CollegeBoard Pre-AP®

Pre-AP® Algebra 2

TEACHER RESOURCES

Units 3 and 4T

ABOUT COLLEGE BOARD

College Board is a mission-driven not-for-profit organization that connects students to college success and opportunity. Founded in 1900, College Board was created to expand access to higher education. Today, the membership association is made up of over 6,000 of the world's leading educational institutions and is dedicated to promoting excellence and equity in education. Each year, College Board helps more than seven million students prepare for a successful transition to college through programs and services in college readiness and college success—including the SAT® and the Advanced Placement Program®. The organization also serves the education community through research and advocacy on behalf of students, educators, and schools.

For further information, visit **www.collegeboard.org**.

PRE-AP EQUITY AND ACCESS POLICY

College Board believes that all students deserve engaging, relevant, and challenging grade-level coursework. Access to this type of coursework increases opportunities for all students, including groups that have been traditionally underrepresented in AP and college classrooms. Therefore, the Pre-AP program is dedicated to collaborating with educators across the country to ensure all students have the supports to succeed in appropriately challenging classroom experiences that allow students to learn and grow. It is only through a sustained commitment to equitable preparation, access, and support that true excellence can be achieved for all students, and the Pre-AP course designation requires this commitment.

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The sentence-writing strategies used in Pre-AP lessons are based upon The Writing Revolution, Inc., a national nonprofit organization that trains educators to implement The Hochman Method, an evidence-based approach to teaching writing. The strategies included in Pre-AP materials are meant to support students' writing, critical thinking, and content understanding, but they do not represent The Writing Revolution's full, comprehensive approach to teaching writing. More information can be found at **www.thewritingrevolution.org**.

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Introduction to Pre-AP Algebra 2

About Pre-AP

Introduction to Pre-AP

Every student deserves classroom opportunities to learn, grow, and succeed. College Board developed Pre-AP® to deliver on this simple premise. Pre-AP courses are designed to support all students across varying levels of readiness. They are not honors or advanced courses.

Participation in Pre-AP courses allows students to slow down and focus on the most essential and relevant concepts and skills. Students have frequent opportunities to engage deeply with texts, sources, and data as well as compelling higher-order questions and problems. Across Pre-AP courses, students experience shared instructional practices and routines that help them develop and strengthen the important critical thinking skills they will need to employ in high school, college, and life. Students and teachers can see progress and opportunities for growth through varied classroom assessments that provide clear and meaningful feedback at key checkpoints throughout each course.

DEVELOPING THE PRE-AP COURSES

Pre-AP courses are carefully developed in partnership with experienced educators, including middle school, high school, and college faculty. Pre-AP educator committees work closely with College Board to ensure that the course resources define, illustrate, and measure grade-level-appropriate learning in a clear, accessible, and engaging way. College Board also gathers feedback from a variety of stakeholders, including Pre-AP partner schools from across the nation who have participated in multiyear pilots of select courses. Data and feedback from partner schools, educator committees, and advisory panels are carefully considered to ensure that Pre-AP courses provide all students with grade-level-appropriate learning experiences that place them on a path to college and career readiness.

PRE-AP EDUCATOR NETWORK

Similar to the way in which teachers of Advanced Placement® (AP®) courses can become more deeply involved in the program by becoming AP Readers or workshop consultants, Pre-AP teachers also have opportunities to become active in their educator network. Each year, College Board expands and strengthens the Pre-AP National Faculty—the team of educators who facilitate Pre-AP Readiness Workshops and Pre-AP Summer Institutes. Pre-AP teachers can also become curriculum and assessment contributors by working with College Board to design, review, or pilot the course resources.

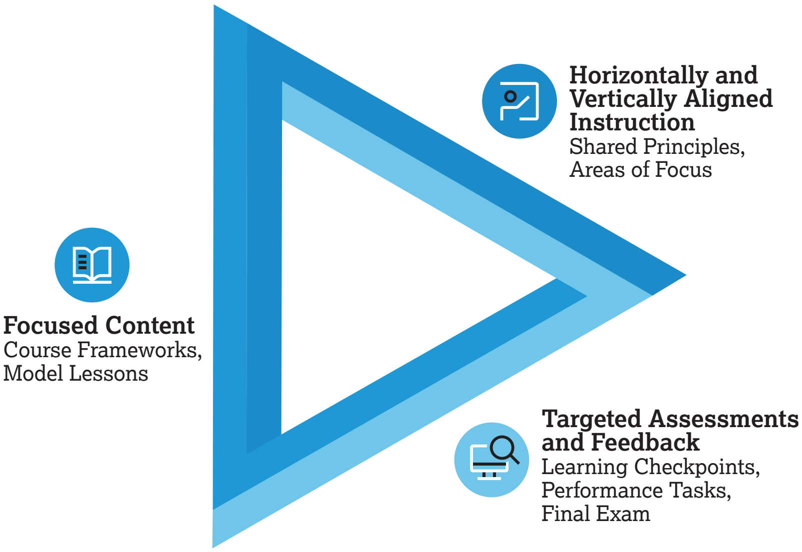
HOW TO GET INVOLVED

Schools and districts interested in learning more about participating in Pre-AP should visit **preap.org/join** or contact us at **preap@collegeboard.org**.

Teachers interested in becoming members of Pre-AP National Faculty or participating in content development should visit **preap.org/national-faculty** or contact us at **preap@collegeboard.org**.

Pre-AP Approach to Teaching and Learning

Pre-AP courses invite all students to learn, grow, and succeed through focused content, horizontally and vertically aligned instruction, and targeted assessments for learning. The Pre-AP approach to teaching and learning, as described below, is not overly complex, yet the combined strength results in powerful and lasting benefits for both teachers and students. This is our theory of action.



FOCUSED CONTENT

Pre-AP courses focus deeply on a limited number of concepts and skills with the broadest relevance for high school coursework and college and career success. The course framework serves as the foundation of the course and defines these prioritized concepts and skills. Pre-AP model lessons and assessments are based directly on this focused framework. The course design provides students and teachers with intentional permission to slow down and focus.

HORIZONTALLY AND VERTICALLY ALIGNED INSTRUCTION

Shared principles cut across all Pre-AP courses and disciplines. Each course is also aligned to discipline-specific areas of focus that prioritize the critical reasoning skills and practices central to that discipline.

SHARED PRINCIPLES

All Pre-AP courses share the following set of research-supported instructional principles. Classrooms that regularly focus on these cross-disciplinary principles allow students to effectively extend their content knowledge while strengthening their critical thinking skills. When students are enrolled in multiple Pre-AP courses, the horizontal alignment of the shared principles provides students and teachers across disciplines with a shared language for their learning and investigation, and multiple opportunities to practice and grow. The critical reasoning and problem-solving tools students develop through these shared principles are highly valued in college coursework and in the workplace.

Close Observation and Analysis

Students are provided time to carefully observe one data set, text image, performance piece, or problem before being asked to explain, analyze, or evaluate. This creates a safe entry point to simply express what they notice and what they wonder. It also encourages students to slow down and capture relevant details with intentionality to support more meaningful analysis, rather than rush to completion at the expense of understanding.

Higher-Order Questioning

Students engage with questions designed to encourage thinking that is elevated beyond simple memorization and recall. Higher-order questions require students to make predictions, synthesize, evaluate, and compare. As students grapple with these questions, they learn that being inquisitive promotes extended thinking and leads to deeper understanding.

Evidence-Based Writing

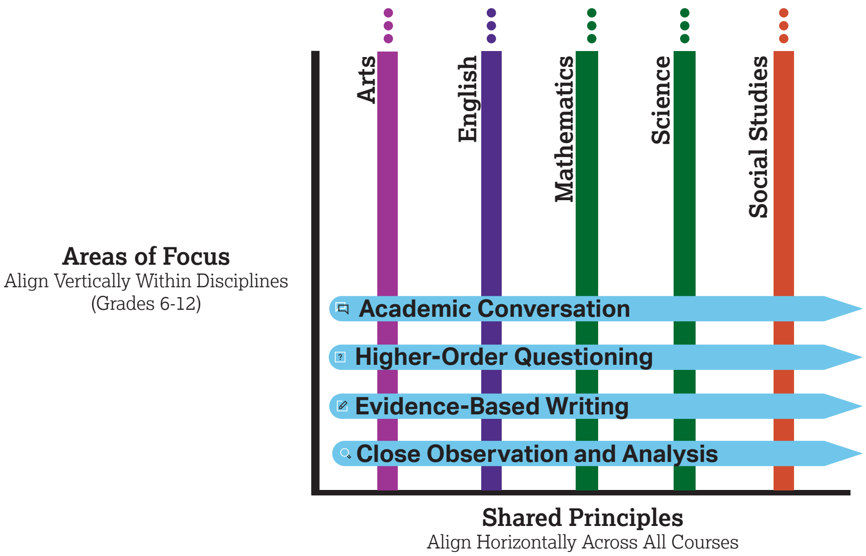
With strategic support, students frequently engage in writing coherent arguments from relevant and valid sources of evidence. Pre-AP courses embrace a purposeful and scaffolded approach to writing that begins with a focus on precise and effective sentences before progressing to longer forms of writing.

Academic Conversation

Through peer-to-peer dialogue, students' ideas are explored, challenged, and refined. As students engage in academic conversation, they come to see the value in being open to new ideas and modifying their own ideas based on new information. Students grow as they frequently practice this type of respectful dialogue and critique and learn to recognize that all voices, including their own, deserve to be heard.

AREAS OF FOCUS

The areas of focus are discipline-specific reasoning skills that students develop and leverage as they engage with content. Whereas the shared principles promote horizontal alignment across disciplines, the areas of focus provide vertical alignment within a discipline, giving students the opportunity to strengthen and deepen their work with these skills in subsequent courses in the same discipline.



For information about the Pre-AP mathematics areas of focus, see page 15.

TARGETED ASSESSMENTS FOR LEARNING

Pre-AP courses include strategically designed classroom assessments that serve as tools for understanding progress and identifying areas that need more support. The assessments provide frequent and meaningful feedback for both teachers and students across each unit of the course and for the course as a whole. For more information about assessments in Pre-AP Algebra 2, see page 49.

Pre-AP Professional Learning

Pre-AP teachers are required to engage in two professional learning opportunities. The first requirement is designed to help prepare them to teach their specific course. There are two options to meet the first requirement: the Pre-AP Summer Institute (Pre-APSI) and the Online Foundational Module Series. Both options provide continuing education units to educators who complete them.

- The Pre-AP Summer Institute is a four-day collaborative experience that empowers participants to prepare and plan for their Pre-AP course. While attending, teachers engage with Pre-AP course frameworks, shared principles, areas of focus, and sample model lessons. Participants are given supportive planning time where they work with peers to begin to build their Pre-AP course plan.
- The Online Foundational Module Series is available to all teachers of Pre-AP courses. This 12- to 20-hour course supports teachers in preparing for their Pre-AP course. Teachers explore course materials and experience model lessons from the student's point of view. They also begin to plan and build their own course so they are ready on day one of instruction.

The second professional learning requirement is to complete at least one of the Online Performance Task Scoring Modules, which offer guidance and practice applying Pre-AP scoring guidelines to student work.

About the Course

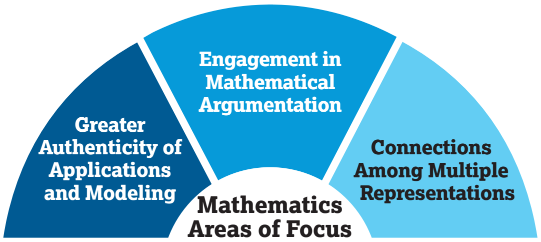
Introduction to Pre-AP Algebra 2

Pre-AP Algebra 2 is designed to optimize students' readiness for college-level mathematics classes. Rather than seeking to cover all topics traditionally included in a standard second-year algebra textbook, this course extends the conceptual understanding of and procedural fluency with functions and data analysis that students developed in their previous mathematics courses. It offers an approach that concentrates on the mathematical content and skills that matter most for college readiness. This approach creates more equitable opportunities for students to take AP STEM courses, especially for those students who are underrepresented in STEM courses and careers. The Pre-AP Algebra 2 Course Framework highlights how to guide students to connect core ideas within and across the units of the course, promoting a coherent understanding of functions.

The components of this course have been crafted to prepare not only the next generation of mathematicians, scientists, programmers, statisticians, and engineers, but also a broader base of mathematically informed citizens who are well equipped to respond to the array of mathematics-related issues that impact our lives at the personal, local, and global levels.

PRE-AP MATHEMATICS AREAS OF FOCUS

The Pre-AP mathematics areas of focus, shown below, are mathematical practices that students develop and leverage as they engage with content. They were identified through educator feedback and research about where students and teachers need the most curriculum support. These areas of focus are vertically aligned to the mathematical practices embedded in other mathematics courses in high school, including AP, and in college, giving students multiple opportunities to strengthen and deepen their work with these skills throughout their educational career. They also support and align to the AP Calculus Mathematical Practices, the AP Statistics Course Skills, and the mathematical practices listed in various state standards.



Greater Authenticity of Applications and Modeling

Students create and use mathematical models to understand and explain authentic scenarios.

Mathematical modeling is a process that helps people analyze and explain the world. In Pre-AP Algebra 2, students explore real-world contexts where mathematics can be used to make sense of a situation. They engage in the modeling process by making choices about what function to use to construct a model, assessing how well the model represents the available data, refining their model as needed, drawing conclusions from their model, and justifying decisions they make through the process.

In addition to mathematical modeling, students engage in mathematics through authentic applications. Applications are similar to modeling problems in that they are drawn from real-world phenomena, but they differ because the applications dictate the appropriate mathematics to use to solve the problem. Pre-AP Algebra 2 balances these two types of real-world tasks.

Engagement in Mathematical Argumentation

Students use evidence to craft mathematical conjectures and prove or disprove them.

Conjecture, reasoning, and proof lie at the heart of the discipline of mathematics. Mathematics is both a way of thinking and a set of tools for solving problems. Pre-AP Algebra 2 students gain proficiency in constructing arguments with definitions of mathematical concepts, reasoning to solve equations, developing skills in using algebra to make sense of data, and crafting assertions using data as evidence and support. Through mathematical argumentation, students learn how to be critical of their own reasoning and the reasoning of others.

Connections Among Multiple Representations

Students represent mathematical concepts in a variety of forms and move fluently among the forms.

Pre-AP Algebra 2 students explore how to weave together multiple representations of function concepts. Every mathematical representation illuminates certain characteristics of a concept while also obscuring other aspects. Throughout the course, students continue to represent mathematical concepts using a variety of forms, allowing them to develop a nuanced understanding of which representations best serve a particular purpose.

PRE-AP ALGEBRA 2 AND CAREER READINESS

The Pre-AP Algebra 2 course resources are designed to expose students to a wide range of career opportunities that depend on algebraic knowledge and skills. Examples include not only field-specific specialty careers such as mathematician and statistician, but also other endeavors where algebraic knowledge is relevant, such as accounting, economics, engineering, and programming.

Career clusters that involve algebra, along with examples of careers in mathematics or related to mathematics, are provided below. Teachers should consider discussing these with students throughout the year to promote motivation and engagement.

Career Clusters Involving Mathematics	
architecture and construction arts, A/V technology, and communications business management and administration finance government and public administration health science information technology marketing STEM (science, technology, engineering, and math) transportation, distribution, and logistics	
Examples of Mathematics Related Careers	**Examples of Algebra 2 Related Careers**
actuary financial analyst mathematician mathematics teacher professor programmer statistician	accountant computer programmer economist electrician engineer health science technician operations research analyst

Source for Career Clusters: "Advanced Placement and Career and Technical Education: Working Together." Advance CTE and the College Board. October 2018. **https://careertech.org/resource/ap-cte-working-together**.

For more information about careers that involve mathematics, teachers and students can visit and explore the College Board's Big Future resources: **https://bigfuture.collegeboard.org/majors/math-statistics-mathematics**.

SUMMARY OF RESOURCES AND SUPPORTS

Teachers are strongly encouraged to take advantage of the full set of resources and supports for Pre-AP Algebra 2, which is summarized below. Some of these resources must be used for a course to receive the Pre-AP Course Designation. To learn more about the requirements for course designation, see details below and on page 70.

COURSE FRAMEWORK

Included in this guide as well as in the *Pre-AP Algebra 2 Teacher Resources*, the course framework defines what students should know and be able to do by the end of the course. It serves as an anchor for model lessons and assessments, and it is the primary document teachers can use to align instruction to course content. **Use of the course framework is required**. *For more details see page 22.*

MODEL LESSONS

Teacher resources, available in print and online, include a robust set of model lessons that demonstrate how to translate the course framework, shared principles, and areas of focus into daily instruction. **Use of the model lessons is encouraged but not required**. *For more details see page 47.*

LEARNING CHECKPOINTS

Accessed through Pre-AP Classroom, these short formative assessments provide insight into student progress. They are automatically scored and include multiple-choice and technology-enhanced items with rationales that explain correct and incorrect answers. **Use of one learning checkpoint per unit is required**. *For more details see page 49.*

PERFORMANCE TASKS

Available in the printed teacher resources as well as on Pre-AP Classroom, performance tasks allow students to demonstrate their learning through extended problem-solving, writing, analysis, and/or reasoning tasks. Scoring guidelines are provided to inform teacher scoring, with additional practice and feedback suggestions available in online modules on Pre-AP Classroom. **Use of each unit's performance task is required**. *For more details see page 51.*

PRACTICE PERFORMANCE TASKS

Available in the student resources, with supporting materials in the teacher resources, these tasks provide an opportunity for students to practice applying skills and knowledge as they would in a performance task, but in a more scaffolded environment. **Use of the practice performance tasks is encouraged but not required**. *For more details see page 51.*

FINAL EXAM

Accessed through Pre-AP Classroom, the final exam serves as a classroom-based, summative assessment designed to measure students' success in learning and applying the knowledge and skills articulated in the course framework. **Administration of the final exam is encouraged but not required**. *For more details see page 65.*

PROFESSIONAL LEARNING

Both the four-day Pre-AP Summer Institute (Pre-APSI) and the Online Foundational Module Series support teachers in preparing and planning to teach their Pre-AP course. **All Pre-AP teachers are required to either attend the Pre-AP Summer Institute or complete the module series. In addition, teachers are required to complete at least one Online Performance Task Scoring module**. *For more details see page 11.*

Course Map

PLAN

The course map shows how components are positioned throughout the course. As the map indicates, the course is designed to be taught over 140 class periods (based on 45-minute class periods), for a total of 28 weeks.

Model lessons are included for approximately 50% of the total instructional time, with the percentage varying by unit. Each unit is divided into key concepts.

TEACH

The model lessons demonstrate how the Pre-AP shared principles and mathematics areas of focus come to life in the classroom.

Shared Principles
Close observation and analysis
Higher-order questioning
Evidence-based writing
Academic conversation

Areas of Focus
Greater authenticity of applications and modeling
Engagement in mathematical argumentation
Connections among multiple representations

ASSESS AND REFLECT

Each unit includes two learning checkpoints and a performance task. These formative assessments are designed to provide meaningful feedback for both teachers and students.

Note: The final exam, offered during a six-week window in the spring, is not represented on the map.

UNIT 1 Modeling with Functions

~35 Class Periods

Pre-AP model lessons provided for 40% of instructional time in this unit

KEY CONCEPT 1.1
Choosing Appropriate Function Models

Learning Checkpoint 1

KEY CONCEPT 1.2
Rate of Change

Performance Task for Unit 1

KEY CONCEPT 1.3
Piecewise-Defined Models

Learning Checkpoint 2

UNIT 2 The Algebra of Functions

~30 Class Periods

Pre-AP model lessons provided for approximately 40% of instructional time in this unit

KEY CONCEPT 2.1
Composing Functions

KEY CONCEPT 2.2
Transforming Functions

Learning Checkpoint 1

KEY CONCEPT 2.3
Inverting Functions

Learning Checkpoint 2

Performance Task for Unit 2

UNIT 3 Function Families

~45 Class Periods

Pre-AP model lessons provided for approximately 20% of instructional time in this unit

KEY CONCEPT 3.1
Exponential and Logarithmic Functions

Learning Checkpoint 1

KEY CONCEPT 3.2
Polynomial and Rational Functions

Performance Task for Unit 3

KEY CONCEPT 3.3
Square Root and Cube Root Functions

Learning Checkpoint 2

UNIT 4T Trigonometric Functions

~30 Class Periods

Pre-AP model lessons provided for approximately 40% of instructional time in this unit

KEY CONCEPT 4T.1
Radian Measure and Sinusoidal Functions

Performance Task for Unit 4T

Learning Checkpoint 1

KEY CONCEPT 4T.2
The Tangent Function and Other Trigonometric Functions

KEY CONCEPT 4T.3
Inverting Trigonometric Functions

Learning Checkpoint 2

UNIT 4M Matrices and Their Applications

~30 Class Periods

Pre-AP model lessons provided for approximately 40% of instructional time in this unit

KEY CONCEPT 4M.1
Geometric Transformations

Learning Checkpoint 1

KEY CONCEPT 4M.2
Solving Systems of Equations with Matrices

KEY CONCEPT 4M.3
Applications of Matrix Multiplication

Learning Checkpoint 2

Performance Task for Unit 4M

Note: Schools can choose to complete either Unit 4T **or** Unit 4M, depending on which unit is the best fit for state or local standards.

Pre-AP Algebra 2 Course Framework

INTRODUCTION

Based on the Understanding by Design® (Wiggins and McTighe) model, the Pre-AP Algebra 2 Course Framework is back mapped from AP expectations and aligned to essential grade-level expectations. The course framework serves as a teacher's blueprint for the Pre-AP Algebra 2 instructional resources and assessments.

The course framework was designed to meet the following criteria:

- **Focused:** The framework provides a deep focus on a limited number of concepts and skills that have the broadest relevance for later high school, college, and career success.
- **Measurable:** The framework's learning objectives are observable and measurable statements about the knowledge and skills students should develop in the course.
- **Manageable:** The framework is manageable for a full year of instruction, fosters the ability to explore concepts in depth, and enables room for additional local or state standards to be addressed where appropriate.
- **Accessible:** The framework's learning objectives are designed to provide all students, across varying levels of readiness, with opportunities to learn, grow, and succeed.

COURSE FRAMEWORK COMPONENTS

The Pre-AP Algebra 2 Course Framework includes the following components:

Big Ideas

The big ideas are recurring themes that allow students to create meaningful connections between course concepts. Revisiting the big ideas throughout the course and applying them in a variety of contexts allow students to develop deeper conceptual understandings.

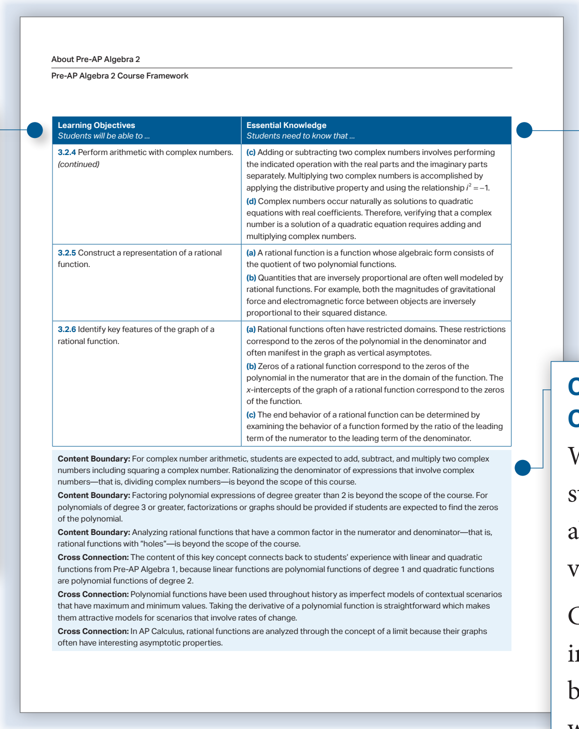
Enduring Understandings

Each unit focuses on a small set of enduring understandings. These are the long-term takeaways related to the big ideas that leave a lasting impression on students. Students build these understandings over time by exploring and applying course content throughout the year.

Key Concepts

To support teacher planning and instruction, each unit is organized by key concepts. Each key concept includes relevant **learning objectives** and **essential knowledge statements** and may also include **content boundary and cross connection statements**. These are illustrated and defined below.

Learning Objectives:
These objectives define what a student needs to be able to do with essential knowledge to progress toward the enduring understandings. The learning objectives serve as actionable targets for instruction and assessment.



About Pre-AP Algebra 2

Pre-AP Algebra 2 Course Framework

Learning Objectives *Students will be able to ...*	Essential Knowledge *Students need to know that ...*
3.2.4 Perform arithmetic with complex numbers. *(continued)*	**(c)** Adding or subtracting two complex numbers involves performing the indicated operation with the real parts and the imaginary parts separately. Multiplying two complex numbers is accomplished by applying the distributive property and using the relationship $i^2 = -1$. **(d)** Complex numbers occur naturally as solutions to quadratic equations with real coefficients. Therefore, verifying that a complex number is a solution of a quadratic equation requires adding and multiplying complex numbers.
3.2.5 Construct a representation of a rational function.	**(a)** A rational function is a function whose algebraic form consists of the quotient of two polynomial functions. **(b)** Quantities that are inversely proportional are often well modeled by rational functions. For example, both the magnitudes of gravitational force and electromagnetic force between objects are inversely proportional to their squared distance.
3.2.6 Identify key features of the graph of a rational function.	**(a)** Rational functions often have restricted domains. These restrictions correspond to the zeros of the polynomial in the denominator and often manifest in the graph as vertical asymptotes. **(b)** Zeros of a rational function correspond to the zeros of the polynomial in the numerator that are in the domain of the function. The x-intercepts of the graph of a rational function correspond to the zeros of the function. **(c)** The end behavior of a rational function can be determined by examining the behavior of a function formed by the ratio of the leading term of the numerator to the leading term of the denominator.

