, interpreting a transition matrix, finding the inverse of a matrix, and using matrix multiplication to solve application problems.

SUPPORTING STUDENTS

This is the last performance task of the course. At this point, students have gained some experience with completing performance tasks but still may need some support to progress through this task. The following are a few possible implementation strategies you can use to help students engage with the task. You should rely on your knowledge of your students and your professional expertise to determine how to provide appropriate scaffolds while maintaining the cognitive demand of the task.

- **Collaboration:** This performance task is an excellent opportunity for students to work with a partner. Students are still learning to solve application problems that involve matrix multiplication and will benefit from academic conversations about how to complete the task.
- **Using Technology:** Students will need to use an online matrix calculator, such as Desmos, to find the inverse of a transformation matrix and perform matrix multiplication for parts (b) through (d). Students will be handling 3×3 matrices in this task and have been able to do this only with technology. Without technology, students will be unable to complete this task.

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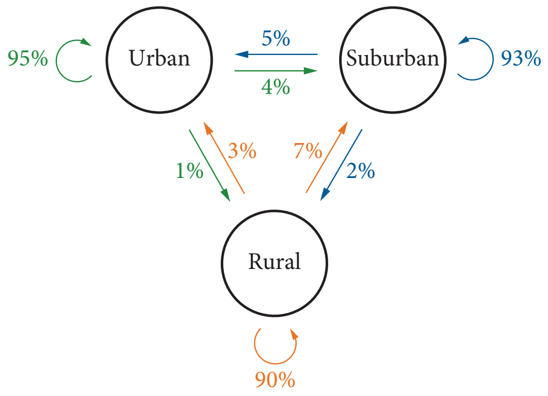
UNIT 4M

SCORING STUDENT WORK

Because this is a performance task and not a practice performance task, it is recommended that you score your students' work, rather than have each student score a classmate's work. Be sure to use the results of the performance task to identify patterns and trends that can inform further instruction about using transition matrices to analyze how members of subgroups move within a population.

Migrating Populations

The annual migration patterns between urban, suburban, and rural areas within a region are shown below. In the year 2021, the urban population, u_0, was 800,000, the suburban population, s_0, was 500,000, and the rural population, r_0, was 40,000.



There are 12 possible points for this performance task.

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Student Stimulus and Part (a)

(a) Write the transition matrix, T, that describes how each member in the region moves between populations and the column matrix, P, to describe the initial sizes of each population.

Sample Solution	Points Possible
The transition matrix that describes how each member in the region moves between populations is $T = \begin{bmatrix} 0.95 & 0.05 & 0.03 \\ 0.04 & 0.93 & 0.07 \\ 0.01 & 0.02 & 0.90 \end{bmatrix}$, and the column matrix is $P = \begin{bmatrix} 800{,}000 \\ 500{,}000 \\ 40{,}000 \end{bmatrix}$.	**2 points maximum** 1 point for expressing the movement of members in the region between the three populations with a 3×3 matrix 1 point for expressing the initial size of each population with a 3×1 matrix

Student Stimulus and Part (b)

(b) What will be the size of each population in 2022? Show the work that leads to your solution.

Sample Solution	Points Possible
The matrix equation to determine the size of each population in 2022 is $\begin{bmatrix} 0.95 & 0.05 & 0.03 \\ 0.04 & 0.93 & 0.07 \\ 0.01 & 0.02 & 0.90 \end{bmatrix} \begin{bmatrix} 800{,}000 \\ 500{,}000 \\ 40{,}000 \end{bmatrix} = \begin{bmatrix} u_1 \\ s_1 \\ r_1 \end{bmatrix}$. This yields $\begin{bmatrix} u_1 \\ s_1 \\ r_1 \end{bmatrix} = \begin{bmatrix} 786{,}200 \\ 499{,}800 \\ 54{,}000 \end{bmatrix}$. In 2022, the urban population was 786,200, the suburban population was 499,800, and the rural population was 54,000.	**3 points maximum** 1 point for expressing the given information as a matrix equation 1 point for determining the size of each population 1 point for interpreting the values of the column matrix as the sizes of the populations in the region

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UNIT 4M

Student Stimulus and Part (c)

(c) Determine the percentage of individuals who will have moved from an urban area to a suburban area 10 years from now. Explain how you arrived at your answer.