Content Boundary: For complex number arithmetic, students are expected to add, subtract, and multiply two complex numbers including squaring a complex number. Rationalizing the denominator of expressions that involve complex numbers—that is, dividing complex numbers—is beyond the scope of this course.

Content Boundary: Factoring polynomial expressions of degree greater than 2 is beyond the scope of the course. For polynomials of degree 3 or greater, factorizations or graphs should be provided if students are expected to find the zeros of the polynomial.

Content Boundary: Analyzing rational functions that have a common factor in the numerator and denominator—that is, rational functions with "holes"—is beyond the scope of the course.

Cross Connection: The content of this key concept connects back to students' experience with linear and quadratic functions from Pre-AP Algebra 1, because linear functions are polynomial functions of degree 1 and quadratic functions are polynomial functions of degree 2.

Cross Connection: Polynomial functions have been used throughout history as imperfect models of contextual scenarios that have maximum and minimum values. Taking the derivative of a polynomial function is straightforward which makes them attractive models for scenarios that involve rates of change.

Cross Connection: In AP Calculus, rational functions are analyzed through the concept of a limit because their graphs often have interesting asymptotic properties.

Essential Knowledge Statements:
Each essential knowledge statement is linked to a learning objective. One or more essential knowledge statements describe the knowledge required to perform each learning objective.

Content Boundary and Cross Connection Statements:
When needed, content boundary statements provide additional clarity about the content and skills that lie within versus beyond the scope of this course.

Cross connection statements highlight important connections that should be made between key concepts within and across the units.

BIG IDEAS IN PRE-AP ALGEBRA 2

While the Pre-AP Algebra 2 framework is organized into four core units of study, the content is grounded in three big ideas, which are cross-cutting concepts that build conceptual understanding and spiral throughout the course. Since these ideas cut across units, they serve as the underlying foundation for the enduring understandings, key concepts, and learning objectives that make up the focus of each unit. A deep and productive understanding in Pre-AP Algebra 2 relies on these three big ideas:

- **Function:** The mathematical concept of function describes a special kind of relationship where each input value corresponds to a single output value. Functions are among the most important objects in modern mathematics. Functions can be constructed to model phenomena that involve quantities such as time, force, and money, among others. These function models of real-world phenomena allow us to discover and investigate patterns among the related quantities. Studying patterns through the lens of a function model provides insights that lead to reasoned predictions and sound decision making.
- **Operations with Functions:** Two functions can be combined, composed, and transformed to form a new function that is a better model of a real-world phenomenon than the original function. Operations on functions include the arithmetic operations of addition, subtraction, multiplication, and division, as well as a special kind of operation, composition. Composition is the process of using the output of one function as the input of another function. When one of the functions in the composition is either a sum or product of a constant and a variable, the composition is referred to as a function transformation because the effect of such an operation on the graph of the function can be described in terms of geometric transformations. A thorough understanding of how to use function operations to construct more complex and nuanced function models is critical to the success of the mathematical modeling process.
- **Inverse Functions:** Solving an equation often relies on undoing an operation with its inverse operation. The processes of inverse operations are formalized as the concept of an inverse function, which expresses the idea that some mathematical relationships can be reversed. If a function defines a way to determine the output value from an input value, an inverse function defines a way to determine the input value from an output value. This implies that patterns observed in the output of one function can be seen in the input of the inverse function. Some of the functions covered in the course, such as exponential and logarithmic functions, are inverses of each other.

OVERVIEW OF PRE-AP ALGEBRA 2 UNITS AND ENDURING UNDERSTANDINGS

<table>
<tr><th>Unit 1: Modeling with Functions</th><th>Unit 2: The Algebra of Functions</th></tr>
<tr><td>Many bivariate data sets can be appropriately modeled by linear, quadratic, or exponential functions because the relationships between the quantities exhibit characteristics similar to those functions.
Mathematical functions almost never perfectly fit a real-world context, but a function model can be useful for making sense of that context.
Average rate of change allows us to understand multifaceted relationships between quantities by modeling them with linear functions.</td><td>Composing functions allows simpler functions to be combined to construct a function model that more appropriately captures the characteristics of a contextual scenario.
Transformations are a special kind of composition. When one of the functions being composed consists only of addition or multiplication, the effects on the other function are straightforward to determine.
An inverse function defines the way to determine the input value that corresponds to a given output value.</td></tr>
<tr><th>Unit 3: Function Families</th><th>Unit 4T: Trigonometric Functions</th></tr>
<tr><td>A function is a special mathematical relationship between two variables that can often be used to make sense of observable patterns in contextual scenarios.
Functions in a family have similar properties, similar algebraic representations, and graphs that share key features.</td><td>Trigonometry connects the study of circles and the study of right triangles.
Real-world contexts that exhibit periodic behavior or circular motion can be modeled by trigonometric functions.</td></tr>
</table>

Unit 1: Modeling with Functions

Suggested timing: Approximately 7 weeks

In the first unit of the course, students build upon their previous experience with linear, quadratic, and exponential functions. These important functions form the foundation upon which other functions introduced in this course are built. Unit 1 focuses on using functions to model real-world data sets and contextual scenarios. This focus on modeling provides authentic opportunities for students to investigate and confirm the defining characteristics of linear, quadratic, and exponential functions while simultaneously reinforcing procedural fluency with these function families.

Throughout Pre-AP Algebra 2, students are expected to take ownership of the mathematics they use by crafting arguments for why one type of function is better than another for modeling a particular data set or contextual scenario. This allows students to develop a deeper understanding of these foundational functions as they drive the mathematical modeling process themselves. This requires a more thorough understanding of modeling than prior Pre-AP mathematics courses, in which students were asked to explain why a given function type was an appropriate model for a given data set or contextual scenario.

ENDURING UNDERSTANDINGS

Students will understand that …

- Many bivariate data sets can be appropriately modeled by linear, quadratic, or exponential functions because the relationships between the quantities exhibit characteristics similar to those functions.
- Mathematical functions almost never perfectly fit a real-world context, but a function model can be useful for making sense of that context.
- Average rate of change allows us to understand multifaceted relationships between quantities by modeling them with linear functions.

KEY CONCEPTS

- 1.1: Choosing Appropriate Function Models – Using linear, quadratic, and exponential functions to make sense of relationships between two quantities
- 1.2: Rate of Change – Using linear functions to make sense of complex relationships
- 1.3: Piecewise-Defined Models – Using functions defined over discrete intervals to make sense of contexts with varied characteristics

KEY CONCEPT 1.1: CHOOSING APPROPRIATE FUNCTION MODELS

Using linear, quadratic, and exponential functions to make sense of relationships between two quantities

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
1.1.1 Identify a function family that would appropriately model a data set or contextual scenario.	**(a)** Linear functions often appropriately model data sets that exhibit a roughly constant rate of change. **(b)** Quadratic functions often appropriately model data sets that exhibit roughly linear rates of change, are roughly symmetric, and have a unique maximum or minimum output value. **(c)** Exponential functions often appropriately model data sets that exhibit roughly constant ratios of output values over equal intervals of input values.
1.1.2 Use residual plots to determine whether a function model appropriately models a data set.	**(a)** The residual for a data point is the deviation between the observed data value and the value predicted by a function model. Graphically, this can be thought of as the vertical segment between the data point and the graph of the function model. **(b)** A residual plot is a scatterplot of values of the independent variable and their corresponding residuals. **(c)** The sign of the residual indicates whether the function model is an overestimate or underestimate of the observed data value. **(d)** An appropriate function model for a data set produces a residual plot with no discernible pattern. Residual plots that display some systematic pattern indicate that there is variation in the data not accounted for in the function model.
1.1.3 Construct a representation of a linear, quadratic, or exponential function both with and without technology.	**(a)** A linear function can be expressed in slope-intercept form to reveal the constant rate of change and the initial value, or in point-slope form to reveal the constant rate of change and one ordered pair that satisfies the relationship. **(b)** A quadratic function can be expressed in vertex form to reveal its maximum or minimum value; in factored form to reveal the zeros of the function, which often correspond to the boundaries of the contextual domain; or in standard form to reveal the initial value. **(c)** An exponential function can be expressed in the form $f(x) = a(1+r)^x$ to reveal the percent change in the output, r, for a one-unit change in the input, or in the form $f(x) = a \cdot b^{\left(\frac{x}{n}\right)}$ to reveal the growth/decay factor, b, over an n-unit change in the input. **(d)** A function within a function family that best fits a data set minimizes the error of the function model, which is often quantified by the sum of the squares of the residuals.

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
1.1.4 Use a function that models a data set or contextual scenario to predict values of the dependent variable.	**(a)** An appropriate model for a bivariate data set can be used to predict values of the dependent variable from a value of the independent variable. **(b)** Functions that model a data set are frequently only useful over their contextual domain.

Content Boundary: A primary focus for this key concept is the use of functions as models for data sets and contextual scenarios. Calculating a function model from a large data set by hand is beyond the scope of the course. The use of technology to determine a function model for a data set is strongly encouraged; however, the analysis arising from a function model is best done by the student.

Cross Connection: In this key concept, students build upon their experience with scatterplots and trend lines from Pre-AP Algebra 1. Through this unit, they see that some data sets are best modeled by linear functions, while other data sets are more appropriately modeled by quadratic or exponential functions.

Cross Connection: Because linear, quadratic, and exponential functions are the most broadly useful functions for making sense of real-world phenomena, developing deep conceptual understanding and procedural fluency supports student success on SAT and access to advanced mathematics courses.

KEY CONCEPT 1.2: RATE OF CHANGE

Using linear functions to make sense of complex relationships

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
1.2.1 Interpret the average rate of change of a function over a given interval, including contextual scenarios.	**(a)** The average rate of change of a function over an interval can be interpreted as the constant rate of change of a linear function with the same change in output for the given change in the input. This constant rate of change can be interpreted graphically as the slope of the line between the points on the ends of the interval. **(b)** The average rate of change of a function f over the interval $[a, b]$ is the ratio $\frac{f(b)-f(a)}{b-a}$. That is, the average rate of change is $\frac{\Delta f(x)}{\Delta x}$.
1.2.2 Predict values of a function using the average rate of change and an input-output pair of a function model.	**(a)** The average rate of change of a function over the interval $[a, b]$ can be used to estimate values of the function within or near the interval. **(b)** The change in the value of $f(x)$ over an interval of width Δx can be determined by the product of the average rate of change of f and Δx.

Cross Connection: This key concept builds directly on students' understanding from Pre-AP Algebra 1 that linear functions have a constant rate of change. That prior knowledge can be leveraged here as students come to see how linear functions are used to make sense of more complex scenarios.

Cross Connection: The concept of average rate of change over increasingly smaller intervals is the basis for understanding the derivative of a function. In a calculus class, students will learn that the average rate of change of a function f over the interval $[x, x+\Delta x]$ is determined by the difference quotient, $\frac{f(x+\Delta x)-f(x)}{\Delta x}$.

KEY CONCEPT 1.3: PIECEWISE-DEFINED MODELS

Using functions defined over discrete intervals to make sense of contexts with varied characteristics

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
1.3.1 Construct a representation of a piecewise-defined function.	**(a)** A piecewise-defined function is a function that is defined on a set of nonoverlapping intervals. **(b)** Data sets or contextual scenarios that demonstrate different characteristics, such as rates of change, over different intervals of the domain would be appropriately modeled by a piecewise-defined function. **(c)** An algebraic representation of a piecewise-defined function consists of multiple algebraic expressions that describe the function over nonoverlapping intervals of the domain. **(d)** The graph of a piecewise-defined function is the set of input-output coordinate pairs that satisfy the function relationship.
1.3.2 Evaluate a piecewise-defined function at specified values of the domain.	**(a)** The output value of a piecewise-defined function for a specific input is determined by applying the algebraic rule for which the input value is defined. **(b)** Output values of a piecewise-defined function can be estimated, and sometimes determined, from a graph of the function.
1.3.3 Construct a representation of an absolute value function.	**(a)** The absolute value function is algebraically denoted as $f(x)=\|x\|$. **(b)** The function $f(x)=\|x\|$ is a piecewise-defined function. If x is nonnegative, then $\|x\| = x$; if x is negative, then $\|x\| = -x$. The graph of $y = f(x)$ consists of $y = x$ for $x \geq 0$ and $y = -x$ for $x < 0$.

Content Boundary: Intervals of real numbers can be expressed in interval notation or in inequality notation. Students are expected to be familiar with reading and writing intervals using both notations. The particular context in which an interval of real numbers is used should determine which notation is more appropriate.

Unit 2: The Algebra of Functions

Suggested timing: Approximately 6 weeks

In Unit 2, students develop a conceptual understanding of the algebra of functions and build procedural fluency with function notation. Students tend to think about transformations of functions and composition of functions as unrelated topics. In this unit, students connect these important concepts to develop a more coherent understanding of functions by first exploring function composition, a new operation that chains functions together in a sequence. Once students understand the power of function composition, they work to see how function transformations are a special case of composition in which a given function is composed with a linear function.

The unit culminates in an exploration of inverses—the mathematical concept of undoing—through inverse operations and inverse functions. Students develop familiarity with inverse operations through their elementary school experiences with addition and multiplication, and their respective inverses, subtraction and division. In this unit, the inverse operation of exponentiating—taking a logarithm—is introduced. From prior coursework, students know that a function associates each input with one output. In this course, students learn that if a function has an inverse function, it associates an output back to its input. By considering inverses as both operations and functions, students develop a deep understanding of this critical concept.

ENDURING UNDERSTANDINGS

Students will understand that …

- Composing functions allows simpler functions to be combined to construct a function model that more appropriately captures the characteristics of a contextual scenario.
- Transformations are a special kind of composition. When one of the functions being composed consists only of addition or multiplication, the effects on the other function are straightforward to determine.
- An inverse function defines the way to determine the input value that corresponds to a given output value.

KEY CONCEPTS

- 2.1: Composing Functions – Chaining functions together in a sequence to construct better function models
- 2.2: Transforming Functions – Exploring how addition and multiplication affect the input or output of a function
- 2.3: Inverting Functions – Making sense of doing and undoing through inverse operations and inverse functions

KEY CONCEPT 2.1: COMPOSING FUNCTIONS

Chaining functions together in a sequence to construct better function models

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
2.1.1 Determine the output value of the composition of two or more functions for a given input value when the functions have the same or different representations.	**(a)** Composing functions is a process in which the output of one function is used as the input of another function. **(b)** Composing functions is generally not a commutative operation. That is, for most functions, the value of $f(g(x))$ is not equal to the value of $g(f(x))$ for a given value of x.
2.1.2 Construct a representation of a composite function when the functions being composed have the same or different representations.	**(a)** Composing two functions, f and g, results in a new function, called the composite function, that can be notated $f \circ g$ where $f \circ g(x) = f(g(x))$. **(b)** An algebraic representation of $f \circ g$ is constructed by substituting every instance of x in the algebraic representation of $f(x)$ with the algebraic representation of $g(x)$. **(c)** A graphical representation of $f \circ g$ can be constructed from the algebraic representation of $f \circ g$ or approximated by plotting some ordered pairs of the form $(x, f(g(x)))$. **(d)** A numerical representation of $f \circ g$ consists of a subset of the ordered pairs that satisfy the relationship and is constructed by directly calculating values of $f(g(x))$ from values of x that are in the domain of g.
2.1.3 Express a given algebraic representation of a function in an equivalent form as the composition of two or more functions.	**(a)** Any function can be expressed as the composition of two or more functions. One of these functions can be the identity function, $f(x) = x$. **(b)** Algebraic techniques, such as factoring, can be used to express an algebraic representation of a function as a composition of functions.

Cross Connection: Students used function composition in Pre-AP Geometry with Statistics when they used sequences of multiple rigid motion and/or similarity transformations to associate one figure with another. In those experiences, the output of one transformation was treated as the input of another transformation. Students learned that changing the order in which the same transformations are applied to a preimage often yields different images. That understanding is reinforced through this key concept when students learn the importance of the order of composing two or more functions.

Cross Connection: Function composition often appears in problems in which the frame of reference for a function model is specified. For example, the output of a function could be the height as measured from the roof instead of the height as measured from the ground, or the input of a function could be the time that has elapsed since noon instead of the time that has elapsed since midnight. Using function composition to change the frame of reference is a valuable technique with applications across mathematics and science courses.

Cross Connection: The work of Learning Objective 2.1.3, understanding how one function can be written as a composition of two or more functions, is a critical first step in understanding the chain rule in AP Calculus. The chain rule defines how to take the derivative of a composite function.

KEY CONCEPT 2.2: TRANSFORMING FUNCTIONS

Exploring how addition and multiplication affect the input or output of a function

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
2.2.1 Compare a function f with an additive transformation of f, that is, $f(x+k)$ or $f(x)+k$.	**(a)** An additive transformation of f is a composition of f with $g(x)=x+k$. That is, $f \circ g(x)=f(x+k)$ and $g \circ f(x)=f(x)+k$. **(b)** The graph of an additive transformation of f is either a vertical or horizontal translation of the graph of f. **(c)** If (a, b) is an input-output pair of f, then $(a-k, b)$ must be an input-output pair of $f(x+k)$. Therefore, the graph of $f(x+k)$ is a horizontal translation of the graph of f by $-k$ units. **(d)** If (a, b) is an input-output pair of f, then $(a, b+k)$ must be an input-output pair of $f(x)+k$. Therefore, the graph of $f(x)+k$ is a vertical translation of the graph of f by k units.
2.2.2 Compare a function f with a multiplicative transformation of f, that is, $f(kx)$ or $k \bullet f(x)$.	**(a)** A multiplicative transformation of f is a composition of f with $g(x)=kx$, where $k \neq 0$. That is, $f \circ g(x)=f(kx)$ and $g \circ f(x)=k \bullet f(x)$. **(b)** The graph of a multiplicative transformation of f is either a vertical or horizontal dilation of the graph of f. When $k<0$, the graph of the multiplicative transformation is also a reflection of the graph of f over one of the axes. **(c)** If (a, b) is an input-output pair of f, then $\left(\frac{1}{k} \bullet a, b\right)$ must be an input-output pair of $f(kx)$. Therefore, the graph of $f(kx)$ is a horizontal dilation of the graph of f by a factor of $\left\lvert\frac{1}{k}\right\rvert$. **(d)** If (a, b) is an input-output pair of f, then (a, kb) must be an input-output pair of $k \bullet f(x)$. Therefore, the graph of $k \bullet f(x)$ is a vertical dilation of the graph by a factor of $\lvert k \rvert$.
2.2.3 Construct a representation of the transformation of a function.	**(a)** A function transformation is a sequence of additive and multiplicative transformations of f. The order in which the transformations are applied matters. **(b)** Changing the reference point for the input or output quantity of a function can be achieved with an additive transformation. **(c)** Converting the unit of measure for an input or output quantity of a function can be achieved with a multiplicative transformation.

Cross Connection: Students used additive and multiplicative transformations in Pre-AP Geometry with Statistics in the form of translations and dilations, respectively. Referring back to related geometric concepts in this key concept will help students make deep connections among the various transformations with which they have worked.

KEY CONCEPT 2.3: INVERTING FUNCTIONS

Making sense of doing and undoing through inverse operations and inverse functions

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
2.3.1 Determine all input values that correspond to a specified output value given a function model on a specified domain.	**(a)** For algebraic representations of an equation, inverse operations, such as squaring/square rooting and cubing/cube rooting, can be used to determine the input values that correspond to a specified output value. **(b)** For algebraic representations of an equation involving an exponential expression, the inverse operation of exponentiating, called taking a logarithm, can be used to determine the input values that correspond to a specified output value. The solution to the equation $b^x = c$ where b and c are both positive and $b \neq 1$, is expressed by $x = \log_b(c)$. **(c)** For graphical representations of an equation, identifying all ordered pairs that lie on the intersection of the line $y = k$ and the graph of $y = f(x)$ provides all input values that correspond with the output value k.
2.3.2 Express exactly or approximate the value of a logarithmic expression as a rational number.	**(a)** An exact value of a logarithm can be determined using laws of exponents. **(b)** If the logarithmic expression $\log_b(c)$ can be expressed exactly as a rational number, then its value is the rational number, x, that makes the equation $b^x = c$ true. **(c)** If the logarithmic expression $\log_b(c)$ cannot be written exactly as a rational number, then its value can be approximated by a rational number x, for which $b^x \approx c$.
2.3.3 Determine a domain over which the inverse function of a specified function is defined.	**(a)** A function f has an inverse function on a specified domain if each output value of f corresponds to exactly one input value in that domain. **(b)** A function f is called invertible on a specified domain if there exists an inverse function, f^{-1}, such that $f(a) = b$ implies $f^{-1}(b) = a$. **(c)** There are multiple ways to restrict the domain of a function so that the function is invertible. The appropriate domain restrictions for making a function invertible may depend on the context.
2.3.4 Construct a representation of the inverse function given a function that is invertible on its domain.	**(a)** A table of values for the inverse function of f consists of all input-output ordered pairs (b, a) such that (a, b) is an input-output ordered pair of f. **(b)** The graph of the inverse function of f is a reflection of the graph of $y = f(x)$ across the line $y = x$. **(c)** The algebraic representation of the inverse function of $y = f(x)$ is determined using inverse operations to express x in terms of y. **(d)** The domain and range of f^{-1} are the range and domain of f, respectively.

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
2.3.5 Verify that one function is an inverse of another function using composition.	**(a)** If f is an inverse function of g, then g is an inverse function of f. **(b)** If f is an inverse function of g, then composing f and g in either order will map each input onto itself. **(c)** The function f is the inverse of the function g if and only if their composition in either order is the identity function, x. That is, $f(g(x)) = x$ and $g(f(x)) = x$.

Content Boundary: In this key concept, students are expected to solve quadratic and exponential functions using their associated inverse operations, taking the square root and taking a logarithm. Students are expected to develop an intuition for exact and approximate values of square roots of real numbers and of logarithms of real numbers by understanding their relationship to their respective inverses.

Content Boundary: The problems and exercises that address Learning Objective 2.3.4 are limited to linear functions and quadratic functions, with their domains appropriately constrained. Determining an inverse function for an exponential function is beyond the scope of this unit but will be expected in Unit 3.

Cross Connection: In Pre-AP Algebra 1, students learned that the expression $\sqrt{a}$ is a notation for the number whose square is a. Similarly, in this key concept, students learn that the expression $\log_b(c)$ is a notation for the exponent of b, such that b raised to that exponent has a value of c. That is, the equations $x = \log_b(c)$ and $b^x = c$ convey the same information about the relationship between the numbers b and c.

Cross Connection: The concepts of doing and undoing are central to many mathematical concepts. Inverse operations allow equations to be solved methodically rather than through an inefficient guess-and-check process. The existence of an inverse function shows that a function relationship between two quantities can be associated in two directions: from input to output and from output to input. In AP Calculus, the Fundamental Theorem of Calculus establishes an inverse relationship between differentiating and integrating a function.

Unit 3: Function Families

Suggested timing: Approximately 9 weeks

Explorations of function families are an important component of any Algebra 2 course because they expand the repertoire of functions students can draw upon to model real-world phenomena. Not all phenomena are appropriately modeled by a linear, quadratic, or exponential function. For example, the gravitational force between two objects is inversely proportional to the square of their distance apart. This relationship would be best modeled with a rational function, one of the functions introduced in Unit 3. Throughout this unit, students learn that a parent function and its transformations form a function family. All functions in the same function family share some properties with each other.

Because function families are traditionally taught as isolated topics, students rarely have time to thoroughly investigate which properties of a function family are maintained by transformations and which are not. Therefore, the key concepts in this unit intentionally focus students' thinking on how function families are related in meaningful ways. The structure of the unit is intended to help students construct a network of connections among these function families. As with all explorations of functions throughout Pre-AP, the emphasis is on contextual scenarios that can be effectively modeled by each function family.

ENDURING UNDERSTANDINGS

Students will understand that …

- A function is a special mathematical relationship between two variables that can often be used to make sense of observable patterns in contextual scenarios.
- Functions in a family have similar properties, similar algebraic representations, and graphs that share key features.

KEY CONCEPTS

- 3.1: Exponential and Logarithmic Functions – Using functions to make sense of multiplicative patterns of change
- 3.2: Polynomial and Rational Functions – Using functions to make sense of sums and quotients of powers
- 3.3: Square Root and Cube Root Functions – Using functions to make sense of inverting quadratic and cubic relationships

KEY CONCEPT 3.1: EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Using functions to make sense of multiplicative patterns of change

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
3.1.1 Construct a representation of an exponential function using the natural base, *e*.	**(a)** The natural base *e*, which is approximately 2.718, is often used as the base in exponential functions that model contextual scenarios involving continuously compounded interest.
3.1.2 Express an exponential function in an equivalent form to reveal properties of the graph and/or the contextual scenario.	**(a)** Any exponential model can be expressed in any base, including the natural base, using properties of exponents and/or function composition and properties of logarithms. **(b)** A horizontal translation of the graph of an exponential function can also be thought of as a vertical dilation of the graph because $f(x)=b^{(x+k)}$ can be expressed as $f(x)=b^x \cdot b^k$, where b^k is a constant. **(c)** A horizontal dilation of the graph of an exponential function is equivalent to a change of the base of the function, because $f(x)=b^{kx}$ can be expressed as $f(x)=\left(b^k\right)^x$, where b^k is a constant.
3.1.3 Construct a representation of a logarithmic function.	**(a)** The graph of a logarithmic function exhibits vertical asymptotic behavior. **(b)** An algebraic representation of a logarithmic function with base *b* is a transformation of $f(x)=\log_b(x)$, where $b \neq 1$ and $b>0$. **(c)** A verbal representation of a logarithmic function describes additive changes in the output corresponding to multiplicative changes in the input. **(d)** Data sets and/or contextual scenarios that exhibit a roughly constant ratio of the inputs for equal additive changes in the outputs are appropriately modeled by logarithmic functions.
3.1.4 Express a logarithmic function in an equivalent form to reveal properties of the graph and/or the contextual scenario.	**(a)** A logarithmic function can be expressed in any base using properties of logarithms. Logarithmic functions are often expressed in base *e*, which is called the natural logarithm and is notated as "ln" rather than $\log_e$. **(b)** A horizontal dilation is equivalent to a vertical translation because $f(x)=\log_b(kx)$ can be rewritten as $f(x)=\log_b(k)+\log_b(x)$, where $\log_b(k)$ is a constant. **(c)** Raising the input of a logarithmic function to a power of *k* results in a vertical dilation of the graph because $f(x)=\log_b\left(x^k\right)$ can be rewritten as $f(x)=k \cdot \log_b(x)$.
3.1.5 Construct a representation of the inverse function of an exponential or logarithmic function.	**(a)** The inverse of the exponential function $f(x)=b^x$ is the logarithmic function $g(x)=\log_b(x)$. **(b)** The inverse of the logarithmic function $f(x)=\log_b(x)$ is the exponential function $g(x)=b^x$.

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
3.1.6 Solve equations involving exponential or logarithmic functions, including those arising from contextual scenarios.	**(a)** Equations involving exponential functions can be solved algebraically by taking a logarithm or have solutions that can be estimated by examining a graph of the function. **(b)** Equations involving logarithmic functions can be solved algebraically by exponentiating or have solutions that can be estimated by examining a graph of the function.

Content Boundary: Because students have familiarity with exponential functions from Pre-AP Algebra 1 and Unit 1 of this course, the exponential functions in this key concept should primarily include the natural base, *e*.

Cross Connection: This key concept provides an opportunity for students to use what they learned about exponential functions in Pre-AP Algebra 1 and function transformations, composition, and inverses in Unit 2 to deepen their understanding of exponential functions, a critically important function family.

Cross Connection: Logarithms are an essential tool in advanced mathematics courses. They are used in AP Statistics to transform a data set into a set that displays a more linear trend than the original one, which makes the data set easier to model. In AP Calculus, logarithms are often used in solving differential equations, especially those involving population growth, which is often exponential.

KEY CONCEPT 3.2: POLYNOMIAL AND RATIONAL FUNCTIONS

Using functions to make sense of sums and quotients of powers

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
3.2.1 Construct a representation of a polynomial function.	**(a)** A polynomial function is a function whose algebraic representation can be expressed as $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $a_n \neq 0$. **(b)** Scenarios involving areas of figures and surface areas and volumes of solids are often well modeled by polynomial functions. **(c)** Data sets and/or contextual scenarios that exhibit maximum or minimum values are sometimes well modeled by polynomial functions. **(d)** Data sets and/or contextual scenarios where equal changes in the input values correspond to approximately constant nth differences in the output values are often well modeled by polynomial functions of degree n.
3.2.2 Express a polynomial function in an equivalent algebraic form to reveal properties of the function.	**(a)** The standard form of a polynomial function reveals the degree of the polynomial, n, which is the highest power of all the terms. **(b)** A linear factor, $(x - a)$, of a polynomial function, p, corresponds to a zero (or root) of p at $x = a$ because $p(a) = 0$. **(c)** A polynomial function factored into a product of linear factors reveals the x-intercepts of the graph of the function, which are the real zeros of the polynomial. The total number of real zeros is at most equal to the degree of the polynomial function.
3.2.3 Identify key features of the graph of a polynomial function.	**(a)** A local maximum or minimum of a nonconstant polynomial function corresponds to the output value of the point at which the function switches between increasing to decreasing in either order. **(b)** Between every two real zeros of a nonconstant polynomial function, there must be at least one input value corresponding to a local maximum or minimum. **(c)** The end behavior of a polynomial function can be determined visually from its graph or by examining the degree of the polynomial and the sign of its leading coefficient. **(d)** If a linear factor $(x - a)$ of a polynomial function has an even power, then the signs of the output values are the same for input values near $x = a$. For these polynomials, the graph will be tangent to the x-axis at $x = a$.
3.2.4 Perform arithmetic with complex numbers.	**(a)** Every complex number has the form $a + bi$ where a and b are real numbers and $i^2 = -1$. The real part of the complex number is a and the imaginary part of the complex number is b. **(b)** The real numbers are a subset of the complex numbers since every real number a is equivalent to the complex number $a + 0i$.

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
3.2.4 Perform arithmetic with complex numbers. *(continued)*	**(c)** Adding or subtracting two complex numbers involves performing the indicated operation with the real parts and the imaginary parts separately. Multiplying two complex numbers is accomplished by applying the distributive property and using the relationship $i^2 = -1$. **(d)** Complex numbers occur naturally as solutions to quadratic equations with real coefficients. Therefore, verifying that a complex number is a solution of a quadratic equation requires adding and multiplying complex numbers.
3.2.5 Construct a representation of a rational function.	**(a)** A rational function is a function whose algebraic form consists of the quotient of two polynomial functions. **(b)** Quantities that are inversely proportional are often well modeled by rational functions. For example, both the magnitudes of gravitational force and electromagnetic force between objects are inversely proportional to their squared distance.
3.2.6 Identify key features of the graph of a rational function.	**(a)** Rational functions often have restricted domains. These restrictions correspond to the zeros of the polynomial in the denominator and often manifest in the graph as vertical asymptotes. **(b)** Zeros of a rational function correspond to the zeros of the polynomial in the numerator that are in the domain of the function. The x-intercepts of the graph of a rational function correspond to the zeros of the function. **(c)** The end behavior of a rational function can be determined by examining the behavior of a function formed by the ratio of the leading term of the numerator to the leading term of the denominator.

Content Boundary: For complex number arithmetic, students are expected to add, subtract, and multiply two complex numbers including squaring a complex number. Rationalizing the denominator of expressions that involve complex numbers—that is, dividing complex numbers—is beyond the scope of this course.

Content Boundary: Factoring polynomial expressions of degree greater than 2 is beyond the scope of the course. For polynomials of degree 3 or greater, factorizations or graphs should be provided if students are expected to find the zeros of the polynomial.

Content Boundary: Analyzing rational functions that have a common factor in the numerator and denominator—that is, rational functions with "holes"—is beyond the scope of the course.

Cross Connection: The content of this key concept connects back to students' experience with linear and quadratic functions from Pre-AP Algebra 1, because linear functions are polynomial functions of degree 1 and quadratic functions are polynomial functions of degree 2.

Cross Connection: Polynomial functions have been used throughout history as imperfect models of contextual scenarios that have maximum and minimum values. Taking the derivative of a polynomial function is straightforward which makes them attractive models for scenarios that involve rates of change.

Cross Connection: In AP Calculus, rational functions are analyzed through the concept of a limit because their graphs often have interesting asymptotic properties.

KEY CONCEPT 3.3: SQUARE ROOT AND CUBE ROOT FUNCTIONS

Using functions to make sense of inverting quadratic and cubic relationships

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
3.3.1 Construct a representation of a square root function.	**(a)** The square root function, $f(x)=\sqrt{x}$, is the inverse of the quadratic function $g(x)=x^2$ over the restricted domain $[0, \infty)$. Therefore, the graph of $y=\sqrt{x}$ resembles the graph of $y=x^2$ for $x \geq 0$ reflected across the line $y=x$. **(b)** The domain of a square root function, a function transformation of $f(x)=\sqrt{x}$, corresponds to the set of input values for which the expression under the radical is nonnegative. **(c)** Real-world scenarios involving distance traveled and elapsed time for free-falling objects are well modeled by square root functions.
3.3.2 Construct a representation of a cube root function.	**(a)** The cube root function, $f(x)=\sqrt[3]{x}$, is the inverse of the cubic function $g(x)=x^3$. Thus, the graph of $y=\sqrt[3]{x}$ resembles the graph of $y=x^3$ reflected across the line $y=x$. **(b)** The domain of a cube root function, a function transformation of $f(x)=\sqrt[3]{x}$, is all real numbers. **(c)** Real-world scenarios involving side lengths of solids with a known volume are well modeled by cube root functions.
3.3.3 Construct a representation of the inverse function of a given quadratic function.	**(a)** The inverse of a quadratic function is a square root function. **(b)** The algebraic representation of the inverse of a quadratic function $y=f(x)$ is determined by first expressing f in vertex form and then using inverse operations to express x in terms of y. **(c)** Since many output values of quadratic functions are each associated with multiple input values, constructing an inverse of a quadratic function requires restricting the domain of the function so the function is invertible. For values of x in the constrained domain of the quadratic function f, $f^{-1}(f(x))=x$.
3.3.4 Solve equations involving square root or cube root functions, including those arising from contextual scenarios.	**(a)** Equations involving square roots and cube roots arising from contextual scenarios can be solved algebraically using inverse operations, such as squaring or cubing, or have solutions that can be estimated by examining an associated graph of the function models. **(b)** Solving equations by squaring can introduce values called extraneous solutions, which are not actual solutions of the equation.

Cross Connection: The square root function appears in a variety of applications, most notably in scenarios involving distance or velocity as a function of time. These types of problems could involve finding the distance between two points in the plane using the Pythagorean theorem or finding a typical distance between a data value and the mean of the data set, called the standard deviation of a data set. In both cases, taking the square root is used to undo squaring a quantity.

Unit 4T: Trigonometric Functions

Suggested timing: Approximately 6 weeks

The final unit in this course provides an exploration of trigonometric functions. Trigonometry is the branch of mathematics that connects two fundamental geometric objects: triangles and circles. In Pre-AP Geometry with Statistics, students learned that the trigonometric ratios relate acute angle measures to ratios of side lengths in right triangles. Algebra 2 extends those relationships to include all real numbers. When the domains of trigonometric functions include angle measures greater than 90°, including greater than 360°, these functions are far more useful in modeling contextual scenarios that involve periodic phenomena, such as the rotation of a ceiling fan, the height of a Ferris wheel car, or the ebb and flow of tides. A sound understanding of trigonometric functions demystifies these common real-world contexts, putting the power of mathematical reasoning at students' command.

Beginning with the introduction of radians as units of angle measure, the unit continues with an investigation of the sine and cosine functions and their transformations, collectively referred to as sinusoidal functions. Students then use what they know about sinusoidal functions and properties of quotients of functions to understand the properties of the tangent function and the three reciprocal trigonometric functions. Finally, students use inverse trigonometric functions to solve problems related to circular and periodic motion. Please note that the Pre-AP three-year mathematics sequence includes trigonometry in Algebra 2 to create a more equitable pathway for students who take Algebra 1 in 9th grade to potentially enroll in AP Calculus AB in 12th grade. If a state's standards do not require trigonometry in Algebra 2, then completing the Pre-AP requirements for Unit 4T are optional for teachers in that state.

ENDURING UNDERSTANDINGS

Students will understand that …

- Trigonometry connects the study of circles and the study of right triangles.
- Real-world contexts that exhibit periodic behavior or circular motion can be modeled by trigonometric functions.

KEY CONCEPTS

- 4T.1: Radian Measure and Sinusoidal Functions – Using circles and triangles to make sense of periodic phenomena
- 4T.2: The Tangent Function and Other Trigonometric Functions – Using quotients of trigonometric functions to define new functions
- 4T.3: Inverting Trigonometric Functions – Using trigonometry to solve equations involving circular motion and periodic phenomena

KEY CONCEPT 4T.1: RADIAN MEASURE AND SINUSOIDAL FUNCTIONS

Using circles and triangles to make sense of periodic phenomena

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
4T.1.1 Use the radian measure of an angle to relate the radius of a circle to the length of the arc subtended by that angle.	**(a)** The radian measure of an angle expresses the ratio of the subtended arc length to the radius of the circle in which it is a central angle. **(b)** An angle that has a measure of 1 radian cuts off an arc length equal to the length of the radius. **(c)** There is a proportional relationship between the radian measure of an angle, the subtended arc length, and the radius length. For a given angle, the ratio of the subtended arc length to the radius length is constant for any radius of the circle in which the angle is a central angle.
4T.1.2 Determine when two angles in the coordinate plane are coterminal.	**(a)** Angle measures can be expressed in radians or in degrees. The relationship between these units is proportional such that $\frac{\text{measure in degrees}}{\text{measure in radians}} = \frac{360}{2\pi}$. **(b)** In the coordinate plane, an angle is in standard position when its vertex is at the origin and one of its rays lies along the positive x-axis. Its other ray is called the terminal ray. **(c)** Two angles in standard position are coterminal if their terminal rays coincide. The amount of rotation of coterminal angles may differ by an integer number of revolutions. **(d)** Positive angle measures indicate that the terminal ray of the angle is constructed by the counterclockwise rotation of the ray about the origin. Negative angle measures indicate the terminal ray of the angle is constructed by a clockwise rotation of the ray about the origin.
4T.1.3 Construct a representation of a sinusoidal function.	**(a)** Periodic phenomena have repeating patterns of output values. Aspects of these phenomena can often be appropriately modeled by sinusoidal functions. **(b)** A unit circle has a radius of 1 unit of measure. The unit of measure should be determined by the context. **(c)** In the context of circular motion, the function $f(\theta) = \sin(\theta)$ relates the measure of an angle in standard position to the vertical displacement from the origin of a point on the unit circle and the function $f(\theta) = \cos(\theta)$ relates the measure of an angle in standard position to the horizontal displacement from the origin of a point on the unit circle. **(d)** Sinusoidal functions include the functions $f(x) = \sin(x)$ and $f(x) = \cos(x)$ and their transformations.

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
4T.1.4 Determine the exact coordinates of any point on a circle centered at the origin.	**(a)** The coordinates of the point at which the terminal ray of an angle in standard position intersects a unit circle with radius r are uniquely determined by the measure of that angle, θ, where $(x, y) = (r\cos(\theta), r\sin(\theta))$. For points on a unit circle, the coordinates are given by $(x, y) = (\cos(\theta), \sin(\theta))$. **(b)** The reference triangle for a given point on a circle is a right triangle whose three vertices are the origin, the point itself, and a point on the x-axis. The reference triangle can be useful in determining the exact coordinates of the given point.
4T.1.5 Identify key characteristics of a sinusoidal function.	**(a)** The amplitude of a sinusoidal function is half the difference between its maximum and minimum values. **(b)** The period of a sinusoidal function is the length of the interval of the input values over which it completes one rotation. The frequency of a sinusoidal function is the number of periods within an interval of length 2π. **(c)** Additive transformations of sinusoidal functions result in vertical or horizontal translations of the graph. A horizontal translation of a sinusoidal function is called a phase shift. **(d)** Multiplicative transformations of sinusoidal functions result in vertical or horizontal dilations of the graph. These transformations can impact the amplitude, period, or frequency of the sinusoidal function.
4T.1.6 Construct a sinusoidal function to model a periodic phenomenon that has a specified frequency, period, amplitude, and phase shift.	**(a)** The smallest interval of the input values over which the maximum or minimum output values start to repeat can be used to determine or estimate the period and frequency of the sinusoidal function model. **(b)** The maximum and minimum output values can be used to determine or estimate the amplitude for a sinusoidal function model. **(c)** The reference point for the input quantity can be used to determine or estimate a phase shift for the sinusoidal function model.
4T.1.7 Solve problems involving trigonometric identities.	**(a)** The Pythagorean theorem for trigonometric functions, $\sin^2(\theta) + \cos^2(\theta) = 1$, can be deduced from the fact that a circle of radius r centered at the origin is the solution set to the equation $x^2 + y^2 = r^2$ and that the coordinates of a point on that circle are given by $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

Content Boundary: The double-angle and half-angle formulas are beyond the scope of the course because of their limited usefulness in developing an understanding of trigonometric functions. The reciprocal identities are beyond the scope of this key concept but are introduced in Key Concept 4T.2. Students are expected to use the Pythagorean identity to solve problems involving squared trigonometric expressions, but verifying trigonometric identities is beyond the scope of the course.

Cross Connection: Embedding an angle in the coordinate plane allows students to connect their Pre-AP Geometry with Statistics experience with the concept of an angle, a union of two rays, to the concept of rotation. The measure of an angle in the coordinate plane quantifies the amount of the rotation about the vertex required of the first ray to coincide with the second ray, while the sign of the angle measure indicates the direction of this rotation.

KEY CONCEPT 4T.2: THE TANGENT FUNCTION AND OTHER TRIGONOMETRIC FUNCTIONS

Using quotients of trigonometric functions to define new functions

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
4T.2.1 Construct a representation of a tangent function.	**(a)** For an angle in standard position, the tangent of the angle's measure is the slope of the terminal ray. **(b)** When the terminal ray is vertical, the slope of the ray is undefined. Therefore, the domain of the tangent function is restricted to exclude all values that correspond to vertical terminal rays. In other words, the domain of $f(\theta)=\tan(\theta)$ excludes all values of θ such that $\theta=\frac{\pi}{2}+k\pi$ for integer values of k. **(c)** When two angles in standard position have measures that differ by π radians, their terminal rays form a line and the rays have the same slope. As such, the tangent function has a period of π. **(d)** Contextual scenarios involving the height of a rising or falling object or slopes of lines are often appropriately modeled with a tangent function.
4T.2.2 Identify key characteristics and values of functions that are defined by quotients of sinusoidal functions.	**(a)** The tangent of angle θ is the slope of the terminal ray, which passes through the points (0, 0) and $(r\cos(\theta), r\sin(\theta))$. Thus, $\tan(\theta)=\frac{\sin(\theta)}{\cos(\theta)}$. **(b)** Secant, cosecant, and cotangent are the names given to trigonometric functions formed by quotients of sinusoidal functions, defined as $\sec(\theta)=\frac{1}{\cos(\theta)}$, $\csc(\theta)=\frac{1}{\sin(\theta)}$, and $\cot(\theta)=\frac{\cos(\theta)}{\sin(\theta)}$.

Content Boundary: Students develop an understanding of the properties of tangent, secant, cosecant, and cotangent functions and their relationships to the sinusoidal functions through this key concept. Verifying trigonometric identities is beyond the scope of the course.

Cross Connection: This key concept connects the slope of a line to the tangent of an angle that the line forms with the horizontal axis. This connection provides an opportunity for students to build a deeper understanding of linear relationships and link trigonometry to the content of Pre-AP Algebra 1. A comprehensive understanding of slope supports students in accessing the concept of the derivative of a function in AP Calculus.

KEY CONCEPT 4T.3: INVERTING TRIGONOMETRIC FUNCTIONS

Using trigonometry to solve equations involving circular motion and periodic phenomena

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
4T.3.1 Construct a representation of an inverse trigonometric function.	**(a)** The inputs and outputs of inverse trigonometric functions are switched from their corresponding trigonometric functions, so the output of an inverse trigonometric function is often interpreted as an angle measure and the input is a value in the range of the corresponding trigonometric function. **(b)** The inverse trigonometric functions arcsine, arccosine, and arctangent (also represented as $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$) are defined by restricting the domain of the sine function to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, restricting the domain of the cosine function to $[0, \pi]$, and restricting the domain of the tangent function to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ so the trigonometric functions are invertible on their restricted domains.
4T.3.2 Solve equations involving trigonometric functions.	**(a)** For algebraic equations involving trigonometric expressions, inverse trigonometric functions can be used to determine the inputs corresponding to a specified output. **(b)** There could be multiple solutions to a trigonometric equation. The exact number of solutions is determined by the context.

Content Boundary: Solving a trigonometric equation for all possible solutions is beyond the scope of this course. Problems and exercises involving trigonometric equations used to model periodic phenomena should be limited to solutions over a finite interval.

Cross Connection: The names of inverse trigonometric functions include the prefix *arc-* because the output of each of these functions is an arc length measured in radians. Knowing why the prefix *arc-* is used can help students connect inverse trigonometric functions to circles, as explored in Key Concept 4T.1.

Pre-AP Algebra 2 Model Lessons

Model lessons in Pre-AP Algebra 2 are developed in collaboration with Algebra 2 educators across the country and are rooted in the course framework, shared principles, and areas of focus. Model lessons are carefully designed to illustrate on-grade-level instruction. Pre-AP strongly encourages teachers to internalize the lessons and then offer the supports, extensions, and adaptations necessary to help all students achieve the lesson goals.

The purpose of these model lessons is twofold:

- **Robust instructional support for teachers:** Pre-AP Algebra 2 model lessons are comprehensive lesson plans that, along with accompanying student resources, embody the Pre-AP approach to teaching and learning. Model lessons provide clear and substantial instructional guidance to support teachers as they engage students in the shared principles and areas of focus.
- **Key instructional strategies:** Commentary and analysis embedded in each lesson highlight not just what students and teachers do in the lesson, but also how and why they do it. This educative approach provides a way for teachers to gain unique insight into key instructional moves that are powerfully aligned with the Pre-AP approach to teaching and learning. In this way, each model lesson works to support teachers in the moment of use with students in their classroom.

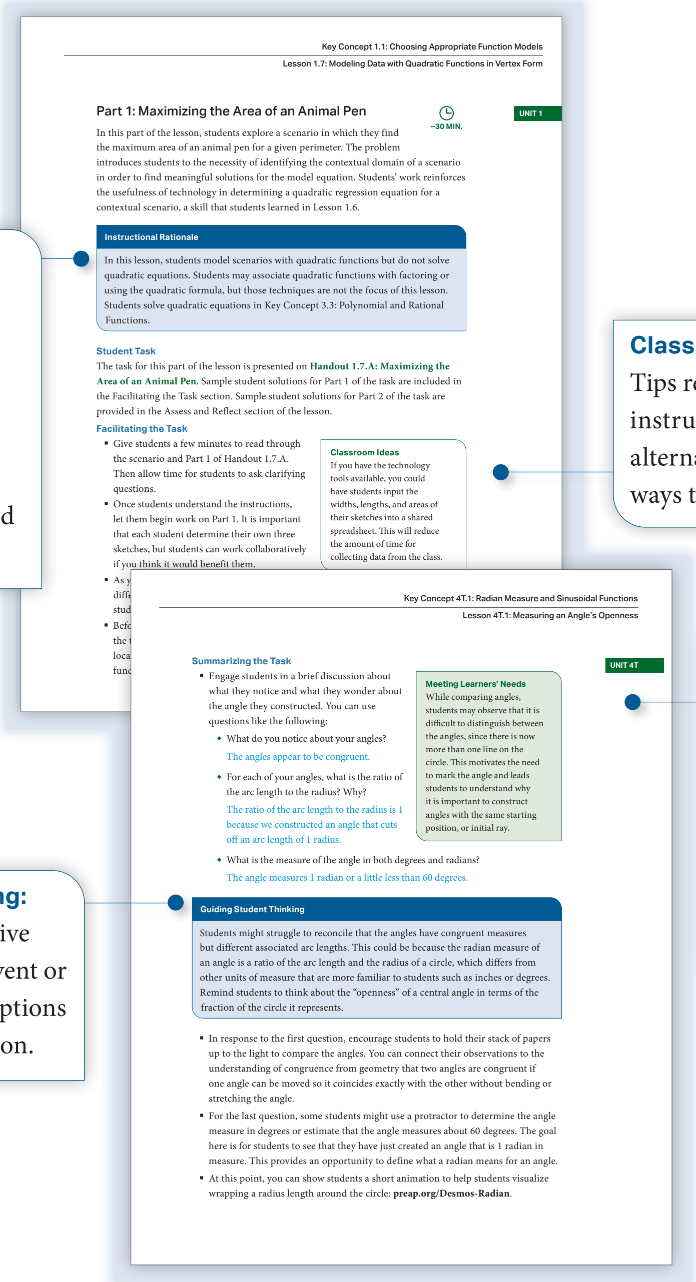
Teachers have the option to use any or all model lessons alongside their own locally developed instructional resources. Model lessons target content areas that tend to be challenging for teachers and students. While the lessons are distributed throughout all four units, they are concentrated more heavily in the beginning of the course to support teachers and students in establishing a strong foundation in the Pre-AP approach to teaching and learning.

SUPPORT FEATURES IN MODEL LESSONS

The following support features recur throughout the Pre-AP Algebra 2 lessons to promote teacher understanding of the lesson design and provide direct-to-teacher strategies for adapting lessons to meet their students' needs:

- Instructional Rationale
- Guiding Student Thinking
- Classroom Ideas
- Meeting Learners' Needs

Instructional Rationale: Insight into the strategic design and purpose of the instructional choices, flow, and scaffolding within the model lesson. Rationales often describe how a concept is continued later in the lesson or unit.



Classroom Ideas: Tips related to the logistics of the instruction, such as suggestions for alternative presentation methods, or ways to alleviate pacing concerns.

Meeting Learners' Needs: Optional differentiation strategies to address diverse learning needs, such as ideas for just-in-time skill building during a lesson or ways to break a task into smaller tasks, if needed, to make it more accessible.

Guiding Student Thinking: Ways to facilitate productive student thinking and prevent or address student misconceptions in critical areas of the lesson.

Pre-AP Algebra 2 Assessments for Learning

Pre-AP Algebra 2 assessments function as components of the teaching and learning cycle. Progress is not measured by performance on any single assessment. Rather, Pre-AP Algebra 2 offers a place to practice, to grow, and to recognize that learning takes time. The assessments are updated and refreshed periodically.