Sample Solution	Points Possible
To find the percentage of individuals who moved from an urban area to a suburban area 10 years from now, we need to raise the one-year transition matrix to the 10th power and then interpret the entry in the resulting matrix that describes movement from an urban area to a suburban area. $$\begin{bmatrix} 0.95 & 0.05 & 0.03 \\ 0.04 & 0.93 & 0.07 \\ 0.01 & 0.02 & 0.90 \end{bmatrix}^{10} \approx \begin{bmatrix} 0.67 & 0.32 & 0.25 \\ 0.26 & 0.58 & 0.36 \\ 0.07 & 0.11 & 0.39 \end{bmatrix}$$ Twenty-six percent of the individuals in the urban area will have moved to the suburban area 10 years from now.	**2 points maximum** 1 point for writing the transition matrix that describes the population after 10 years *Scoring note*: The student does not have to show the simplified matrix to receive the first point. 1 point for interpreting the transition matrix taken to the 10th power to determine the percentage of individuals who moved from an urban area to a suburban area

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Student Stimulus and Part (d)

(d) How many years will it take for the sizes of the populations in this region to stabilize? Explain how you determined that the sizes of the populations were stable.

Sample Solution	Points Possible
Using my calculator, I found that the sizes of the three populations in the region stabilize after 124 years. From 124 years and on, the sizes of the three populations do not change. *Scoring note*: Students should come within four years of the population stabilizing in 124 years. That is, any answer from 120 years to 128 years is acceptable.	**2 points maximum** 1 point for determining a reasonable year at which the populations stabilize 1 point for stating the process for determining when the populations stabilized

Student Stimulus and Part (e)

(e) Assuming that the migration trends observed of the populations have been continuing at the same rate, what was the size of each population in 2020? Explain how you arrived at your solution.

Sample Solution	Points Possible
I have to first find the inverse of the transition matrix, T^{-1}, and then multiply it by the matrix representing the initial sizes of the populations, P. $T^{-1} \cdot P = \begin{bmatrix} 814{,}984 \\ 500{,}755 \\ 24{,}261 \end{bmatrix}$ In 2020, the urban population was 814,984, the suburban population was 500,755, and the rural population was 24,261.	**3 points maximum** 1 point for explaining how to arrive at the solution 1 point for writing the matrix equation *Scoring note*: Students do not need to show the inverse of the matrix in the explanation of their work. 1 point for interpreting the solutions in context

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UNIT 4M

PROVIDING FEEDBACK ON STUDENT WORK

After scoring your students' work, it is important to identify trends in their responses to inform further instruction. These trends should include topics that students consistently displayed mastery of, as well as conceptual errors that students commonly made. Possible trends and suggested guidance for each part of the task follow, although the patterns you observe in your classroom may differ.

(a) If students struggle to express the transition diagram as a matrix, they may need additional instruction on connecting the numeric information in a transition diagram to the entries within the transition matrix. Labeling the columns and rows of the matrix with the variable representing each population helps provide a visual when expressing the values from the diagram in matrix form. You may also want to show the corresponding equations that represent this scenario and use the matrix to remind students how its entries relate to the transition diagram.