LEARNING CHECKPOINTS

Based on the Pre-AP Algebra 2 Course Framework, the learning checkpoints require students to examine data, models, diagrams, and short texts—set in authentic contexts—in order to respond to a targeted set of questions that measure students' application of the key concepts and skills from the unit. All eight learning checkpoints are automatically scored, with results provided through feedback reports that contain explanations of all questions and answers as well as individual and class views for educators. Teachers also have access to assessment summaries on Pre-AP Classroom, which provide more insight into the question sets and targeted learning objectives for each assessment.

The following tables provide a synopsis of key elements of the Pre-AP Algebra 2 learning checkpoints.

Format	Two learning checkpoints per unit Digitally administered with automated scoring and reporting Questions target both concepts and skills from the course framework
Time Allocated	Designed for one 45-minute class period per assessment
Number of Questions	10–12 questions per assessment ▪ 7–9 four-option multiple choice ▪ 3–5 technology-enhanced questions

Domains Assessed	
Learning Objectives	Learning objectives within each key concept from the course framework
Areas of Focus	Three skill categories aligned to the Pre-AP mathematics areas of focus are assessed regularly across all eight learning checkpoints: ▪ greater authenticity of applications and modeling ▪ engagement in mathematical argumentation ▪ connections among multiple representations

Question Styles	Question sets consist of two or three questions that focus on a single stimulus, such as a diagram, graph, or table. Questions embed mathematical concepts in real-world contexts. *Please see page 66 for a sample question set that illustrates the types of questions included in Pre-AP learning checkpoints and the Pre-AP final exam.*

PERFORMANCE TASKS

Each unit includes one performance-based assessment designed to evaluate the depth of student understanding of key concepts and skills that are not easily assessed in a multiple-choice format.

These tasks, developed for high school students across a broad range of readiness levels, are accessible while still providing sufficient challenge and the opportunity to practice the analytical skills that will be required in AP mathematics courses and for college and career readiness. Teachers participating in the official Pre-AP Program will receive access to online learning modules to support them in evaluating student work for each performance task.

Format	One performance task per unit May be administered online or in print ▪ If administered online, then a score report is available. Educator scored using scoring guidelines
Time Allocated	Approximately 45 minutes or as indicated
Number of Questions	An open-response task with multiple parts

Domains Assessed	
Key Concepts	Key concepts and prioritized learning objectives from the course framework
Skills	Three skill categories aligned to the Pre-AP mathematics areas of focus: ▪ greater authenticity of applications and modeling ▪ engagement in mathematical argumentation ▪ connections among multiple representations

PRACTICE PERFORMANCE TASKS

A practice performance task in each unit provides students with the opportunity to practice applying skills and knowledge in a context similar to a performance task, but in a more scaffolded environment. These tasks include strategies for adapting instruction based on student performance and ideas for modifying or extending tasks based on students' needs.

SAMPLE PERFORMANCE TASK AND SCORING GUIDELINES

The following task and set of scoring guidelines are representative of what students and educators will encounter on the performance tasks. (The example below is a practice performance task in Unit 2.)

PRACTICE PERFORMANCE TASK

Using Transformations to Model a Lion's Location

LEARNING OBJECTIVES

2.2.1 Compare a function f with an additive transformation of f, that is, $f(x+k)$ or $f(x)+k$.

2.2.2 Compare a function f with a multiplicative transformation of f, that is, $f(kx)$ or $k \cdot f(x)$.

2.2.3 Construct a representation of the transformation of a function.

PRACTICE PERFORMANCE TASK DESCRIPTION

In this practice performance task, students explore a real-world scenario in which scientists track the distance a lion travels in a day. Students write a function to model the lion's distance from a radio receiver and then use function transformation to write a new function that converts the units of measure for both the independent and dependent variables. This practice performance task provides students with an opportunity to explore additive and multiplicative transformations in a real-world context. Transformations that are applied to the output of a function can be thought of as a composition of functions in which the output of the original function becomes the input of the transformation function. In transformations that are applied to the input of a function, by contrast, the output of the transformation function becomes the input of the original function.

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Engagement in Mathematical Argumentation
- Connections Among Multiple Representations

SUGGESTED TIMING

~45 minutes

MATERIALS

- calculator (optional)

HANDOUT

- Unit 2 Practice Performance Task: Using Transformations to Model a Lion's Location

AP Connections

This performance task supports AP preparation through alignment to the following AP Calculus Course Skills:

- **2.B** Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.
- **3.E** Provide reasons or rationales for solutions and conclusions.

ELICITING PRIOR KNOWLEDGE

The goal of this task is for students to demonstrate their understanding of using transformations and composition to construct a function that models a contextual scenario.

- To begin, introduce students to the practice performance task. An excerpt from the student handout of the task is shown in the Scoring Student Work section.
- To prepare students to engage in the task, you could ask the following warm-up problems:
  - Consider a general function $y = f(x)$. Compare the transformation $f(x)+1$ to the function $y = f(x)$.

    The transformation $f(x)+1$ is an additive transformation of *f*. It has the effect of adding 1 to the output of *f*. It can also be interpreted as a composition of *f* and $g(x) = x+1$ such that $f(x)+1 = g(f(x))$. The graph of $y = f(x)+1$ looks like the graph of $y = f(x)$ translated vertically up by 1 unit.
  - Compare the transformation $f(x+1)$ to the function $y = f(x)$.

    The transformation $f(x+1)$ is an additive transformation of *f*. It has the effect of adding 1 to each input value before *f* is applied. It can also be interpreted as a composition of *f* and $g(x) = x+1$ such that $f(x+1) = f(g(x))$. The graph of $y = f(x+1)$ looks like the graph of $y = f(x)$ translated horizontally to the left by 1 unit.
  - Compare the transformation $2f(x)$ to the function $y = f(x)$.

    The transformation $2f(x)$ is a multiplicative transformation of *f*. It has the effect of multiplying each output of *f* by 2. It can also be interpreted as a composition of *f* and $g(x) = 2x$ such that $2f(x) = g(f(x))$. The graph of $y = 2f(x)$ looks like the graph of $y = f(x)$ vertically dilated by a factor of 2.
  - Compare the transformation $f(2x)$ to the function $y = f(x)$.

    The transformation $f(2x)$ is a multiplicative transformation of *f*. It has the effect of multiplying each input value by 2 before *f* is applied. It can also be interpreted as a composition of *f* and $g(x) = 2x$ such that $f(2x) = f(g(x))$. The graph of $y = f(2x)$ looks like the graph of $y = f(x)$ dilated horizontally by a factor of $\frac{1}{2}$.

Meeting Learners' Needs

If you find that students need more support to engage with the warm-up problems, you can provide them with a simple, specific function like $f(x) = x^2$ or $f(x) = |x|$.

Meeting Learners' Needs

Students may find it helpful to make a table of values for each function to make sense of the order in which the transformation functions are composed and to keep track of the relationships between the inputs and outputs.

- If students struggle with the warm-up problems, it could indicate that they are not yet fully prepared to engage in the practice performance task. You may find it beneficial to provide a just-in-time review of the concepts critical for success with this task: function composition and transformations of functions.

SUPPORTING STUDENTS

Here are some possible implementation strategies you could use to help support students in engaging with the task.

- **Previewing the Task:** To support students in identifying key features of the problem, you could display the introductory text and graph and read them with the entire class. This particular practice performance task includes a great deal of information that students must read and analyze. Allow some time for students to ask clarifying questions about anything they read or observe, which may decrease the likelihood that they start working on the task with incorrect interpretations of the given information.
- **Collaboration:** To encourage students to engage in academic conversations in mathematics, you could have them work in pairs to complete the task. It is not recommended that students work in groups of three or more. While there is ample work and enough potential discussion areas for two students, some students in groups of more than two may not have an opportunity to engage meaningfully in all parts of the task.
- **Chunking the Task:** To support students who struggle with time management or may be overwhelmed by large tasks, you could chunk the task into several parts. For this task, parts (a) and (b) could be completed together or separately. Parts (c) and (d) would be best completed together, and part (e) could be completed separately from the other parts.
- **Iteration Support:** If you choose to chunk the task, then be sure to spend a few moments discussing the components of the solution with students during each teacher-facilitated check. Focus on what revisions, if any, they could make to their solution to craft a more complete response. Parts (a) and (b) of the task require a written explanation, which students sometimes forget or neglect to include in their response.

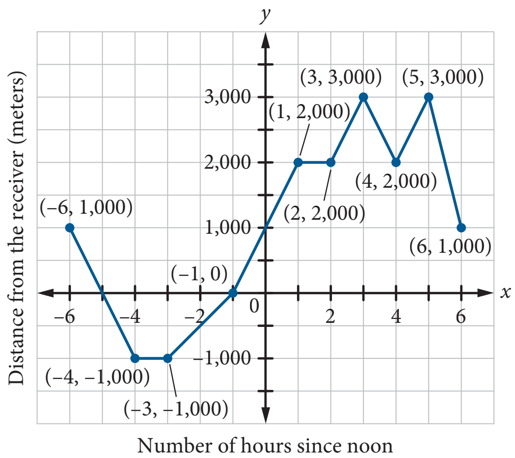
SCORING STUDENT WORK

Whether you decide to have students score their own solutions, have students score their classmates' solutions, or score the solutions yourself, you should use the results of the practice performance task to inform further instruction.

Using Transformations to Model a Lion's Location

African wildlife ecologists study lion populations' behaviors, movements, and interactions in order to develop appropriate conservation plans to better protect these declining populations. This work can be challenging as these large carnivores utilize vast territories across many African countries, such as Botswana, Namibia, Zambia, and Zimbabwe. To monitor the lions' movements, ecologists tag the animals with high-tech collars equipped with radio transmitters that send signals to a stationary receiver, which allows their movements to be tracked and plotted.

One particular ecologist tracks the location of a lion relative to the receiver over the course of one day. From her data, she assigns the number of hours since noon (12 p.m.) as the independent variable and the lion's distance from the receiver (in meters) as the dependent variable. She records the lion's distance from the receiver at different times and constructs a graph of a function model, f, shown in the following figure:



Her colleague also constructs a model based on observations of the same lion over the same time period but using different units of measure. So that the two ecologists can compare their models, the first ecologist needs to construct transformations of her function so that the unit of measure for the independent variable is the number of hours since midnight (12 a.m.) and the unit of measure for the dependent variable is kilometers.

There are 12 possible points for this practice performance task.

Student Stimulus and Part (a)

(a) Consider the transformation of the unit of measure for the dependent variable described in the third paragraph. Use the points on the graph of f to write an algebraic representation of a transformation of f such that the dependent variable (the output) of the transformation of f is the distance of the lion from the receiver in kilometers. Explain whether the transformation is an additive or multiplicative transformation. What effect would this transformation have on the graph of f?

Sample Solution	Points Possible
Constructing a transformation of f so that the unit of measure of its dependent variable is kilometers requires dividing the output of f by 1,000. As a result, the output of the transformation yields values in kilometers rather than in meters. The algebraic representation of the transformation of f is $\frac{1}{1{,}000}f(x)$. It is a multiplicative transformation because the transformation involves multiplying the values of the dependent variable by $\frac{1}{1{,}000}$ (or, equivalently, dividing by 1,000). This transformation would have the effect of vertically dilating the graph of f by a factor of $\frac{1}{1{,}000}$. *Scoring note:* Students can receive the third point if they correctly describe the effect on the graph of their incorrect transformation.	**3 points maximum** 1 point for providing a correct algebraic representation of a transformation of f 1 point for providing a correct explanation of why the transformation is a multiplicative transformation 1 point for providing a correct description of the effect the transformation would have on the graph of f

Student Stimulus and Part (b)

(b) Consider the transformation of the unit of measure for the independent variable described in the third paragraph. Use the points given on the graph of f to write an algebraic representation of a transformation of f such that the independent variable (the input) of the transformation is the number of hours since midnight. Explain whether the transformation is an additive or multiplicative transformation. What effect would this transformation have on the graph of f?

Sample Solution	Points Possible
Constructing a transformation of f so that the unit of measure of its independent variable is the number of hours since midnight requires subtracting 12 from the input of the transformation so that it corresponds with the input of f. The algebraic representation of the transformation of f is $f(x-12)$. For example, the ordered pair $(0, 1{,}000)$ for f means that at noon, the lion is 1,000 meters from the receiver. For the transformation of function f, the independent variable should be the number of hours since midnight. Because noon is 12 hours since midnight, an input of 12 should be associated with an output of 1,000 so that the ordered pair (12, 1,000) should be a solution to the transformation of f. Evaluating the function $f(x-12)$ at $x=12$ yields the same output as $x=0$: $f(12-12)=f(0)=1{,}000$. This is an additive transformation because the independent variable is transformed by addition. The transformation would have the effect of horizontally translating the graph of f to the right 12 units. *Scoring note:* Students can receive the third point if they correctly describe the effect on the graph of their incorrect transformation.	**3 points maximum** 1 point for providing a correct algebraic representation of a transformation of f 1 point for providing a correct explanation of why the transformation is an additive transformation 1 point for providing a correct description of the effect the transformation would have on the graph of f

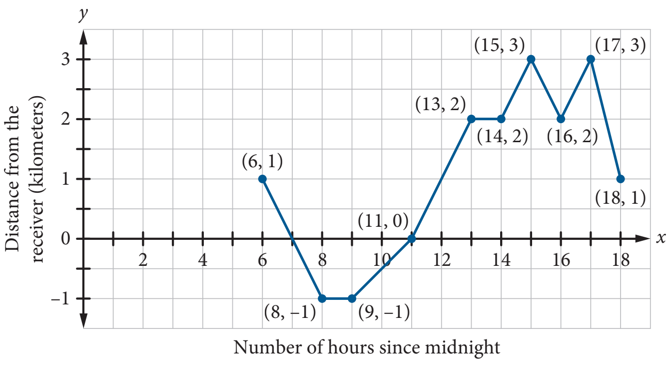
Student Stimulus and Part (c)

(c) Let g be a function that models the relationship between the number of hours since midnight and the lion's distance from the receiver in kilometers. Use your transformations from parts (a) and (b) to write an algebraic representation of g in terms of f.

Sample Solution	Points Possible
An algebraic representation of g that uses the transformations of f from parts (a) and (b) is $g(x)=\frac{1}{1{,}000}f(x-12)$. *Scoring note:* Students can receive this point if they correctly combine the transformations they wrote in parts (a) and (b), even if one or both of those transformations are incorrect.	**1 point maximum** 1 point for combining the transformations of f from parts (a) and (b) to form function g

Student Stimulus and Part (d)

(d) Sketch a graph of the function g that you constructed in part (c) using the coordinates given on the graph of f.

Sample Solution	Points Possible
 A graph of g is shown in the figure. *Scoring note:* Students can receive full credit for this part if their graph for function g corresponds to their algebraic representation of g from part (c), even if the algebraic representation is incorrect.	**2 points maximum** 1 point for representing the horizontal translation 1 point for representing the vertical dilation

Student Stimulus and Part (e)

(e) Identify the intercepts and interpret their meaning in the graph of g. How does this meaning compare to the meaning of the x-intercepts of the graph of f?

Sample Solution	Points Possible
The x-intercepts of the graph of g are $(7,\ 0)$ and $(11,\ 0)$. This means that at 7 hours since midnight, or 7 a.m., and at 11 hours since midnight, or 11 a.m., the lion is 0 kilometers from the receiver. The values of the x-intercepts are different for the graph of f, but the meanings are the same. The x-intercepts for the graph of f are $(-5,\ 0)$ and $(-1,\ 0)$. These correspond to 5 hours before noon, or 7 a.m., and one hour before noon, or 11 a.m. The lion is 0 meters from the receiver at these times.	**3 points maximum** 1 point for identifying the x-intercepts of the graph of g 1 point for providing a correct interpretation of the meaning of the x-intercepts of g with correct units 1 point for providing a correct comparison of the x-intercepts for the graphs of f and g

PROVIDING FEEDBACK ON STUDENT WORK

After scoring your students' work, it is important to identify trends in their responses to inform further instruction. These trends should include topics that students consistently displayed mastery of, as well as conceptual errors that students commonly made. Possible trends and suggested guidance for each part of the task follow, although the patterns you observe in your classroom may differ.

(a) If students have difficulty identifying whether they should multiply or divide by 1,000 to convert from meters to kilometers, it could be helpful to have them make a table of values with equivalent measurements in both units to look for a pattern. Also, students may find a graphing utility to be a useful tool for identifying the effect a multiplicative transformation has on the graph of a function.

Teacher Notes and Reflections

(b) If students have difficulty identifying whether they should add or subtract 12 to the input of the function transformation, it may be helpful to have them make a table of values with different times of day represented using both units of measure. Also, students may find a graphing utility to be a useful tool for identifying the effect of an additive transformation on the graph of a function.

Teacher Notes and Reflections

(c) Students may have difficulty using function composition to combine the two transformed functions into a single transformed function model. They may benefit from creating a table of values with columns for hours since midnight, hours since noon, distance in meters, and distance in kilometers to see how the desired input (number of hours since midnight) can be associated with the desired output (distance in kilometers) using the given data.

Teacher Notes and Reflections

(d) If students cannot produce a correct graph by analyzing the graph of f and their algebraic representation of the function g, they may benefit from using a dynamic graphing utility to look for patterns in the effect that different transformations have on the graph of function f.

Teacher Notes and Reflections

(e) Some students could experience difficulty interpreting the meaning of the x-intercepts of the graphs because they do not see the connections between the two representations. These students may benefit from some additional practice interpreting a graph generated from a different contextual scenario.

Teacher Notes and Reflections

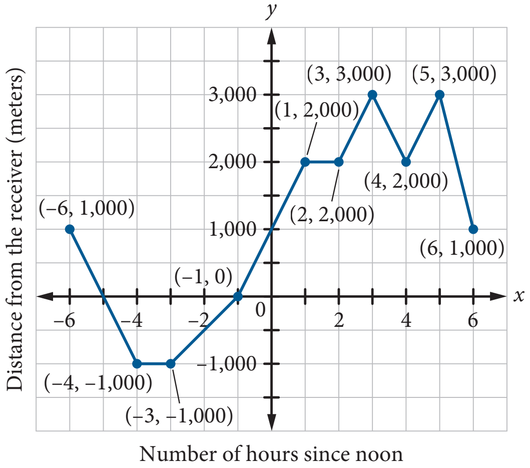
Assure students that converting their score into a percentage does not provide an accurate measure of how they performed on the task. You can use the following suggested score interpretations with students to discuss their performance.

Points Received	How Students Should Interpret Their Score
11 or 12 points	"I know all of these algebraic concepts really well. This is top-level work. (A)"
8 to 10 points	"I know all of these algebraic concepts well, but I made a few mistakes. This is above-average work. (B)"
5 to 7 points	"I know some of these algebraic concepts well, but not all of them. This is average-level work. (C)"
2 to 4 points	"I know only a little bit about these algebraic concepts. This is below-average work. (D)"
0 or 1 point	"I don't know much about these algebraic concepts at all. This is not passing work. (F)"

Using Transformations to Model a Lion's Location

African wildlife ecologists study lion populations' behaviors, movements, and interactions in order to develop appropriate conservation plans to better protect these declining populations. This work can be challenging as these large carnivores utilize vast territories across many African countries, such as Botswana, Namibia, Zambia, and Zimbabwe. To monitor the lions' movements, ecologists tag the animals with high-tech collars equipped with radio transmitters that send signals to a stationary receiver, which allows their movements to be tracked and plotted.

One particular ecologist tracks the location of a lion relative to the receiver over the course of one day. From her data, she assigns the number of hours since noon (12 p.m.) as the independent variable and the lion's distance from the receiver (in meters) as the dependent variable. She records the lion's distance from the receiver at different times and constructs a graph of a function model, *f*, shown in the following figure:



Her colleague also constructs a model based on observations of the same lion over the same time period but using different units of measure. So that the two ecologists can compare their models, the first ecologist needs to construct transformations of her function so that the unit of measure for the independent variable is the number of hours since midnight (12 a.m.) and the unit of measure for the dependent variable is kilometers.

(a) Consider the transformation of the unit of measure for the dependent variable described in the third paragraph. Use the points on the graph of f to write an algebraic representation of a transformation of f such that the dependent variable (the output) of the transformation of f is the distance of the lion from the receiver in kilometers. Explain whether the transformation is an additive or multiplicative transformation. What effect would this transformation have on the graph of f?

(b) Consider the transformation of the unit of measure for the independent variable described in the third paragraph. Use the points given on the graph of f to write an algebraic representation of a transformation of f such that the independent variable (the input) of the transformation is the number of hours since midnight. Explain whether the transformation is an additive or multiplicative transformation. What effect would this transformation have on the graph of f?

(c) Let g be a function that models the relationship between the number of hours since midnight and the lion's distance from the receiver in kilometers. Use your transformations from parts (a) and (b) to write an algebraic representation of g in terms of f.

(d) Sketch a graph of the function g that you constructed in part (c) using the coordinates given on the graph of f.

(e) Identify the intercepts and interpret their meaning in the graph of g. How does this meaning compare to the meaning of the x-intercepts of the graph of f?

FINAL EXAM

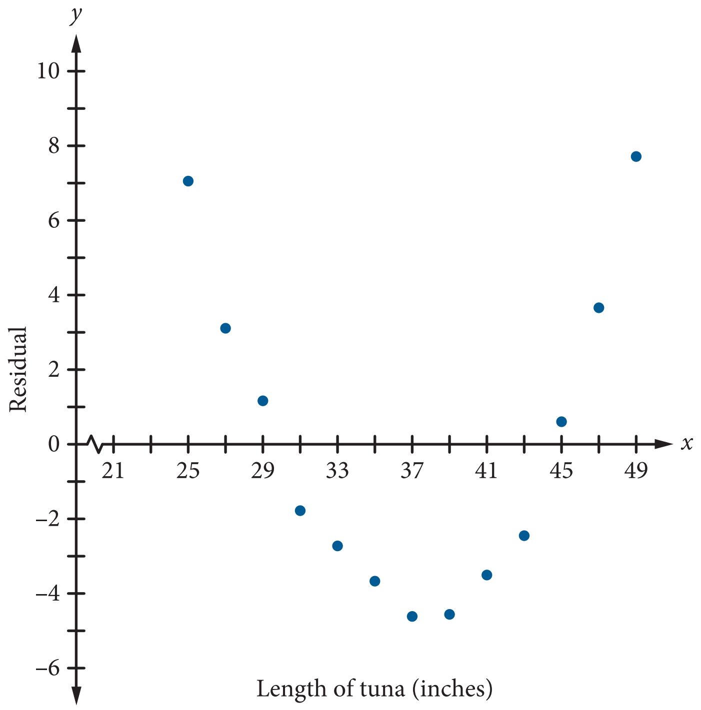
Starting in the school year of 2023–24, Pre-AP Algebra 2 will include a final exam featuring multiple-choice and technology-enhanced questions as well as an open-response question. The final exam will be a summative assessment designed to measure students' success in learning and applying the knowledge and skills articulated in the Pre-AP Algebra 2 Course Framework. The final exam's development will follow best practices such as multiple levels of review by educators and experts in the field for content accuracy, fairness, and sensitivity. The questions on the final exam will be pretested, and the resulting data will be collected and analyzed to ensure that the final exam is fair and represents an appropriate range of the knowledge and skills of the course.

The final exam will be delivered on a secure digital platform in a classroom setting. Educators will have the option of administering the final exam in a single extended session or in two shorter consecutive sessions to accommodate a range of final exam schedules.

Multiple-choice and technology-enhanced questions will be delivered digitally and scored automatically with detailed score reports available to educators. This portion of the final exam will build on the question styles and formats of the learning checkpoints; thus, in addition to their formative purpose, the learning checkpoints provide practice and familiarity with the final exam. The open-response question, modeled after the performance tasks, will be delivered as part of the digital final exam but scored locally by educators.

SAMPLE ASSESSMENT QUESTIONS

The following questions are representative of what students and educators will encounter on the learning checkpoints and final exam.



A particular fishing company records the length, in inches, and the weight, in pounds, of the yellowfin tuna that they catch. To determine if there is a linear relationship between the length and weight, the company constructs a regression equation, whose residual plot is shown in the figure. Which of the following conclusions about the relationship between the length and weight of the tuna is best supported using the information presented in the residual plot?

(A) A linear model is appropriate for the relationship between the length and weight because the residual plot displays a quadratic pattern.

(B) A linear model is appropriate for the relationship between the length and weight because the residual plot displays no observable pattern.

(C) A linear model is not appropriate for the relationship between the length and weight because the residual plot displays a pattern.

(D) A linear model is not appropriate for the relationship between the length and weight because the residual plot displays no observable pattern.

Assessment Focus

In problem 1, students evaluate the appropriateness of using a linear equation to model the relationship between two variables by determining if the associated residual plot for the function model displays a discernable pattern.

Correct Answer: C

Learning Objective:

1.1.2 Use residual plots to determine whether a function model appropriately models a data set.

Area of Focus: Engagement in Mathematical Argumentation

The function h is defined as $h(x) = \sqrt{x^2 + 1}$. If h can be formed by the composition of functions f and g such that $h(x) = f(g(x))$, then which TWO of the following pairs of functions could be f and g?

(A) $f(x) = x^2 + 1$ and $g(x) = \sqrt{x}$

(B) $f(x) = \sqrt{x+1}$ and $g(x) = x^2$

(C) $f(x) = x^2$ and $g(x) = \sqrt{x+1}$

(D) $f(x) = \sqrt{x-1}$ and $g(x) = x^2 + 2$

Assessment Focus

In problem 2, students decompose a composite function into two individual functions. The multiple responses required to correctly answer the question reinforces for students that there are multiple correct ways to decompose a function.