$$\begin{aligned} 0.95u_0 + 0.05s_0 + 0.03r_0 &= u_1 \\ 0.04u_0 + 0.93s_0 + 0.07r_0 &= s_1 \\ 0.01u_0 + 0.02s_0 + 0.90r_0 &= r_1 \end{aligned}$$

$$\begin{array}{c} \\ \mathbf{U} \\ \mathbf{S} \\ \mathbf{R} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{U} & \mathbf{S} & \mathbf{R} \end{array} \\ \begin{bmatrix} 0.95 & 0.05 & 0.03 \\ 0.04 & 0.93 & 0.07 \\ 0.01 & 0.02 & 0.90 \end{bmatrix} \end{array} \qquad \begin{bmatrix} UU & SU & RU \\ US & SS & RS \\ UR & SR & RR \end{bmatrix}$$

Teacher Notes and Reflections

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(b) Students may struggle to set up the matrix equation to determine the size of each subgroup of a population. Model for students how to set up the transition matrix and the column matrix representing the initial amounts before applying matrix multiplication. You can also reiterate the form of a matrix equation so that students can recall how to assign values to each entry of the matrix.

Teacher Notes and Reflections

(c) Students may struggle to recall that raising a transition matrix to a power is equivalent to describing how members of a subgroup move within a population over the number of time-steps equal to that power. Students may also need to be reminded that in this case we are not interested in the sizes of each subgroup at 10 years, just the numeric description of their movement between subgroups.

Teacher Notes and Reflections

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(d) Since this problem should be solved with technology, students may need additional practice exploring how to change the exponent for a matrix when using Desmos to determine when the sizes of each subgroup stabilize.

Teacher Notes and Reflections

(e) Since students are finding the sizes of the subgroups in the previous year—or one time-step back—they may need to be reminded that they must invert the transition matrix, or equivalently, raise the transition matrix to the exponent -1, and then multiply it by the column matrix that represents the existing sizes of the subgroups.

Teacher Notes and Reflections

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Try to assure students that converting their score into a percentage does not provide an accurate measure of how they performed on the task. You can use the following suggested score interpretations with students to discuss their performance.

Points Received	How Students Should Interpret Their Score
11 or 12 points	"I know all of these algebraic concepts really well. This is top-level work. (A)"
8 or 10 points	"I know all of these algebraic concepts well, but I made a few mistakes. This is above-average work. (B)"
5 or 7 points	"I know some of these algebraic concepts well, but not all of them. This is average-level work. (C)"
3 or 4 points	"I know only a little bit about these algebraic concepts. This is below-average work. (D)"
0 or 2 points	"I don't know much about these algebraic concepts at all. This is not passing work. (F)"

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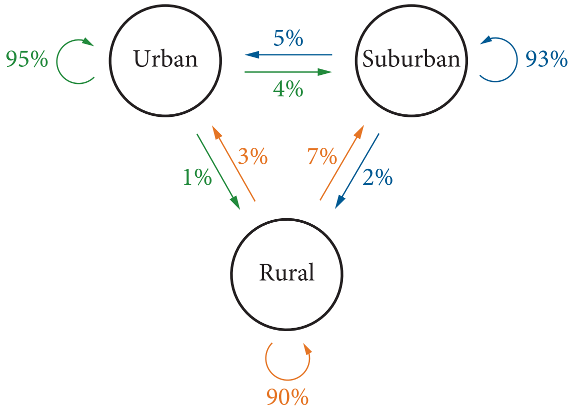
PERFORMANCE TASK

Migrating Populations

The annual migration patterns between urban, suburban, and rural areas within a region are shown below. In the year 2021, the urban population, u_0, was 800,000, the suburban population, s_0, was 500,000, and the rural population, r_0, was 40,000.



(a) Write the transition matrix, T, that describes how each member in the region moves between populations and the column matrix, P, to describe the initial sizes of each population.

(b) What will be the size of each population in 2022? Show the work that leads to your solution.

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(a) Determine the percentage of individuals who will have moved from an urban area to a suburban area 10 years from now. Explain how you arrived at your answer.

(b) How many years will it take for the sizes of the populations in this region to stabilize? Explain how you determined that the sizes of the populations were stable.

(c) Assuming that the migration trends observed of the populations have been continuing at the same rate, what was the size of each population in 2020? Explain how you arrived at your solution.