Correct Answer: B and D

Learning Objective:

2.1.3 Express a given algebraic representation of a function in an equivalent form as the composition of two or more functions.

Area of Focus: Connections Among Multiple Representations

Let the functions f and g be defined as follows: $f(x) = \log_b(x)$ and $g(x) = \log_b(x - c)$, where c is a positive real number. Which of the following statements correctly describes the relationship between the graphs of $y = f(x)$ and $y = g(x)$?

(A) The graph of $y = g(x)$ coincides with the graph of $y = f(x)$ for all values of $x > c$.

(B) The graph of $y = g(x)$ is a dilation of the graph of $y = f(x)$ by a factor of c.

(C) The graph of $y = g(x)$ is a vertical translation of the graph of $y = f(x)$ down c units.

(D) The graph of $y = g(x)$ is a horizontal translation of the graph of $y = f(x)$ to the right c units.

Assessment Focus

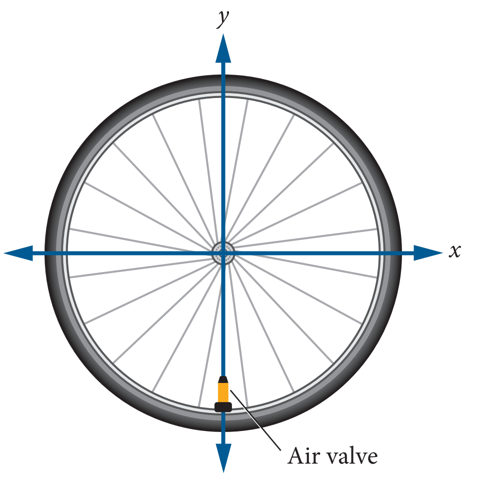
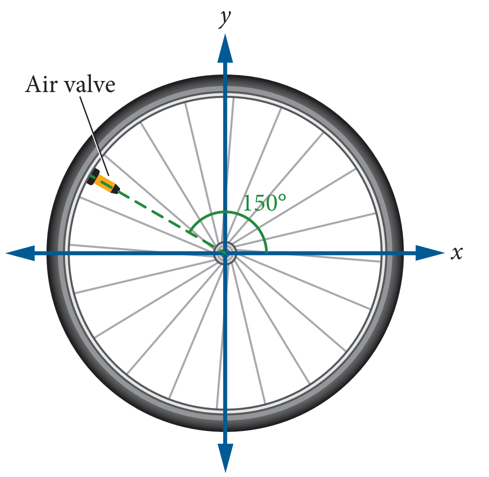
In problem 3, students compare the algebraic forms of two related logarithmic functions to determine the relationship between their graphs.

Correct Answer: D

Learning Objective:

3.1.3 Construct a representation of a logarithmic function.

Area of Focus: Connections Among Multiple Representations

Starting Position	Final Position
	

A bicycle tire with a diameter of 700 millimeters is shown at its starting and final positions. At the start of a bicycle ride, the front tire's air valve is positioned at the bottom of the tire. After riding the bicycle, the final position of the air valve forms an angle in standard position, relative to the horizontal axis of the bicycle tire, that measures 150°. Which of the following expressions represents the height of the air valve from the ground?

(A) $350\cos\left(\frac{5\pi}{6}\right)$

(B) $350+350\cos\left(\frac{5\pi}{6}\right)$

(C) $350\sin\left(\frac{5\pi}{6}\right)$

(D) $350+350\sin\left(\frac{5\pi}{6}\right)$

Assessment Focus

In problem 4, students are expected to apply their knowledge of the relationship between the coordinates of a point on a circle and a related central angle to answer a question in a real-world context.

Correct Answer: D

Learning Objective:

4T.1.4 Determine the exact coordinates of any point on a circle centered at the origin.

Area of Focus: Greater Authenticity of Applications and Modeling

Pre-AP Algebra 2 Course Designation

Schools can earn an official Pre-AP Algebra 2 Course Designation by meeting the requirements summarized below. Pre-AP Course Audit Administrators and teachers will complete a Pre-AP Course Audit process to attest to these requirements. All schools offering courses that have received a Pre-AP Course Designation will be listed in the Pre-AP Course Ledger, in a process similar to that used for listing authorized AP courses.

PROGRAM REQUIREMENTS

- The school ensures that Pre-AP course frameworks and assessments serve as the foundation for all sections of the course at the school. This means that the school must not establish any barriers (e.g., test scores, grades in prior coursework, teacher or counselor recommendation) to student access and participation in the Pre-AP Algebra 2 coursework.
- Teachers have read the most recent *Pre-AP Algebra 2 Course Guide*.
- Teachers administer each performance task and at least one of two learning checkpoints per unit.
- Teachers and at least one administrator per site complete a Pre-AP Summer Institute or the Online Foundational Module Series. Teachers complete at least one Online Performance Task Scoring Module.
- Teachers align instruction to the Pre-AP Algebra 2 Course Framework and ensure their course meets the curricular requirements summarized below.
- The school ensures that the resource requirements summarized below are met.
- Please note if a state's standards do not require trigonometry in Algebra 2, then completing the Pre-AP requirements for Unit 4T are optional for teachers in that state. These requirements include aligning instruction to the Unit 4T learning objectives in the course framework, and completing the learning checkpoints and performance task assessments.

CURRICULAR REQUIREMENTS

- The course provides opportunities for students to develop an understanding of the Pre-AP Algebra 2 key concepts and skills articulated in the course framework through the four units of study.
- The course provides opportunities for students to engage in the Pre-AP shared instructional principles.
  - close observation and analysis
  - evidence-based writing

- higher-order questioning
- academic conversation

- The course provides opportunities for students to engage in the three Pre-AP mathematics areas of focus. The areas of focus are:
  - greater authenticity of applications and modeling
  - engagement in mathematical argumentation
  - connections among multiple representations
- The instructional plan for the course includes opportunities for students to continue to practice and develop disciplinary skills.
- The instructional plan reflects time and instructional methods for engaging students in reflection and feedback based on their progress.
- The instructional plan reflects making responsive adjustments to instruction based on student performance.

RESOURCE REQUIREMENTS

- The school ensures that participating teachers and students are provided computer and internet access for completion of course and assessment requirements.
- Teachers should have consistent access to a video projector for sharing web-based instructional content and short web videos.

Accessing the Digital Materials

Pre-AP Classroom is the online application through which teachers and students can access Pre-AP instructional resources and assessments. The digital platform is similar to AP Classroom, the online system used for AP courses.

Pre-AP coordinators receive access to Pre-AP Classroom via an access code delivered after orders are processed. Teachers receive access after the Pre-AP Course Audit process has been completed.

Once teachers have created course sections, students can enroll in them via a join code. When both teachers and students have access, teachers can share instructional resources with students, assign and score assessments, and complete online learning modules; students can view resources shared by the teacher, take assessments, and receive feedback reports to understand progress and growth.

Unit 3

Unit 3
Function Families

Overview

SUGGESTED TIMING: APPROXIMATELY 9 WEEKS

Explorations of function families are an important component of any Algebra 2 course because they expand the repertoire of functions students can draw upon to model real-world phenomena. Not all phenomena are appropriately modeled by a linear, quadratic, or exponential function. For example, the gravitational force between two objects is inversely proportional to the square of their distance apart. This relationship would be best modeled with a rational function, one of the functions introduced in Unit 3. Throughout this unit, students learn that a parent function and its transformations form a function family. All functions in the same function family share some properties with each other.

Because function families are traditionally taught as isolated topics, students rarely have time to thoroughly investigate which properties of a function family are maintained by transformations and which are not. Therefore, the key concepts in this unit intentionally focus students' thinking on how function families are related in meaningful ways. The structure of the unit is intended to help students construct a network of connections among these function families. As with all explorations of functions throughout Pre-AP, the emphasis is on contextual scenarios that can be effectively modeled by each function family.

Technology Note: There are many different technologies available for students to use to construct scatterplots, determine regression equations, and create graphs of functions. Because facility with technology tools is an integral part of developing a deep conceptual understanding of the mathematical concepts in this course, you should select the technology tool with which you are most comfortable from the tools that are available to your students. The model lessons use Desmos, which is a free, online graphing tool available at **Desmos.com**. Different technology tools use different methods for calculating regression equations and for graphing functions. Where relevant, the model lessons alert you to possible discrepancies between the output displayed by Desmos and the output displayed by other common technology tools.

ENDURING UNDERSTANDINGS

This unit focuses on the following enduring understandings:

- A function is a special mathematical relationship between two variables that can often be used to make sense of observable patterns in contextual scenarios.
- Functions in a family have similar properties, similar algebraic representations, and graphs that share key features.

KEY CONCEPTS

This unit focuses on the following key concepts:

- 3.1: Exponential and Logarithmic Functions
- 3.2: Polynomial and Rational Functions
- 3.3: Square Root and Cube Root Functions

UNIT RESOURCES

The tables below outline the resources provided by Pre-AP for this unit.

Lessons for Key Concept 3.1: Exponential and Logarithmic Functions			
Lesson Title	**Learning Objectives Addressed**	**Suggested Timing**	**Areas of Focus**
3.1: Problem Set for Exponential Functions	3.1.1, 3.1.2	~45 minutes	Connections Among Multiple Representations, Engagement in Mathematical Argumentation, Greater Authenticity of Applications and Modeling
3.2: Introduction to the Logarithm Function	3.1.3, 3.1.4, 3.1.5	~125 minutes	Greater Authenticity of Applications and Modeling, Connections Among Multiple Representations

Lessons for Key Concept 3.1: Exponential and Logarithmic Functions			
Lesson Title	**Learning Objectives Addressed**	**Suggested Timing**	**Areas of Focus**
3.3: Connecting Properties of Logarithms with Transformations of the Graph of the Parent Logarithm Function	3.1.3, 3.1.4	~150 minutes	Connections Among Multiple Representations, Engagement in Mathematical Argumentation
3.4: Applications of Logarithms	3.1.4, 3.1.6	~90 minutes	Engagement in Mathematical Argumentation, Greater Authenticity of Applications and Modeling
	All learning objectives from this key concept are addressed with the provided materials.		

Practice Performance Task for Unit 3 (~45 minutes)
This practice performance task assesses learning objectives from Key Concept 3.1.

Learning Checkpoint 1: Key Concept 3.1 (~45 minutes)
This learning checkpoint assesses learning objectives from Key Concept 3.1. For sample items and learning checkpoint details, visit Pre-AP Classroom.

UNIT 3

Lessons for Key Concept 3.2: Polynomial and Rational Functions

Lesson Title	Learning Objectives Addressed	Suggested Timing	Areas of Focus
3.5: A Field Guide to Polynomial Functions	3.2.1, 3.2.2, 3.2.3	~45 minutes	Engagement in Mathematical Argumentation, Connections Among Multiple Representations

The following Key Concept 3.2 learning objectives are not addressed in Pre-AP lessons. They should be addressed in teacher-developed materials.

- Learning Objectives: 3.2.4, 3.2.5, 3.2.6

Performance Task for Unit 3 (~45 minutes)
This performance task assesses learning objectives from Key Concept 3.2.

Lessons for Key Concept 3.3: Square Root and Cube Root Functions
There are no provided Pre-AP lessons for this key concept. As with all key concepts, this key concept is addressed in a learning checkpoint.
All Key Concept 3.3 learning objectives should be addressed with teacher-developed materials.

Learning Checkpoint 2: Key Concepts 3.2 and 3.3 (~45 minutes)
This learning checkpoint assesses learning objectives from Key Concepts 3.2 and 3.3. For sample items and learning checkpoint details, visit Pre-AP Classroom.

LESSON 3.1

Problem Set for Exponential Functions

LEARNING OBJECTIVES

3.1.1 Construct a representation of an exponential function using the natural base, *e*.

3.1.2 Express an exponential function in an equivalent form to reveal properties of the graph and/or the contextual scenario.

LESSON OVERVIEW

CONTENT FOCUS

This problem set is intended to provide students with some additional practice opportunities for working with exponential functions that use the natural base, *e*. The natural base is often used by scientists and mathematicians to model scenarios that involve continuous growth or decay. The problems in this set require students to manipulate the algebraic representations of exponential functions and express them in equivalent forms with different bases. Several of these problems also require students to construct an exponential function to model a contextual scenario and use the function to answer questions about the scenario.

AREAS OF FOCUS

- Connections Among Multiple Representations
- Engagement in Mathematical Argumentation
- Greater Authenticity of Applications and Modeling

SUGGESTED TIMING

~60 minutes

MATERIALS

- graphing utility such as **Desmos.com**

HANDOUT

Lesson

- 3.1: Exponential Function Models

LESSON DESCRIPTION

This problem set is intended to provide students with some additional practice opportunities for working with exponential functions that use the natural base, *e*. The natural base is often used by scientists and mathematicians to model scenarios that involve continuous growth or decay. The problems in this set require students to manipulate the algebraic representations of exponential functions and express them in equivalent forms with different bases. Several of these problems also require students to construct an exponential function to model a contextual scenario and use the function to answer questions about the scenario.

FORMATIVE ASSESSMENT GOAL

This lesson prepares students to complete the following formative assessment activity.

> Ishi performs an experiment to investigate how quickly a certain strain of bacteria reproduces in a controlled environment. She plates 10 bacteria colonies in a Petri dish at the start of the experiment, which she identifies as Day 0. On Day 4 of the experiment, she observes that there are 20 bacteria colonies in the Petri dish. She reasons that the population of bacteria is doubling every 4 days. She predicts that if the bacteria growth continues with the same growth factor, then there should be 40 bacteria colonies on Day 8. Assuming that the bacteria grow continuously, write an exponential function of the form $f(x) = a \cdot e^{kx}$ to model the number of bacteria colonies in the Petri dish after x days. Then use the function to estimate the number of bacteria colonies, to the nearest whole number, in the Petri dish on Day 10 and Day 20 of the experiment. Explain your reasoning.

Problem Set for Exponential Functions

~60 MIN.

This problem set is intended to provide students with some additional practice opportunities for working with exponential functions that use the natural base, *e*. The natural base is often used by scientists and mathematicians to model scenarios that involve continuous growth or decay. The problems in this set require students to manipulate the algebraic representations of exponential functions and express them in equivalent forms with different bases. Several of these problems also require students to construct an exponential function to model a contextual scenario and use the function to answer questions about the scenario.

Student Task

The student task is provided on **Handout 3.1: Exponential Function Models**. Solutions to the problem set are provided in the Assess and Reflect section of the lesson. Some instructional guidance is given in the Facilitating the Task section.

Facilitating the Task

- It is not intended that students complete the entire handout in one class period. Rather, the problem set is chunked into four parts that you can use at appropriate intervals during instruction involving exponential functions with the natural base, *e*.

Instructional Rationale

Part A of the handout is designed to help students see the connection between exponential functions written with the natural base and the growth or decay factor of the function. Expressing an exponential function with the natural base is useful in scientific contexts, but it can obscure some of the key features of the function.

- If you notice students struggling to answer problems 1 and 2, you may need to prompt them to remember some properties of exponents. You can use questions like the following:
  - Suppose I asked you to simplify the expression $\left(x^2\right)^5$. What would you do with the exponents? How do you know?

    I would multiply the exponents because a power raised to a power is equal to the base raised to the product of the exponents. The simplified expression is x^{10}.

- What if I asked you to simplify the expression $\left(x^4\right)^a$ where a is a constant? How would you deal with the exponents? How do you know?

  Assuming that a is a constant, then the expression would equal the base raised to the product of the exponents, or x^{4a}. This is because a power raised to a power is equal to the base raised to the product of the exponents.

- How can we use the power-to-a-power property of exponents to write an expression equivalent to e^{3t}? How do you know?

  Because $3t$ is a product, we could write the expression e^{3t} as a power raised to a power, so $e^{3t} = \left(e^3\right)^t$ or $e^{3t} = \left(e^t\right)^3$.

- It is important for students to answer problem 3 in Part A so that they develop a general formula for determining the growth or decay factor of an exponential function expressed with the natural base.

Instructional Rationale

Part B of the handout is designed to help students observe that a horizontal translation of an exponential function can also be expressed equivalently as a vertical dilation. This is an important relationship for students to establish because it will be useful when they explore connections between vertical translations and horizontal dilations of logarithmic functions in Lesson 3.3: Connecting Properties of Logarithms with Transformations of the Graph of the Parent Logarithm Function.

- If you notice students struggling to answer problems 4 and 5, you may need to prompt them to remember some exponent rules. You can ask questions like the following:
  - Suppose I asked you to simplify the expression $x^2 \cdot x^5$. To express this using a single base, what would you do with the exponents? How do you know?

    I would add the exponents because the product of expressions with the same base is equal to the same base raised to the sum of the exponents. The simplified expression is x^7.

  - What if I asked you to simplify the expression $x^4 \cdot x^a$ where a is a constant? To express this using a single base, what would you do with the exponents? How do you know?

    Assuming that a is a constant, then you would add the exponents to get x^{4+a}. This is because the product of expressions with the same base is equal to the same base raised to the sum of the exponents.

- How can we use the product property of exponents to rewrite e^{x+5}? How do you know?

  Because $x+5$ is a sum, we could write the expression e^{x+5} as the product of two powers with the same base. This means that $e^{x+5} = e^x \cdot e^5$.

Instructional Rationale

Part C of the handout is designed to help students explore how to express an exponential function with base *b* as an exponential function with base *e*. In this part of the problem set, students solve exponential equations with base *e*. As in Lesson 2.3, students need to understand that taking a logarithm is the inverse operation of exponentiating. It is not necessary to introduce students to the term *natural logarithm* or to the notation "ln". It is sufficient for students to take the logarithm base *e* on both sides of these equations to solve them. The natural logarithm is introduced in Lesson 3.2: Introduction to the Logarithm Function.

- For problems 7, 8, and 9, students may need some support to solve equations involving *e*. You may find it helpful to ask them some questions about the operations they used in Lesson 2.3 to solve equations involving exponentiating.
  - Suppose I asked you to solve the equation $2^x = 5$ for x. What would you do? Why would you do that?

    To solve the equation, I would take the logarithm base 2 of both sides, since taking a logarithm base 2 is the inverse operation of exponentiating with base 2.

  - How could you use that same method to solve an equation like $e^x = 5$? What is the same and what is different about this equation and the equation $2^x = 5$?

    Using the same method, I could take the logarithm base *e* of both sides of the equation because taking a logarithm base *e* is the inverse operation of exponentiating with base *e*. It is the same because both 2 and *e* are real numbers, but it feels different because *e* is irrational.

- It is important for students to answer problem 10 so that they can articulate the relationship between exponential functions with various bases. It is possible to express any exponential function in an equivalent form with any other base. That is, if *f* has the form $f(x) = b^x$, it can be expressed equivalently in the form $f(x) = c^{kx}$ where the value of k is $\log_c(b)$. Said another way, $f(x) = b^x = c^{x\log_c(b)}$.

Instructional Rationale

Part D of the handout provides students with an opportunity to practice using the methods they developed in Part C to model contextual scenarios with exponential functions using base *e*.

- For problems 11, 12, and 13, students may find it easier to write an exponential function in the form $f(x) = a \cdot b^x$ before converting it to the form $f(x) = a \cdot e^{kx}$ where the value of k is $\log_e(b)$. You may find it helpful to post the technique for converting the base of an exponential function in a central location to support students in completing this part of the handout successfully.
- In problem 13, students may be confused by their answer of approximately 11.747°C, misinterpreting it to mean the model suggests the water in a cup is that cold at room temperature. You may have to point them back to the scenario and have them closely observe that the y-values in this data set represent the difference between the temperature of the water in a cup and the room temperature, not the temperature of the water itself.

Summarizing the Task

- Be sure to debrief each part of the handout with students. There are a few interesting mathematical conclusions, listed in the bullets that follow, that students should draw from this problem set, which should not function solely as practice exercises.
    - In Part A of the handout, students use properties of exponents to develop a general rule for determining the growth or decay factor of an exponential function expressed with the natural base, *e*. This is a useful skill because it reinforces how to utilize properties of exponents for different problem-solving situations. Additionally, it gives students some insight into the structure and key features of exponential functions with base *e*.
    - In Part B of the handout, students explore the relationship between horizontal translations and vertical dilations of exponential functions. As in Part A, this subset of problems gives students some insight into the structure and key features of exponential functions. Understanding that some transformations are equivalent for exponential functions will be useful in future lessons when students explore transformations of logarithmic functions.

- In Part C of the handout, students investigate how to express exponential functions using other bases. It is not necessary for students to memorize any formulas associated with changing the base of an exponential function. Rather, students should observe the adaptability of exponential function forms. This part of the handout also introduces students to the logarithm base *e*, or the natural logarithm. This can serve as a helpful preview of the more formal investigations they perform in later lessons.
- Part D of the handout gives students an opportunity to use exponential functions with base *e* to model contextual scenarios. These scenarios also appeared in Unit 1. You could choose to compare the models that students constructed in Unit 1 to the ones they construct here to show that they are equivalent.

Assess and Reflect on the Lesson

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

Ishi performs an experiment to investigate how quickly a certain strain of bacteria reproduces in a controlled environment. She plates 10 bacteria colonies in a Petri dish at the start of the experiment, which she identifies as Day 0. On Day 4 of the experiment, she observes that there are 20 bacteria colonies in the Petri dish. She reasons that the population of bacteria is doubling every 4 days. She predicts that if the bacteria growth continues with the same growth factor, then there should be 40 bacteria colonies on Day 8. Assuming that the bacteria grow continuously, write an exponential function of the form $f(x) = a \cdot e^{kx}$ to model the number of bacteria colonies in the Petri dish after x days. Then use the function to estimate the number of bacteria colonies, to the nearest whole number, in the Petri dish on Day 10 and Day 20 of the experiment. Explain your reasoning.

Because the number of bacteria colonies increases continuously and doubles every 4 days, we can determine that the growth factor per day is $2^{\frac{1}{4}} \approx 1.189$. This means that the situation can be modeled by the function $P(t) = 10e^{t\log_e(1.189)}$, where t is the number of days since the beginning of the experiment and $P(t)$ is the number of bacteria colonies. Using the function, we estimate that the number of bacteria colonies on Day 10 of the experiment is $P(10) = 10e^{10\bullet\log_e(1.189)} \approx 56$ and the number of bacteria colonies on Day 20 of the experiment is $P(20) = 10e^{20\bullet\log_e(1.189)} \approx 319$.

HANDOUT

To supplement the information within the body of the lesson, additional answers and guidance on the handout are provided below.

Handout 3.1: Exponential Function Models

PART A

1. The function $f(t) = e^{3t}$ can be expressed in the form $f(t) = \left(e^3\right)^t \approx 20.086^t$. This means that the growth factor of $f(t) = e^{3t}$ is about 20.
2. The function $f(t) = e^{-\frac{t}{4}}$ can be expressed in the form $f(t) = \left(e^{-\frac{1}{4}}\right)^t \approx 0.779^t$. This means that the decay factor of $f(t) = e^{-\frac{t}{4}}$ is about 0.779.
3. A function of the form $f(x) = e^{kx}$ can be expressed as $f(x) = \left(e^k\right)^x$ where $b = e^k$ is a constant. This means the growth or decay factor of $f(x) = e^{kx}$ can be approximated by a decimal representation of e^k.

PART B

4. The function $g(x)=e^{x+5}$ is a horizontal translation of $f(x)=e^x$ by 5 units to the left. The function g can be expressed in the form $g(x)=e^x \cdot e^5 \approx 148.413e^x$, which is a vertical dilation of $f(x)=e^x$ by a scale factor of approximately 148.413. This means that the graph $g(x)=e^{x+5} \approx 148.413e^x$ has a y-intercept of $(0,\ 148.413)$.
5. The function $g(x)=e^{x-1}$ is a horizontal translation of $f(x)=e^x$ by 1 unit to the right. The function g can be expressed in the form $g(x)=e^x \cdot e^{-1} \approx 0.368e^x$, which is a vertical dilation of $f(x)=e^x$ by a scale factor of approximately 0.368. This means that the graph $g(x)=e^{x-1} \approx 0.368e^x$ has a y-intercept of $(0,\ 0.368)$.
6. The function $g(x)=e^{x-c}$ is a horizontal translation of $f(x)=e^x$ by c units to the right if c is positive, and by $|c|$ units to the left if c is negative. The function g can be expressed as $g(x)=a \cdot e^x$, where $a=e^{-c}$, which is a vertical dilation of $f(x)=e^x$ by a scale factor of a. This means that the graph of $g(x)=e^{x-c}$ has a y-intercept of $\left(0,\ e^{-c}\right)$.

PART C

7. The function $h(n)=2^n$ can be expressed in the form $h(n)=e^{n \cdot \log_e(2)}$. To find the value of k we can solve the equation $2^n = e^{kn}$ for k using properties of equality and logarithms, as follows:

$$\begin{aligned} 2^n &= e^{kn} \\ 2 &= e^k \\ \log_e(2) &= \log_e\left(e^k\right) \\ \log_e(2) &= k \end{aligned}$$

8. The function $h(n)=0.25^n$ can be expressed in the form $h(n)=e^{n \cdot \log_e(0.25)}$. To find the value of k we can solve the equation $0.25^n = e^{kn}$ for k using properties of equality and logarithms, as follows:

$$\begin{aligned} 0.25^n &= e^{kn} \\ 0.25 &= e^k \\ \log_e(0.25) &= \log_e\left(e^k\right) \\ \log_e(0.25) &= k \end{aligned}$$

9. The function $h(n) = 2 \bullet 3^n$ can be expressed in the form $h(n) = 2 \bullet e^{n \bullet \log_e(3)}$. The value of a is 2, and the value of k is found by solving the equation $3^n = e^{kn}$ for k using properties of equality and logarithms, as follows:

$$3^n = e^{kn}$$
$$3 = e^k$$
$$\log_e(3) = \log_e\left(e^k\right)$$
$$\log_e(3) = k$$

10. It is possible to express any exponential function of the form $h(n) = a \bullet b^n$ in the form $h(n) = a \bullet e^{kn}$, where $\log_e(b) = k$. That is, $h(n) = a \bullet b^n = a \bullet e^{n\log_e(b)}$. The value of k can be determined by solving the equation $b^x = e^{kx}$ for k with properties of equality and logarithms, as follows:

$$b^x = e^{kx}$$
$$b = e^k$$
$$\log_e(b) = \log_e\left(e^k\right)$$
$$\log_e(b) = k$$

PART D

11. This scenario can be expressed as $f(x) = 20{,}000(0.99)^x$, where x is the number of hours that water has been leaking from the pool and $f(x)$ is the number of gallons of water left in the pool after x hours. Using the method from Part C, f can be expressed as $f(x) = 20{,}000e^{x\log_e(0.99)}$. There are approximately $f(48) = 20{,}000e^{48\log_e(0.99)} \approx 12{,}345.80$ gallons of water left in the pool after water has been leaking from it for 48 hours.

12. The growth factor of the Bronco for the 5-year period can be determined by the ratio $\dfrac{47{,}035}{23{,}400} \approx 2.01$. This means that the growth factor for 1 year is $(2.01)^{\frac{1}{5}} = 1.150$. Using the method from Part C, we can express the value of the Bronco as a function of the number of years since 2012 as $V(t) = 23{,}400e^{t\log_e(1.15)}$, where t is the number of years since 2012 and $V(t)$ is the value of the Bronco after t years. We can use function V to predict that the value of the Bronco in 2021 is $V(9) = 23{,}400e^{9\log_e(1.15)} \approx 82{,}318.31$ dollars.

13. Using the regression command $y_1 \sim a \bullet e^{kx_1}$ in Desmos, the regression equation for the data set is $y = 63.029 \bullet e^{-0.028x}$. The regression equation can be used to predict the difference between the temperature of the water in a cup and the room temperature after 60 minutes: $63.029 \bullet e^{-0.028(60)} \approx 11.747°\text{C}$. Because the room temperature is 21.7°C, we can estimate that the water temperature is about 33.447°C.

LESSON 3.2

Introduction to the Logarithm Function

LEARNING OBJECTIVES

3.1.3 Construct a representation of a logarithmic function.

3.1.4 Express a logarithmic function in an equivalent form to reveal properties of the graph and/or the contextual scenario.

3.1.5 Construct a representation of the inverse function of an exponential or logarithmic function.

LESSON OVERVIEW

CONTENT FOCUS

This lesson expands on the understanding of logarithms that students developed at the end of Unit 2 to define logarithmic functions as inverses of exponential functions. The mathematical motivation for this definition is that all exponential equations can be solved if we understand exponential functions as inverses of logarithmic functions. The lesson provides students with opportunities to explore the relationship between exponential and logarithmic functions numerically, graphically, and algebraically.

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Connections Among Multiple Representations

SUGGESTED TIMING

~125 minutes

LESSON SEQUENCE

- This lesson is part of a lesson sequence (~365 minutes total) with Lessons 3.2 through 3.4.

MATERIALS

- graphing utility such as **Desmos.com**

HANDOUTS

Lesson

- 3.2.A: The Doubling Coin, Part 1
- 3.2.B: The Doubling Coin, Part 2

Practice

- 3.2.C: Practice with Logarithmic Functions in Context

LESSON DESCRIPTION

Part 1: Investigating a Doubling Coin

In this part of the lesson, students explore the idea of a coin whose value doubles in a predictable way. Students use tabular, graphic, and algebraic representations of this concept to determine which option would yield the greatest value on a specified day. Students observe that there are certain circumstances in which choosing the option of the Doubling Coin is preferable to choosing a fixed amount of $1,000,000, and vice versa.

Part 2: Revisiting the Doubling Coin
In this part of the lesson, students extend their exploration of the Doubling Coin scenario by analyzing two related scenarios to determine how graphs and inverse functions can be useful in solving similar problems. The lesson concludes with students creating a function that models the number of days required for the Doubling Coin to have a specified value. This leads to defining the logarithmic function as the inverse of the exponential function.

FORMATIVE ASSESSMENT GOAL

This lesson prepares students to complete the following formative assessment activity.

The population of a town was estimated to be 15,000 people in the year 2020. Researchers predict that each year after 2020, the population of the town will increase by 2%. The population of the town is modeled by a function f, defined as $f(t) = 15{,}000e^{0.0198t}$, where $f(t) = P$ is the population and t is the number of years since 2020.

(a) Determine the value of $f(3)$ and explain what it means in the context of this situation.

(b) Write a logarithmic function, g, whose input is the population of the town, P, and whose output is the number of years since 2020.

(c) Use your logarithmic function from part (b) to determine the value of $g(30{,}000)$ and explain what it means in the context of this situation.

Part 1: Investigating the Doubling Coin

~25 MIN.

In this part of the lesson, students explore the idea of a coin whose value doubles in a predictable way. Students use tabular, graphic, and algebraic representations of this concept to determine which option would yield the greatest value on a specified day. Students observe that there are certain circumstances in which choosing the option of the Doubling Coin is preferable to choosing a fixed amount of $1,000,000, and vice versa.

Instructional Rationale

In Unit 1 and earlier in Unit 3, students observed that exponential functions grow or decay very quickly. In fact, their growth rate accelerates as input values increase. This is colloquially known as *exponential growth*. A well-known mathematics problem asks if it is preferable to have a penny whose value doubles every day for 30 days or a flat $1,000,000. In this part of the lesson, we use the idea of a "Doubling Coin" to explore the length of time needed for a growing value to reach several different specified amounts of money.

Student Task

The student task for this part of the lesson is provided on **Handout 3.2.A: The Doubling Coin, Part 1**. In this task, students write an exponential function to model a scenario involving a coin whose value doubles every day. Answers to the handout are provided in the Assess and Reflect section of the lesson.

Facilitating the Task

- Before distributing Handout 3.2.A, engage students in a brief discussion about the idea of a coin whose value doubles every day. You can ask questions like the ones that follow. These questions should be answered without students doing any formal calculations, relying instead on their instinct about the scenario:
  - Suppose that you have a coin whose value doubles every day. That is, on the day you get the coin, Day 0, its value is $1. On Day 1, its value is $2. On Day 2, its value is $4, and so on, doubling in value with each additional day. Would you prefer a Doubling Coin for 5 days or a single payment of $1,000,000?

    I would prefer $1,000,000, since 2 doubled four times must be less than 1,000,000.
  - Would you prefer (a) a Doubling Coin for 10 days or (b) a single payment of $1,000,000?

    I would prefer $1,000,000, since 2 doubled nine times must be less than 1,000,000.

- Would you prefer (a) a Doubling Coin for 15 days or (b) a single payment of $1,000,000?

  I would prefer $1,000,000, since 2 doubled fourteen times must be less than 1,000,000.

- Let students know that they are going to develop a model for this scenario to try to determine the number of days required for the value of the Doubling Coin to exceed $1,000,000.
- You can distribute Handout 3.2.A to students and allow them to work in pairs to answer the questions.
- As you circulate around the room, be sure that students have access to a graphing utility and are recording the responses to the task on the handout.
- If necessary, you can let students know that they can refer to the graph of $D(t)$ and the table of values to answer problem 3.

Summarizing the Task

- Once students have had some time to complete the student task, have pairs of students share their responses to problem 3.
- If students need some prompting, you can ask some debriefing questions like the following:
  - How did you know when to choose the value of the Doubling Coin over the flat amount of $1,000,000?

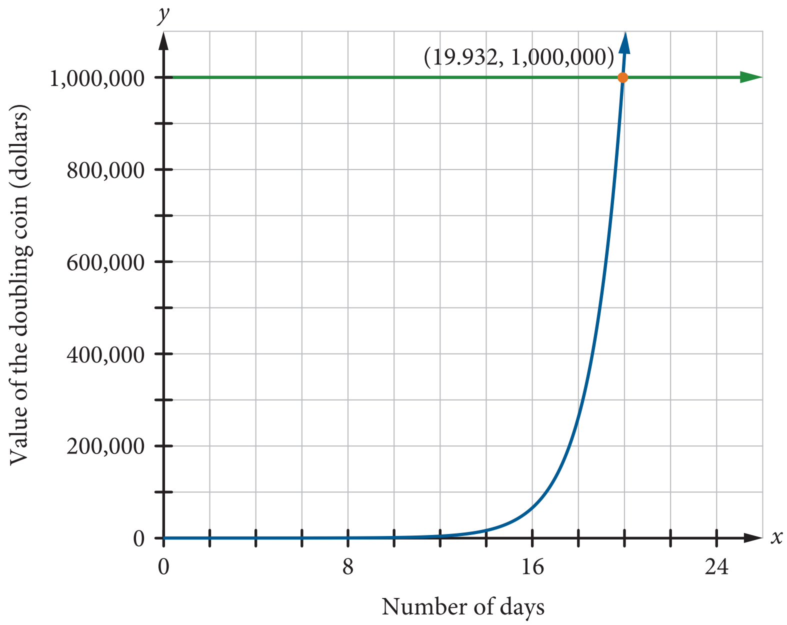
    I used trial and error to determine that at 20 days, the value of the Doubling Coin was greater than $1,000,000.

  - Using a table of values, on which day is the value of the Doubling Coin greater than $1,000,000?

    At 20 days, the value of the Doubling Coin is more than $1,000,000.

  - How can we use the graph to determine when the value of the Doubling Coin is greater than $1,000,000?

    We can look for where the graph of $y = D(t)$ intersects the line $y = 1{,}000{,}000$ to determine when the value of the Doubling Coin exceeds $1,000,000. The intersection point of the two graphs has a t-coordinate of 19.932, which means that on Day 19 the value of the Doubling Coin is less than $1,000,000 and on Day 20 the value of the Doubling Coin is greater than $1,000,000.



- How can you set up an equation that makes use of logarithms to solve the problem?

  We can set up the equation $2^t = 1{,}000{,}000$ and use properties of logarithms to solve for t. By taking the logarithm base 2 of both sides of the equation, we get $t = \log_2(1{,}000{,}000)$, which is approximately equal to 19.932 days.

- By the end of this part of the lesson, students have modeled the value of the Doubling Coin with an exponential function and recognized that taking a logarithm of both sides can solve the equation for the exponent, t. This recognition sets them up to formalize the process of taking logarithms to express exponential functions as equivalent logarithmic functions in the next part of the lesson.

Part 2: Revisiting the Doubling Coin

~30 MIN.

In this part of the lesson, students extend their exploration of the Doubling Coin scenario by analyzing two related scenarios to determine how graphs and inverse functions can be useful in solving similar problems. The lesson concludes with students creating a function that models the number of days required for the Doubling Coin to have a specified value. This leads to defining the logarithmic function as the inverse of the exponential function.

Student Task

The student task for this part of the lesson is provided on **Handout 3.2.B: The Doubling Coin, Part 2**. In this task, students compare different values of the Doubling Coin to determine the number of days it takes for the Doubling Coin to have a specified value. Answers for the handout are provided in the Assess and Reflect section of the lesson.

Facilitating the Task

- Begin by letting students know that this task is an extension of their previous task. In this extension, they develop a function that they can use to determine the number of days to wait for the Doubling Coin to have a specified value.
- Provide students with Handout 3.2.B and allow the same pairs of students to closely observe the task and ask clarifying questions.
- As you circulate around the room while students work, you can remind students that they can use tables of values, graphs, and functions to help them answer problems 1 and 2.
- If students struggle to get started, you might find it helpful to ask questions like the following:
  - How can you use the solution method you used on Handout 3.2.A to help you solve these problems?
  - How did you determine the number of days it takes for the Doubling Coin to have a value of $1,000,000?

Summarizing the Task

- As students reach problem 6, bring the class together to discuss some patterns they used to determine the number of days required for the Doubling Coin to be worth each of the four different values: $1,000,000, $1,000, $50,000, and $125,000.
- Students may be a little unsure of how to answer problem 6 on Handout 3.2.B. You can lead students through a discussion about defining the logarithmic function while debriefing the task.
- Patterns observed when setting up and solving exponential equations suggest defining a new type of function that is the inverse of the exponential function. To help students make this connection, ask them questions like the following:
  - What equations did you set up to determine the answers to problems 1, 2, and 3?

    The equations were $2^t = 1{,}000$, $2^t = 50{,}000$, and $2^t = 125{,}000$.
  - What was the same and what was different about the equations you set up?

    Each of the equations included 2^t, but each was equal to a different constant value.

- How did you rewrite each of these as logarithms to solve for t, the number of days it takes to reach a specified constant value?

  The logarithmic equations were $t = \log_2(1{,}000)$, $t = \log_2(50{,}000)$, and $t = \log_2(125{,}000)$.

- What was the same and what was different about the logarithms you set up?

  Each of the logarithms included $\log_2$, but each had a different argument.

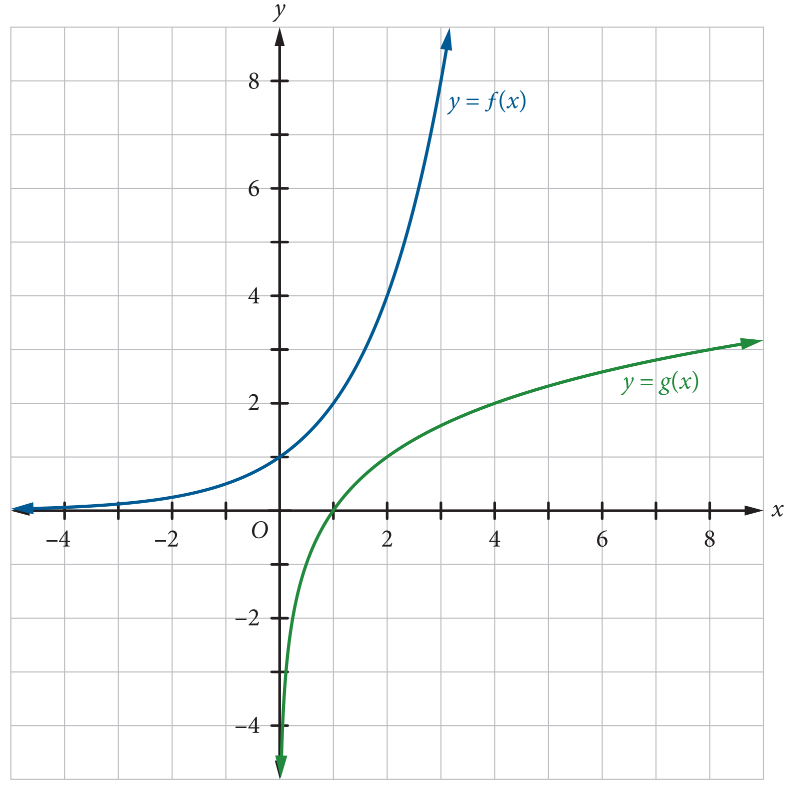
- What exponential equation could we set up to determine the number of days it would take for the Doubling Coin to be worth D dollars? Use your work from the previous problems to establish a pattern.

  Based on the pattern in the previous problems, we could set up the equation $2^t = D$.

- What logarithmic equation could we set up to determine the number of days it would take for the Doubling Coin to be worth D dollars? Use your work from the previous problems to establish a pattern.

  Based on the pattern in the previous problems, we could set up the equation $t = \log_2(D)$.

- At this point, let students know that just as raising a number to a power leads to exponential functions, so does taking a logarithm to solve an exponential equation lead to logarithmic functions. A logarithmic function has the form $f(x) = \log_b(x)$, where b is a positive real number. Logarithmic functions and exponential functions are inverses.
- At this point, it is valuable to have students graph the functions $f(x) = 2^x$ and $g(x) = \log_2(x)$ to compare their features and observe their properties. You can have students use a graphing utility, like Desmos, to graph the functions. Students should produce graphs that look like the following:



- Ask students basic questions like the following about the relationship between these two functions and have partners share their observations.
  - What about the graphs of these functions confirms that they are inverses?

    The functions are reflections across the line $y = x$.
  - What is the domain of $f(x) = 2^x$ and how does that relate to the range of $g(x) = \log_2(x)$?

    The domain of *f* is all real numbers, so the range of *g* is all real numbers.
  - What is the range of $f(x) = 2^x$ and how does that relate to the domain of $g(x) = \log_2(x)$?

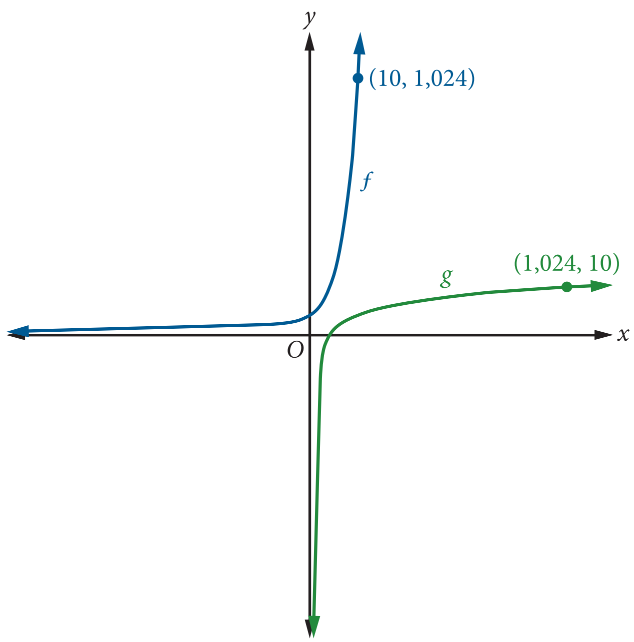
    The range of *f* is all positive real numbers, so the domain of *g* is all positive real numbers.
  - What is the *y*-intercept of $y = f(x)$ and how does it relate to the *x*-intercept of $y = g(x)$?

    The *y*-intercept of $y = f(x)$ is $(0, 1)$, so the *x*-intercept of $y = g(x)$ is $(1, 0)$.

- The graph of $f(x) = 2^x$ increases very quickly. How does this relate to the rate at which the graph $g(x) = \log_2(x)$ grows?

  The graph of t also increases, but at a slower rate.

- You can reinforce for students the differing rates of change for exponential and logarithmic functions by zooming out on the graph and comparing the coordinates $(10, 1{,}024)$ on $y = f(x)$ and $(1{,}024,\ 10)$ on $y = g(x)$.



- Examining the graphs of these two functions helps reinforce for students the relationships between inverse functions that they learned in the previous unit.
- This is a good opportunity to provide students with some time to engage in practice problems that reinforce the relationship between exponential and logarithmic functions. **Handout 3.2.C: Practice with Logarithmic Functions in Context** has three contexts that students can explore.

UNIT 3

Assess and Reflect on the Lesson

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

The population of a town was estimated to be 15,000 people in the year 2020. Researchers predict that each year after 2020, the population of the town will increase by 2%. The population of the town is modeled by a function *f*, defined as $f(t) = 15{,}000e^{0.0198t}$, where $f(t) = P$ is the population and t is the number of years since 2020.

(a) Determine the value of $f(3)$ and explain what it means in the context of this situation.

$f(3) \approx 15{,}918$. This means that the approximate population of the town in 2023 is expected to be 15,918 people.

(b) Write a logarithmic function, *g*, whose input is the population of the town, *P*, and whose output is the number of years since 2020.

A logarithmic function, *g*, whose input is the population of the town, *P*, and whose output is the number of years since 2020 is $g(P) = \frac{1}{0.0198}\ln\left(\frac{P}{15{,}000}\right)$.

(c) Use your logarithmic function from part (b) to determine the value of $g(30{,}000)$ and explain what it means in the context of this situation.

$g(30{,}000) \approx 35$. This means that if the population continues to increase by 2% each year, the population of the town can be expected to double about 35 years after 2020, which is the year 2055.

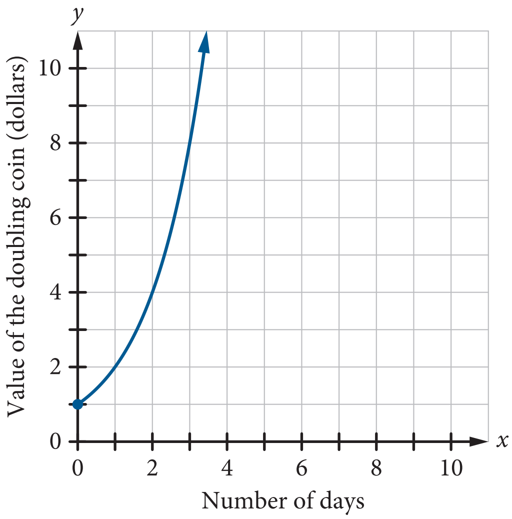
HANDOUTS

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 3.2.A: The Doubling Coin, Part 1

1. We can use the function $f(t) = 2^t$, where t is the number of days and $D = f(t)$ is the value in dollars. An exponential function is appropriate for this situation because the value of the Doubling Coin is twice its previous value each day. This means that the ratio of the values of the Doubling Coin on consecutive days will be constant.

2. A graph of the function $f(t) = 2^t$ is shown in the following figure:



3. One way is to continue to substitute values for t until the value of $f(t)$ meets or exceeds 1,000,000. Another way is to use a graphing utility to extend the y-axis of the graph to determine the value of t for which the graph of $y = f(t)$ intersects the line $y = 1{,}000{,}000$. Similarly, another way is to set up the equation $2^t = 1{,}000{,}000$ and take a logarithm base 2 of both sides to solve for t.

Handout 3.2.B: The Doubling Coin, Part 2

1. The Doubling Coin would have a value of $1,000 after about 10 days. This is because $2^{10} = 1{,}024$.
2. The Doubling Coin would have a value of $50,000 after about 15 days. This is because $2^{15} = 32{,}768$ and $2^{16} = 65{,}536$.
3. The Doubling Coin would have a value of $125,000 after about 17 days. This is because $2^{17} = 131{,}072$.
4. It is possible to use a graph to solve problem 1 by determining where the graphs of $y = f(t)$ and $y = 1{,}000$ intersect. It is possible to use a graph to solve problem 2 by determining where the graphs of $y = f(t)$ and $y = 50{,}000$ intersect. It is possible to use a graph to solve problem 3 by determining where the graphs of $y = f(t)$ and $y = 125{,}000$ intersect.
5. The exponential equation $2^t = 1{,}000$ can be solved for t using properties of logarithms to determine the number of days at which the Doubling Coin will have a value of $1,000. This method yields $t = \log_2(1{,}000)$. The exponential equation $2^t = 50{,}000$ can be solved for t using properties of logarithms to determine the number of

days at which the Doubling Coin will have a value of \$50,000. This method yields $t = \log_2(50{,}000)$. The exponential equation $2^t = 125{,}000$ can be solved for t using properties of logarithms to determine the number of days at which the Doubling Coin will have a value of \$125,000. This method yields $t = \log_2(125{,}000)$.

6. The exponential equation $2^t = D$ can be solved for t using properties of logarithms to determine the number of days at which the Doubling Coin is worth D dollars. This method yields $t = \log_2(D)$. We can use this to define a function g, defined by $g(D) = \log_2(D)$, where $t = g(D)$ is the number of days at which the coin has the specified value, D.

Handout 3.2.C: Practice with Logarithmic Functions

1. (a) $A(5) \approx 552.04$, which means that 5 years after the investment was made, the balance of the account is about \$552.04.

   (b) The equation $500(1.02)^n = 1{,}000$ can be solved to determine the number of years it would take for the balance to be \$1,000. Solving the equation using properties of equality and logarithms yields $n \approx 35.003$, which means it would take about 35 years for the account balance to be twice the original amount.

Instructional Rationale

Students may not yet be familiar with the notation "ln" for the logarithm base e, also called the natural logarithm. You can introduce this notation to students at this point if you feel it would help them, but it is not necessary. As in Lesson 3.1, students need to understand that e is a real, irrational number, and it can be used as the base of a logarithm. It would be acceptable for students to use the equivalent notation $\log_e$ in place of ln.

2. (a) $f(0) = 10$, which means that on Day 0, the start of the experiment, the number of bacteria colonies is 10. $f(3) \approx 20.544$, which means that on Day 3 of the experiment the number of bacteria colonies is about 20. $f(7) \approx 53.656$, which means that on Day 7 of the experiment the number of bacteria colonies is about 53. $f(10) \approx 110.232$, which means that on Day 10 of the experiment the number of bacteria colonies is about 110. $f(14) \approx 287.892$, which means that on Day 14 of the experiment the number of bacteria colonies is about 288.

   (b) The logarithmic function $g(P) = \dfrac{1}{0.24}\ln\left(\dfrac{P}{10}\right)$, where P is the number of bacteria colonies and $g(P)$ is the number of days it will take for the number of bacteria colonies to equal P, describes this situation.

3. (a) The exponential function $f(x) = 1{,}000(2)^{\frac{1}{1.5}x}$, where $f(x)$ is the number of transistors that can be placed on an integrated circuit and x is the number of years after 1970, describes this situation.

(b) The logarithmic function $g(T) = 1.5\log_2\left(\frac{T}{1{,}000}\right)$, where T is the number of transistors that can be placed on a circuit and $x = g(T)$ is the number of years since 1970, describes this situation.

(c) The value of $g(1{,}000) = 0$, which means that in 1970, 1,000 transistors could be placed on an integrated circuit.

(d) The value of $g(3{,}000{,}000) \approx 17.326$, which means in about 17 years after 1970, or 1987, 3,000,000 transistors could be placed on an integrated circuit.

(e) We can use function g to estimate the year when there will be 50 billion transistors on an integrated circuit. The value of $g(50{,}000{,}000{,}000) \approx 38.363$, which means that Moore's Law predicts that in about 38 years after 1970, or 2008, 50 billion transistors could be placed on an integrated circuit.

UNIT 3

LESSON 3.3

Connecting Properties of Logarithms with Transformations of the Graph of the Parent Logarithm Function

AREAS OF FOCUS

- Connections Among Multiple Representations
- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~150 minutes

LESSON SEQUENCE

- This lesson is part of a lesson sequence (~365 minutes total) with Lessons 3.2 through 3.4.

MATERIALS

- graphing utility such as **Desmos.com**
- graph paper (optional)

HANDOUT

Practice

- 3.3: Using Properties of Logarithms

LEARNING OBJECTIVES

3.1.3 Construct a representation of a logarithmic function.

3.1.4 Express a logarithmic function in an equivalent form to reveal properties of the graph and/or the contextual scenario.

LESSON OVERVIEW

CONTENT FOCUS

This lesson connects the properties of logarithms—the product rule, the quotient rule, and the power rule—with various transformations of the graph of the parent logarithm function. The transformations of the graph are algebraically justified by the properties of exponents. Through a Desmos activity, students investigate how transformations of logarithmic functions compare to the graph of the parent logarithmic function. The activity allows students to quickly and easily change the parameters of a logarithmic function to identify how different types of transformations affect its graph. Students then use algebra to rewrite the logarithmic function in an equivalent form to explain the effect of the transformation.

LESSON DESCRIPTION

Part 1: Exploring Transformations of Logarithms

This part of the lesson focuses on the graphs of basic logarithm functions with various bases. Students examine several graphs and make some observations about key features of the graphs of logarithm functions. Their preliminary observations prepare them to connect the algebraic and graphic representations of logarithmic functions in the later parts of this lesson.

Part 2: Exploring the Product Rule
In this part of the lesson, students use Desmos to compare the graph of the function $g(x)=\log_2(cx)$, where c is a positive real number, to the parent function $f(x)=\log_2(x)$. Through their investigation, students observe that the graph of $g(x)=\log_2(cx)$ is identical to the graph of $f(x)=\log_2(x)$ with a vertical translation of $\log_2(c)$ units. This observation leads students to the product rule for logarithms: $\log_b(cx)=\log_b(x)+\log_b(c)$.

Part 3: Exploring the Power Rule
In this part of the lesson, students use Desmos to compare the graph of the function $k(x)=\log_2\left(x^d\right)$, where d is a positive real number, to the parent function $f(x)=\log_2(x)$. Through their investigation, students observe that the graph of $k(x)=\log_2\left(x^d\right)$ is identical to the graph of $f(x)=\log_2(x)$ with a vertical dilation by a factor of d. This observation leads students to the power rule for logarithms: $\log_b\left(x^d\right)=d\log_b(x)$.

Part 4: Exploring the Quotient Rule
In this part of the lesson, students use Desmos to compare the graph of the function $h(x)=\log_2\left(\frac{x}{c}\right)$, where c is a positive real number, to the parent function $f(x)=\log_2(x)$. Through their investigation, students observe that the graph of $h(x)=\log_2\left(\frac{x}{c}\right)$ is identical to the graph of $f(x)=\log_2(x)$ with a vertical translation of $\log_2(c)$ units. This observation leads students to the quotient rule for logarithms: $\log_b\left(\frac{x}{c}\right)=\log_b(x)-\log_b(c)$.

FORMATIVE ASSESSMENT GOAL

This lesson prepares students to complete the following formative assessment activity.

Use properties of logarithms to write the expression $\log_b\left(\frac{m^2}{\sqrt[5]{n}}\right)$ as combinations of $\log_b(m)$ and $\log_b(n)$. Identify the properties you use to justify your steps in rewriting the expression.

Part 1: Exploring the Graph of a Logarithmic Function

~15 MIN.

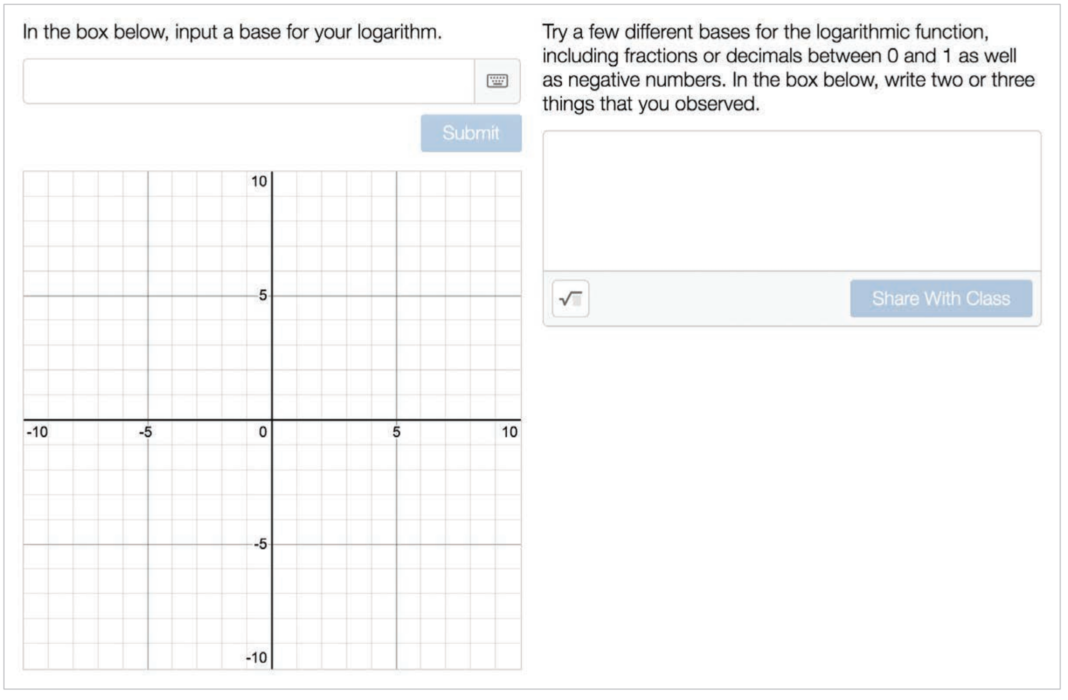
This part of the lesson focuses on the graphs of basic logarithm functions with various bases. Students examine several graphs and make some observations about key features of the graphs of logarithm functions. Their preliminary observations prepare them to connect the algebraic and graphic representations of logarithmic functions in the later parts of this lesson.

Student Task

Begin the Desmos activity. In this task, you will identify key features of the graph of a logarithmic function.

Facilitating the Task

- Begin by ensuring that each student has access to the Desmos activity. You will need to assign the activity to your class or create a single session code. The activity is located at **preap.org/Desmos-Logarithms**.
- Students need to work only on Screen 1 of the activity in this part of the lesson.
- As you circulate around the room, be sure that students try several different bases for their logarithmic function, including fractions or decimals between 0 and 1 as well as negative numbers. Encourage them to identify features such as the intercepts, the asymptotic behavior, the domain and range, and the general shape of the graph.
- Students should capture their observations in the textbox on the right of the screen.



Summarizing the Task

- As students finish their observations of the graph of logarithmic functions with different bases, pause the activity so that you can engage students in a discussion about what they observed.
- If students are reluctant to share their observations, you can prompt them with some questions like the following:
  - What did you notice about the x-intercepts of the graphs of $f(x) = \log_b(x)$ for any value of the base, b?

    All the logarithmic functions, regardless of the value of the base, have an x-intercept of 1. That means the coordinate $(1, 0)$ is a point on all the graphs.
  - Why does it make sense that the coordinate $(1, 0)$ would be on all the logarithmic graphs you explored?

    The coordinate $(1, 0)$ corresponds to the statement $\log_b(1) = 0$ for any nonzero value of b. The related exponential equation is $b^0 = 1$, which is true for any nonzero value of b.
  - Were there any bases that did not have an associated graph?

    If the base was negative, 0, or 1, then there was no graph.
  - Why would a negative base not have an associated graph?

    If the base was negative, then the logarithmic equation $y = \log_b(x)$ would have a related exponential equation of $b^y = x$. If b is negative, there is no pattern in the relationship between x and y.
  - What did you notice about the shape (or behavior) of the graph of $f(x) = \log_b(x)$ when b was greater than 1?

    The graph of the function increased as x increased, and the graph curved toward the y-axis as the value of x got closer to 0.
  - What happened to the graph when the base was a number between 0 and 1?

    The graph of the function decreased as x increased, and the graph curved toward the y-axis as the value of x got closer to 0.
  - What did you notice about the shape of the graph as the value of the base increased?

    The greater the value of b, the “flatter” the graph of the function. In other words, the greater the value of b, the slower the rate of change of the function.

- What did you notice about the domain and range of logarithmic functions? Why do you think those observations are true?

  Regardless of the value of b, a logarithmic function always has a domain of positive real numbers and a range of all real numbers. This is consistent with properties of its inverse function, an exponential function. The domain of an exponential function is all real numbers and the range is positive real numbers. Because exponential functions and logarithmic functions are inverses, the domain and range switch.

Guiding Student Thinking

Some students may be unsure whether the range of the logarithmic function is all real numbers, since the graphs appear to level out as the value of x increases. You can ask students to choose a value that they think the function would not exceed. You can then display a graph of a logarithmic function, such as $f(x) = \log_2(x)$, and a horizontal line at the value the students suggest. By adjusting the x-axis of the graph, you can show students that the graph of the function intersects the horizontal line because it does not ever "level out." It may be helpful to connect the graphical representation to an algebraic representation. That is, if our logarithmic function can be expressed as $y = \log_2(x)$, then its inverse can be expressed as $2^y = x$. Because it is possible to substitute any real number value for y, the equation $2^y = x$ will always have a real number solution x. Therefore, the range of the logarithmic function is all real numbers.

- At this time, you could have students begin a vocabulary graphic organizer about logarithmic functions. A blank organizer can be found on Handout 1.1. By the end of this lesson, students will be able to add the properties of logarithms to their organizer. You can post a completed vocabulary graphic organizer for logarithmic functions on your classroom's word wall for easy reference for students.

Part 2: Exploring the Product Rule

~30 MIN.

In this part of the lesson, students use Desmos to compare the graph of the function $g(x) = \log_2(cx)$, where c is a positive real number, to the parent function $f(x) = \log_2(x)$. Through their investigation, students observe that the graph of $g(x) = \log_2(cx)$ is identical to the graph of $f(x) = \log_2(x)$ with a vertical translation of $\log_2(c)$ units. This observation leads students to the product rule for logarithms: $\log_b(cx) = \log_b(x) + \log_b(c)$.

Instructional Rationale

This part of the lesson uses the parent function $f(x) = \log_2(x)$ so that it is more apparent to students how the value of c in the multiplicative transformation $g(x) = \log_2(cx)$ affects the graph. That is, because g can be written as $g(x) = \log_2(c) + \log_2(x)$, it is easier to determine the value of $\log_2(c)$ and the magnitude of the vertical translation when c is a power of 2. Using a base of 2 also means that the values of c can be kept relatively small and students are still able to gather a lot of information.

Student Task

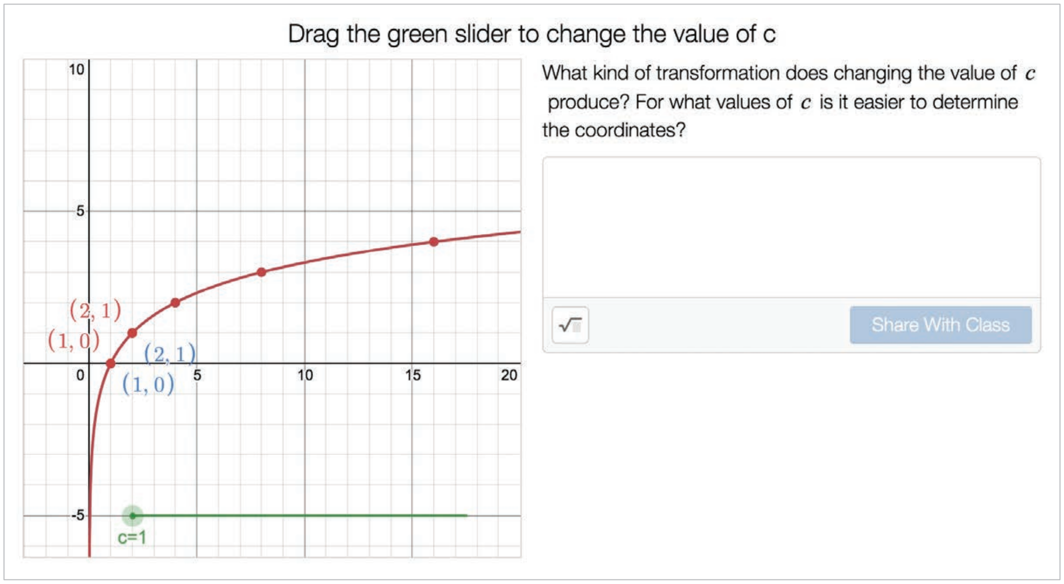
Continue with the Desmos session from Part 1. On Screen 2, you and your partner will investigate transformations of the function $g(x) = \log_2(cx)$ by comparing it to the parent function $f(x) = \log_2(x)$. Slide the green dot on the screen to change the value of c and observe the resulting changes to the graph. The coordinates of two points of the transformed function are shown on the graph. Make note of the values of c that result in integer values for the coordinates of these points.

Facilitating the Task

- Before starting the task, prepare students by asking them a few questions to get them thinking about transformations of a function. You can use questions like the following:
  - Consider the parent function $y = f(x)$. What kind of transformation would the function $y = f(cx)$ be? How would the graphs differ?

    This is a multiplicative transformation applied to the input of the function, which would produce a horizontal dilation of the graph of the parent function.
  - Consider the parent function $f(x) = \log_2(x)$. What kind of transformation would you expect the function $g(x) = \log_2(cx)$ to be? How would you expect the graphs to differ?

    This is a multiplicative transformation applied to the input of the function, which should produce a horizontal dilation of the graph of the parent function.
- Let students know that they will investigate the graph of $g(x) = \log_2(cx)$ for different values of c. You may want to let them know that on Screen 2 of the activity, the parent function $f(x) = \log_2(x)$ is displayed in blue while the transformed function $g(x) = \log_2(cx)$ is displayed in red. The green slider bar on the graph controls the value of c. Now have students work in pairs on Screen 2 of the Desmos activity.



- Have students observe that the coordinates of two specified points on the graph are displayed. Be sure that students make note of the values of c where the coordinates of these points have integer values. They should identify that the labeled coordinates have integer values for both x and y when $c = 1, 2, 4, 8, 16$, and 32.
- As you circulate around the room, listen as students try to make sense of the transformation. They should identify this transformation as a horizontal dilation but observe that it also appears to be a vertical translation. At this point, it is better to let students puzzle through this without help from you.

Summarizing the Task

- As students finish their observations on Screen 2 of the activity, be sure to pause the session so that you can engage students in a discussion about what they noticed. This discussion is necessary for students to understand how the Desmos activity connects to the product rule for logarithms.
- In the debrief of this part of the lesson, you may find it helpful to create a table of values that can be used to compare the value of c with the corresponding coordinates of the points on the graph and the description of the transformation. If you choose to construct the table of values, have students share their observations so that you can complete the table as a class.

Value of c	Image of (1, 0) Under the Transformation	Image of (2, 1) Under the Transformation	Description of Transformation
1	$(1,0)$	$(2,1)$	Parent function
2	$(1,1)$	$(2,2)$	Vertical translation up by 1 unit
4	$(1,2)$	$(2,3)$	Vertical translation up by 2 units
8	$(1,3)$	$(2,4)$	Vertical translation up by 3 units
16	$(1,4)$	$(2,5)$	Vertical translation up by 4 units
32	$(1,5)$	$(2,6)$	Vertical translation up by 5 units

- Ask some students what they notice about the relationship among the values of c, powers of 2, and the magnitude of the vertical translation. The following questions provide scaffolding to focus students' attention on the eventual observation that the magnitude of the vertical translation can be determined by the power of the base.
  - What do you notice about the values of c for which the points on the graph have coordinates with integer values?

    Those values of c are powers of 2.
  - If you rewrote those values as powers of 2, how would you express them?

    The values of c would be: $1 = 2^0$, $2 = 2^1$, $4 = 2^2$, $8 = 2^3$, $16 = 2^4$, and $32 = 2^5$.
  - Suppose that we wanted to express the powers of 2 using logarithms instead of exponents. How would we express them?

    We would rewrite the expression $1 = 2^0$ as $\log_2(1) = 0$, the expression $2 = 2^1$ as $\log_2(2) = 1$, the expression $4 = 2^2$ as $\log_2(4) = 2$, the expression $8 = 2^3$ as $\log_2(8) = 3$, the expression $16 = 2^4$ as $\log_2(16) = 4$, and the expression $32 = 2^5$ as $\log_2(32) = 5$.
  - Examine the logarithmic expressions that we wrote. How do they compare to the magnitudes of the vertical translations we observed in the activity?

    The values of the logarithmic expressions are equal to the magnitudes of the vertical translations.
- At this point, you may need to engage in some direct instruction to help students connect these concepts. You can work through an example or two with your students to help them connect a vertical translation to a logarithmic expression.

Some steps and two examples follow:

Explanation	Example 1	Example 2
These are our example functions with different bases (values of c).	$g(x) = \log_2(8x)$	$g(x) = \log_2(32x)$
We can express each function using vertical translations. Because the graphs coincide, the functions are equivalent.	$\log_2(8x) = \log_2(x) + 3$	$\log_2(32x) = \log_2(x) + 5$
We can express the constant using logarithms because $\log_2(8) = 3$ and $\log_2(32) = 5$.	$\log_2(8x) = \log_2(x) + \log_2(8)$	$\log_2(32x) = \log_2(x) + \log_2(32)$

- The examples show specific instances of the product rule for logarithms. That is, the logarithm of a product can also be expressed as the sum of the logarithms of the factors of the argument. Written algebraically, this can be expressed as $\log_b(cx) = \log_b(x) + \log_b(c)$.
- The most important takeaway from this part of the lesson is that for logarithmic functions, a horizontal dilation can also be expressed as a vertical translation because of the product rule for logarithms: $\log_b(cx) = \log_b(x) + \log_b(c)$.

Guiding Student Thinking

Another interesting takeaway from this part of the lesson is that logarithms can be used to express multiplication as addition. This is one of the critical properties of logarithms that made them so useful to mathematicians and scientists for centuries. In the 1600s, astronomer and mathematician John Napier realized that addition was significantly easier than multiplication, especially with very large numbers. Astronomy was a major field of scientific investigation at that time, and working with large numbers was a routine part of an astronomer's work. To more efficiently work with large numbers, Napier created a system, later modified by Henry Briggs, that could express multiplication as addition. This innovation saved mathematicians and scientists thousands of hours of computation time.
In Lesson 3.4: Applications of Logarithms, students investigate how modern mathematicians and scientists use logarithms to express an exponential scale with a linear scale. This modern use is closely related to the use developed by Napier.

Part 3: Exploring the Power Rule

~30 MIN.

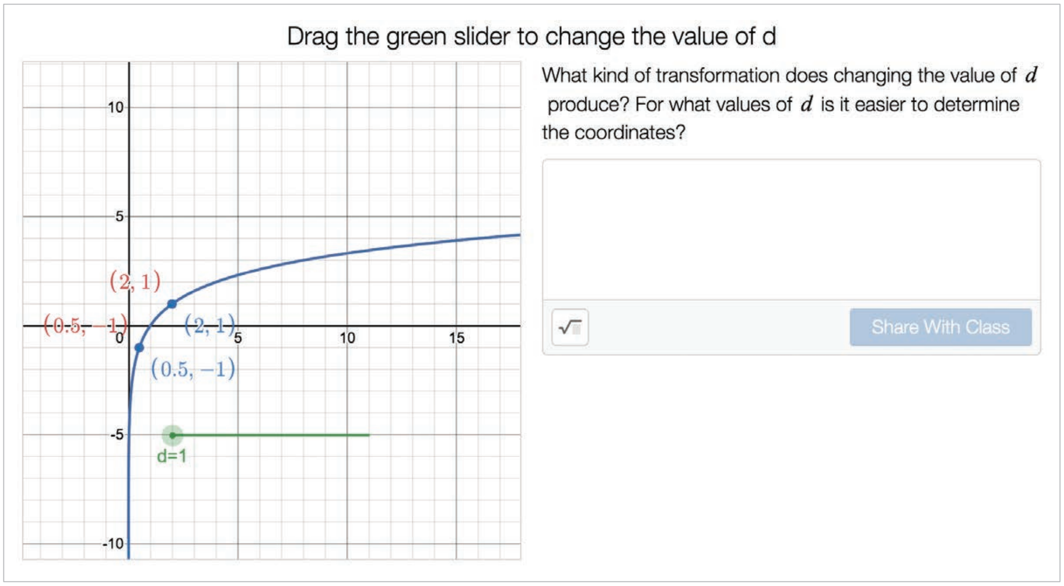
In this part of the lesson, students use Desmos to compare the graph of the function $k(x)=\log_2\left(x^d\right)$, where d is a positive real number, to the parent function $f(x)=\log_2(x)$. Through their investigation, students observe that the graph of $k(x)=\log_2\left(x^d\right)$ is identical to the graph of $f(x)=\log_2(x)$ with a vertical dilation by a factor of d. This observation leads students to the power rule for logarithms: $\log_b\left(x^d\right)=d\log_b(x)$.

Student Task

Continue with the Desmos session from Part 2. On Screen 3, you and your partner will investigate transformations of the function $k(x)=\log_2\left(x^d\right)$ by comparing it with the parent function $f(x)=\log_2(x)$. Slide the green dot on the screen to change the value of d and observe the resulting changes to the graph. As before, the coordinates of two points of the transformed function are shown on the graph. Make note of the values of d that result in integer values for the coordinates of these points.

Facilitating the Task

- Before starting the task, prepare students by asking them a few questions to get them thinking about transformations of a function. You can use questions like the following:
    - Consider the parent function $y = f(x)$. What kind of transformation would the function $y = f\left(x^d\right)$ be? How would the graphs be different?
- Students may not be sure how to respond to the questions because this may be the first time they are investigating the exponentiation of the input of a function. Allow students to make some conjectures and then record them in a central location. You can return to the conjectures when you debrief this part of the lesson.
- Let students know that they will investigate the graph of $k(x)=\log_2\left(x^d\right)$ for different values of d. You may want to let them know that on Screen 3 of the activity, the parent function $f(x)=\log_2(x)$ is displayed in blue, while the transformation $k(x)=\log_2\left(x^d\right)$ is displayed in red. The green slider bar on the graph controls the value of d. Now have students work in pairs on Screen 3 of the Desmos activity.



- Have students observe that the coordinates of two specified points on the graph are displayed. Be sure that students make note of the values of *d* where the coordinates of these points have integer values. They should identify that the labeled coordinates have integer values for both *x* and *y* for all integer values of *d*.
- As you circulate around the room, listen as students try to make sense of the transformation. They should identify this transformation to be a vertical dilation.

Summarizing the Task

- As students finish their observations on Screen 3 of the activity, be sure to pause the session so that you can engage students in a discussion about what they noticed. This discussion is necessary for students to understand how the Desmos activity connects to the power rule for logarithms.
- In the debrief of this part of the lesson, you may find it helpful to create a table of values that can be used to compare the value of *d* with the corresponding coordinates of the points on the graph and the description of the transformation. If you choose to construct the table of values, have students share their observations so that you can complete the table as a class.

Value of d	Image of $\left(\frac{1}{2}, -1\right)$ Under the Transformation	Image of (2, 1) Under the Transformation	Description of Transformation
1	$\left(\frac{1}{2}, -1\right)$	$(2, 1)$	Parent function
2	$\left(\frac{1}{2}, -2\right)$	$(2, 2)$	Vertical dilation by a factor of 2
3	$\left(\frac{1}{2}, -3\right)$	$(2, 3)$	Vertical dilation by a factor of 3
4	$\left(\frac{1}{2}, -4\right)$	$(2, 4)$	Vertical dilation by a factor of 4
5	$\left(\frac{1}{2}, -5\right)$	$(2, 5)$	Vertical dilation by a factor of 5
6	$\left(\frac{1}{2}, -6\right)$	$(2, 6)$	Vertical dilation by a factor of 6

- Ask some students what they notice about the relationship between the value of d and the magnitude of the vertical dilation.
  - What do you notice about the values of d for which the points on the graph have coordinates with integer values?

    Those values of d are whole numbers.
  - What kind of transformation does $k(x) = \log_2\left(x^d\right)$ appear to be?

    It looks like a vertical dilation.
  - How does the value of d correspond with the magnitude of the vertical dilation?

    The value of d is the magnitude of the vertical dilation of the parent function $f(x) = \log_2(x)$.
- At this point, you may need to engage in some direct instruction to help students connect these concepts. You can work through an example or two with your students to help them connect a vertical dilation and a logarithmic expression. Some steps and two examples follow:

Explanation	Example 1	Example 2
These are our example functions with different exponents (values of d).	$k(x) = \log_2(x^2)$	$k(x) = \log_2(x^5)$
Because the graphs coincide, the function with the exponent expression as an argument and the function expressed as a vertical dilation are equivalent.	$\log_2(x^2) = 2\log_2(x)$	$\log_2(x^5) = 5\log_2(x)$
We can express the exponent expression in each argument as a product.	$\log_2(x \cdot x) = 2\log_2(x)$	$\log_2(x \cdot x \cdot x \cdot x \cdot x) = 5\log_2(x)$
Using the product rule, we know that we can rewrite the log expression on the left as a sum.	$\log_2(x) + \log_2(x) = 2\log_2(x)$	$\log_2(x) + \log_2(x) + \log_2(x)$ $+\log_2(x) + 1 = 5\log_2(x)$

- The statements in the last row of the table are true because it is possible to express a sum as the addition of like terms. These examples show specific applications of the power rule for logarithms. That is, the logarithm of a base raised to an exponent can also be expressed as a product of the value of the exponent and the logarithm of the base. Written algebraically, this is expressed as $\log_b(x^d) = d\log_b(x)$.
- At this point, you can engage students in a brief thought experiment related to negative exponents.
    - Suppose we had a logarithmic function whose argument was a base raised to a negative exponent, such as $k(x) = \log_2(x^{-1})$. If we use the power rule to rewrite the function, what would it look like and what kind of transformation would we expect the graph to exhibit?

      We could rewrite the function as $k(x) = -1 \cdot \log_2(x)$. This suggests that the transformation would be a reflection across the x-axis.

- Suppose we had a logarithmic function whose argument was a base raised to a negative exponent, such as $k(x) = \log_2(x^{-2})$. If we use the product rule to rewrite the function, what would it look like and what kind of transformation would we expect the graph to exhibit?

  We could rewrite the function as $k(x) = -2 \cdot \log_2(x)$. This suggests that the transformation would be a reflection across the x-axis and a vertical dilation.

- You may want to take a moment to have students confirm that the graphs of $k(x) = \log_2(x^{-1})$ and $k(x) = -1 \cdot \log_2(x)$ coincide.
- At this point it is important to conclude that the power rule for logarithms, $\log_b(x^d) = d\log_b(x)$, is true for all real number values of d.

Guiding Student Thinking

Some students who are familiar with using negative exponents to express reciprocals might suggest rewriting the function $k(x) = \log_2(x^{-1})$ as $k(x) = \log_2\left(\frac{1}{x}\right)$. This is a correct way to rewrite the function in an equivalent form, and it is important to validate this approach. You can have students graph the two different versions of the function to confirm that they are equivalent. To use the power rule to make sense of the graph of $k(x) = \log_2(x^{-1})$, it is necessary to express the function as $k(x) = -\log_2(x)$. In Part 4 of the lesson, students have an opportunity to connect the product rule and the power rule to develop the quotient rule for logarithms.

- The most important takeaway from this part of the lesson is that for logarithmic functions, an argument of a logarithm consisting of a base raised to a power can also be expressed as a vertical dilation because of the power rule for logarithms: $\log_b(x^d) = d\log_b(x)$.

Part 4: Exploring the Quotient Rule

~30 MIN.

In this part of the lesson, students use Desmos to compare the graph of the function $h(x) = \log_2\left(\frac{x}{c}\right)$, where c is a positive real number, to the parent function $f(x) = \log_2(x)$. Through their investigation, students observe that the graph of $h(x) = \log_2\left(\frac{x}{c}\right)$ is identical to the graph of $f(x) = \log_2(x)$ with a vertical translation of $\log_2(c)$ units. This observation leads students to the quotient rule for logarithms: $\log_b\left(\frac{x}{c}\right) = \log_b(x) - \log_b(c)$.

Student Task

Continue with the Desmos session from Part 3. On Screen 4, you and your partner will investigate transformations of the function $h(x) = \log_2\left(\frac{x}{c}\right)$ by comparing it to the parent function $f(x) = \log_2(x)$. Slide the green dot on the screen to change the value of c and observe the resulting changes to the graph. As on the previous two screens, the coordinates of two points of the transformed function are shown on the graph. Make note of the values of c that result in integer values for the coordinates of these points.

Facilitating the Task

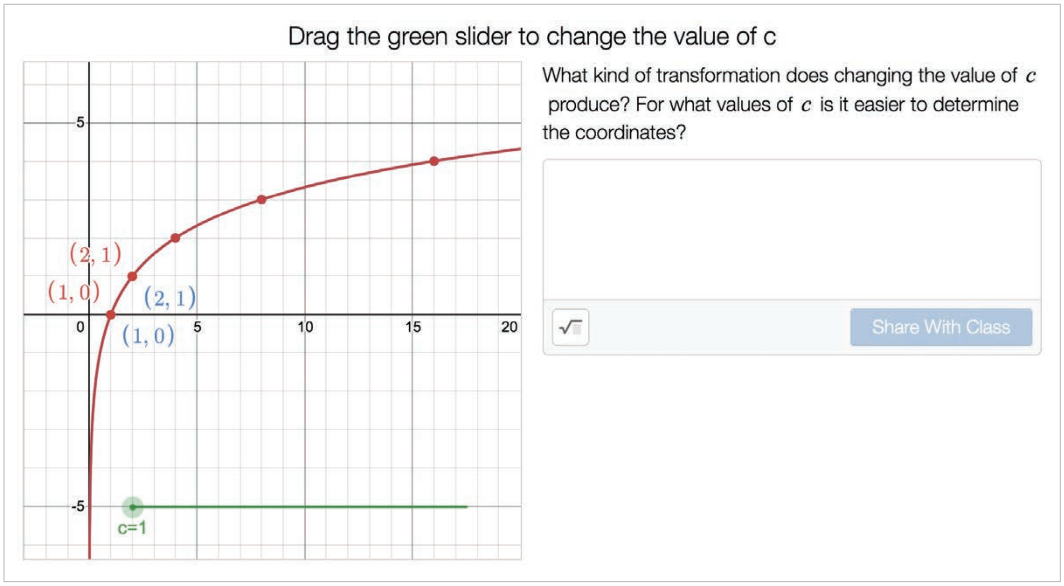
- Before assigning the student task, take students through an exploration of the quotient rule for logarithms.
    - Consider a function $y = f\left(\frac{x}{c}\right)$, where c is a constant. How could we express the argument of the function as a product?

      Since c is a constant, that means that we could express $\frac{x}{c}$ as either $x \cdot c^{-1}$ or $\frac{1}{c} \cdot x$.
    - Consider a function $h(x) = \log_2\left(\frac{x}{c}\right)$, where c is a constant. How could we rewrite the function so that the argument is a product?

      We could rewrite this function as $h(x) = \log_2(x \cdot c^{-1})$ or as $h(x) = \log_2\left(\frac{1}{c} \cdot x\right)$.
    - How could we use the properties of logarithms to express this function in an equivalent form?

      Using the product rule, we can rewrite $h(x) = \log_2\left(\frac{1}{c} \cdot x\right)$ as $h(x) = \log_2\left(\frac{1}{c}\right) + \log_2(x)$. Using both the product rule and the power rule, we can rewrite $h(x) = \log_2(x \cdot c^{-1})$ as $h(x) = \log_2(x) - \log_2(c)$.
    - What kind of transformation would you expect the graph of $h(x) = \log_2\left(\frac{x}{c}\right)$ to exhibit when you compare it to the parent function $f(x) = \log_2(x)$?

      Both of these forms of h suggest that h will differ from f by a vertical translation of some kind.
- Let students know that they will investigate the graph of $h(x) = \log_2\left(\frac{x}{c}\right)$ for different values of c. You may want to let them know that on Screen 4 of the activity, the parent function $f(x) = \log_2(x)$ is displayed in blue while the transformation $h(x) = \log_2\left(\frac{x}{c}\right)$ is displayed in red. The green slider bar on the graph controls the value of c. Now have students work in pairs on Screen 4 of the Desmos activity.



- Have students observe that the coordinates of two specified points on the graph are displayed. Be sure that students make note of the values of c where the coordinates of these points have integer values. They should identify that the labeled coordinates have integer values for both x and y when $c = 1,\ 2,\ 4,\ 8,\ 16,$ or 32.
- As you circulate around the room, listen as students try to make sense of the transformation. They should identify this transformation to be a horizontal dilation but observe that it also appears to be a vertical translation. At this point, it is better to let students puzzle through this on their own.

Summarizing the Task

- As students finish their observations on Screen 4 of the Desmos activity, be sure to pause the session so that you can engage students in a discussion about what they noticed. This discussion is necessary for students to understand how the Desmos activity connects to the product rule for logarithms.
- In the debrief of this part of the lesson, you may find it helpful to create a table of values that can be used to compare the value of c with the corresponding coordinates of the points on the graph and the description of the transformation. If you choose to construct the table of values, have students share their observations so that you can complete the table as a class.

Value of c	Image of (1, 0) Under the Transformation	Image of (2, 1) Under the Transformation	Description of Transformation
1	$(1, 0)$	$(2, 1)$	Parent function
2	$(1, -1)$	$(2, 0)$	Vertical translation down by 1 unit
4	$(1, -2)$	$(2, -1)$	Vertical translation down by 2 units
8	$(1, -3)$	$(2, -2)$	Vertical translation down by 3 units
16	$(1, -4)$	$(2, -3)$	Vertical translation down by 4 units
32	$(1, -5)$	$(2, -4)$	Vertical translation down by 5 units

- Ask students what they notice about the relationship among the values of c, powers of 2, and the magnitudes of the vertical translation. The questions below provide scaffolding to focus students' attention on the eventual observation that the magnitude of the vertical translation can be determined by the power of the base.
  - What do you notice about the values of c for which the points on the graph have integer values?

    Those values of c are powers of 2.
  - If you rewrote those values as powers of 2, how would you express them?

    The values of c would be: $1 = 2^0$, $2 = 2^1$, $4 = 2^2$, $8 = 2^3$, $16 = 2^4$, and $32 = 2^5$.
  - Suppose that we wanted to express the powers of 2 using logarithms instead of exponents. How would we express them?

    We would rewrite the expression $1 = 2^0$ as $\log_2(1) = 0$, the expression $2 = 2^1$ as $\log_2(2) = 1$, the expression $4 = 2^2$ as $\log_2(4) = 2$, the expression $8 = 2^3$ as $\log_2(8) = 3$, the expression $16 = 2^4$ as $\log_2(16) = 4$, and the expression $32 = 2^5$ as $\log_2(32) = 5$.
  - Examine the logarithmic expressions that we wrote. How do they compare to the magnitudes of the vertical translations we observed in the activity?

    The values of the logarithmic expressions are equal to the magnitudes of the vertical translations.
- At this point, you may need to engage in some direct instruction to help students connect these concepts. You can work through an example or two with your students, connecting a vertical translation to a logarithmic expression. Some steps and two examples follow:

Explanation	Example 1	Example 2
These are our example functions with different bases (values of c).	$g(x) = \log_2\left(\frac{x}{8}\right)$	$g(x) = \log_2\left(\frac{x}{32}\right)$
We can express each function using the vertical translations. Because the graphs coincide, the functions are equivalent.	$\log_2\left(\frac{x}{8}\right) = \log_2(x) - 3$	$\log_2\left(\frac{x}{32}\right) = \log_2(x) - 5$
We can express the constant using logarithms because $\log_2(8) = 3$ and $\log_2(32) = 5$.	$\log_2\left(\frac{x}{8}\right) = \log_2(x) - \log_2(8)$	$\log_2\left(\frac{x}{32}\right) = \log_2(x) - \log_2(32)$

- The examples show specific instances of the quotient rule for logarithms. That is, the logarithm of a quotient can also be expressed as the difference of the logarithms of the dividend and divisor. Written algebraically, this can be expressed as $\log_b\left(\frac{x}{c}\right) = \log_b(x) - \log_b(c)$.
- At this point, you can have students return to their ideas from the thought experiment from Part 3:
  - Think back to our thought experiment about negative exponents. Suppose we had a logarithmic function whose argument had a negative exponent, such as $k(x) = \log_2(x^{-1})$. How could we rewrite the argument as a quotient?

    We could rewrite the function as $k(x) = \log_2\left(\frac{1}{x}\right)$.
  - If we use the quotient rule for logarithms, how would we express $k(x) = \log_2\left(\frac{1}{x}\right)$ in an equivalent form?

    We could rewrite the function as $k(x) = \log_2(1) - \log_2(x)$. This function can be simplified because $\log_2(1) = 0$. Therefore $k(x) = \log_2(1) - \log_2(x)$ is equivalent to $k(x) = -\log_2(x)$.
- The most important takeaway from this part of the lesson is that for logarithmic functions, a horizontal dilation can also be expressed as a vertical translation because of the quotient rule for logarithms: $\log_b\left(\frac{x}{c}\right) = \log_b(x) - \log_b(c)$.
- You may want to provide students with some time to practice using the properties of logarithms. **Handout 3.3: Using Properties of Logarithms** has several exercises that they can use to develop some fluency with these laws.

Assess and Reflect on the Lesson

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

Use properties of logarithms to write the expression $\log_b\left(\frac{m^2}{\sqrt[5]{n}}\right)$ as combinations of $\log_b(m)$ and $\log_b(n)$. Identify the properties you use to justify your steps in rewriting the expression.

The expression $\log_b\left(\frac{m^2}{\sqrt[5]{n}}\right)$ can be written in an equivalent form as $2\log_b(m)-\frac{1}{5}\log_b(n)$. This form can be determined using the following steps and properties of logarithms.

Quotient rule: $\log_b\left(\frac{m^2}{\sqrt[5]{n}}\right)=\log_b\left(m^2\right)-\log_b\left(\sqrt[5]{n}\right)=\log_b\left(m^2\right)-\log_b\left(n^{\frac{1}{5}}\right)$

Power rule: $\log_b\left(m^2\right)-\log_b\left(n^{\frac{1}{5}}\right)=2\log_b(m)-\frac{1}{5}\log_b(n)$

HANDOUT

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 3.3: Using Properties of Logarithms

1. (a) Product Rule: $\log_b(cx)=\log_b(x)+\log_b(c)$
   (b) Power Rule: $\log_b\left(x^d\right)=d\log_b(x)$
   (c) Quotient Rule: $\log_b\left(\frac{x}{c}\right)=\log_b(x)-\log_b(c)$
2. (a) This is a true statement because $\log_2\left(\left(\frac{x}{7}\right)^5\right)=5\log_2\left(\frac{x}{7}\right)$ by the power rule. Then $5\log_2\left(\frac{x}{7}\right)=5(\log_2(x)-\log_2(7))$ by the quotient rule. Then $5(\log_2(x)-\log_2(7))=5\log_2(x)-5\log_2(7)$ by the distributive property.
   (b) This is not a true statement because it does not correctly use the product rule property. One way to fix the statement would be to write $\log_3(5(7x))$ as $\log_3(5)+\log_3(7x)$, which could be further expanded to $\log_3(5)+\log_3(7)+\log_3(x)$.

(c) This is a true statement because $4\log_6(x) - (\log_6(3) + \log_6(y)) = \log_6(x^4) - \log_6(3y)$ by the power rule and the product rule. The quotient rule can then be used to express $\log_6(x^4) - \log_6(3y)$ as $\log_6\left(\frac{x^4}{3y}\right)$.

(d) This is not a true statement because it does not correctly use the product rule. It is not possible to express $\log_4(6+x)$ in an equivalent form using properties of logarithms. It is possible to express $\log_4(6x)$ as $\log_4(6) + \log_4(x)$.

(e) This is a true statement because $\log_{10}\left(\frac{x^5y^2}{10}\right) = \log_{10}(x^5y^2) - \log_{10}(10)$ by the quotient rule. The product rule can be used to write $\log_{10}(x^5y^2) - \log_{10}(10)$ as $\log_{10}(x^5) + \log_{10}(y^2) - \log_{10}(10)$. Then the power rule can be used to express $\log_{10}(x^5) + \log_{10}(y^2) - \log_{10}(10)$ as $5\log_{10}(x) + 2\log_{10}(y) - \log_{10}(10)$. Because $\log_{10}(10) = 1$, this can be expressed as $5\log_{10}(x) + 2\log_{10}(y) - 1$.

3. (a) $\ln(12) = \ln(4 \cdot 3) = \ln(4) + \ln(3) = \ln(2^2) + \ln(3) = 2\ln(2) + \ln(3)$

(b) $\ln\left(\frac{3}{4}\right) = \ln(3) - \ln(4) = \ln(3) - \ln(2^2) = \ln(3) - 2\ln(2)$

(c) $\ln\left(8\sqrt{3}\right) = \ln(8) + \ln\left(\sqrt{3}\right) = \ln\left(2^3\right) + \ln\left(3^{\frac{1}{2}}\right) = 3\ln(2) + \frac{1}{2}\ln(3)$

4. (a) $2\ln(3) + \ln(7) = \ln(3^2) + \ln(7) = \ln(9) + \ln(7) = \ln(9 \cdot 7) = \ln(63)$

(b) $\log_2(6) - 3\log_2(5) = \log_2(6) - \log_2(5^3) = \log_2\left(\frac{6}{5^3}\right) = \log_2\left(\frac{6}{125}\right)$

(c)
$$\begin{aligned} -2\log_3(5) + 4 &= -2\log_3(5) + 4\log_3(3) \\ &= -2\log_3(5) + \log_3(3^4) \\ &= \log_3(5^{-2}) + \log_3(3^4) \\ &= \log_3(5^{-2} \cdot 3^4) \\ &= \log_3\left(\frac{1}{25} \cdot 81\right) \\ &= \log_3\left(\frac{81}{25}\right) \end{aligned}$$

5. (a) The function $f(x) = \log_{10}(5x)$ can be expressed as $\underline{f(x) = \log_{10}(5) + \log_{10}(x)}$. Function f is a vertical translation **up** of the function $g(x) = \log_{10}(x)$ by $\underline{\log_{10}(5)}$ units.

(b) The function $f(x) = \ln(3x^2)$ can be expressed as $\underline{f(x) = \ln(3) + 2\ln(x)}$. Function f is a vertical dilation of the function $g(x) = \ln(x)$ by a factor of $\underline{2}$ followed by a vertical translation **up** by $\underline{\ln(3) \text{ units}}$.

(c) The function $f(x) = \log_2(3x + 12)$ can be expressed as $\underline{f(x) = \log_2(3(x+4)) = \log_2(3) + \log_2(x+4)}$. Function f is a horizontal translation to the **right** of $g(x) = \log_2(x)$ by $\underline{4 \text{ units}}$ and a vertical translation **up** by $\underline{\log_2(3) \text{ units}}$.

(d) The function $f(x) = \ln\left(\dfrac{3}{x^2}\right)$ can be expressed as $\underline{f(x) = \ln(3) - 2\ln(x)}$. Function f is a vertical dilation of the function $g(x) = \ln(x)$ by a factor of $\underline{2}$, followed by a reflection across the ***x*-axis**, and then a vertical translation **up** by $\underline{\ln(3) \text{ units}}$.

LESSON 3.4

Applications of Logarithmic Functions

LEARNING OBJECTIVES

3.1.4 Express a logarithmic function in an equivalent form to reveal properties of the graph and/or the contextual scenario.

3.1.6 Solve equations involving exponential or logarithmic functions, including those arising from contextual scenarios.

LESSON OVERVIEW

CONTENT FOCUS

This lesson provides students with an opportunity to experience both the mathematical and practical utility of logarithmic functions. Students explore several contextual examples that demonstrate how logarithms are used to "tame" very large or very small numbers. The key feature of logarithms is that they can be used to turn very large or very small numbers into more reasonable quantities. This feature is often employed in practical applications where a quantity grows or decays by a constant factor. Logarithms map the values onto a linear scale, which makes the quantities involved far easier to comprehend.

AREAS OF FOCUS

- Engagement in Mathematical Argumentation
- Greater Authenticity of Applications and Modeling

SUGGESTED TIMING

~90 minutes

LESSON SEQUENCE

- This lesson is part of a lesson sequence (~365 minutes total) with Lessons 3.2 and 3.3.

MATERIALS

- calculator
- graph paper
- graphing utility such as **Desmos.com**

HANDOUTS

Lesson

- 3.4.A: Exploring the Richter Scale

Practice

- 3.4.B: Applications of Logarithmic Functions

LESSON DESCRIPTION

Part 1: Exploring the Utility of Logarithms

Students previously explored taking a logarithm as the inverse operation of exponentiating and as a function with interesting properties. In this part of the lesson, students investigate the key feature of logarithms: they provide a method of expressing very large or very small numbers as more reasonable quantities. That is, logarithms are a tool that scientists and mathematicians use to convey exponential growth with a linear scale. Students analyze a scatterplot that appears to be linear but observe that it has been produced using data that displays an exponential relationship. This helps students see that the relative magnitudes of very large or very small numbers are not immediately evident by inspection.

Part 2: Linearizing Exponential Data
In this part of the lesson, students explore what happens when a data set that exhibits exponential behavior is linearized by taking a logarithm of the dependent variable values. This process is useful when the values are extremely large or extremely small and when there is an exponential relationship between the quantities. Mathematicians and scientists often use logarithms to make the range of values of one or both quantities in a data set smaller, which makes them easier to manipulate. At the end of this part of the lesson, students connect the parameters of a linear model they determined using the transformed dependent variable values with the parameters of the original exponential function. The goal of this part of the lesson is for students to develop an understanding of how logarithms are useful in scientific contexts, and not for them to develop mastery of linearizing a data set.

Part 3: Understanding the Richter Scale
In this part of the lesson, students examine the Richter scale as a real-world application of logarithms. The Richter scale quantifies the magnitude of an earthquake by comparing the relative intensity of a given earthquake to the smallest measurable earthquake. The Richter scale uses logarithms to express the wide-ranging intensities with smaller numbers. This example also provides students with a chance to see how the laws of logarithms can be used to solve applied mathematics problems.

FORMATIVE ASSESSMENT GOAL

This lesson prepares students to complete the following formative assessment activity.

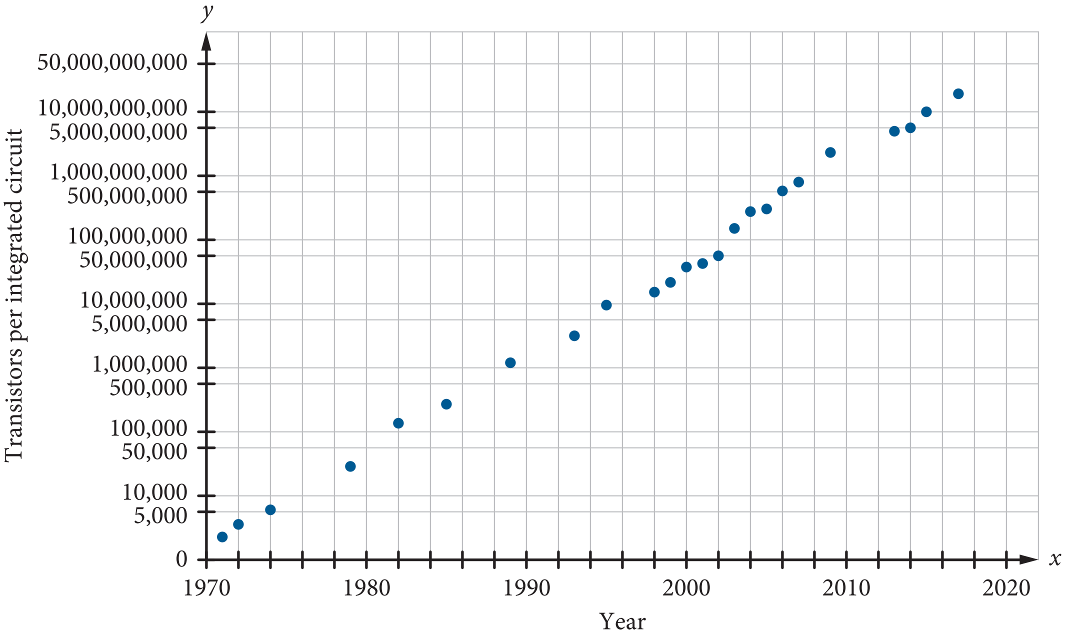
In 2011, the Tohoku earthquake off the coast of Japan caused a tsunami (a massive wave of ocean water that impacts land) and the eventual meltdown of the Fukushima Daiichi nuclear power plant. This magnitude 9.1 earthquake was the largest ever recorded in Japan and the fourth largest in modern history. In 2015, there was a magnitude 7.8 earthquake near Nepal. Use the formula $M = \log_{10}\left(\frac{I}{S}\right)$, which relates the magnitude of an earthquake, M, to the intensity of the standard earthquake, S, and the intensity of the earthquake in question, I, to determine how many times more powerful the 2011 Tohoku earthquake was than the 2015 Nepal earthquake. Show your work and explain your answer.

Part 1: Exploring the Utility of Logarithms

~15 MIN.

Students previously explored taking a logarithm as the inverse operation of exponentiating and as a function with interesting properties. In this part of the lesson, students investigate the key feature of logarithms: they provide a method of expressing very large or very small numbers as more reasonable quantities. That is, logarithms are a tool that scientists and mathematicians use to convey exponential growth with a linear scale. Students analyze a scatterplot that appears to be linear but observe that it has been produced using data that displays an exponential relationship. This helps students see that the relative magnitudes of very large or very small numbers are not immediately evident by inspection.

Student Task



- What do you notice and what do you wonder about the graph?
- What kind of function would you use to model the data? What evidence do you have to support your claim?

Facilitating the Task

- Share the image in a location where all students can see it. Then allow students some time to share what they notice and what they wonder about the graph. At this point, just record in a central location everything students state without addressing the correctness of their observations, answering any questions, or asking any follow-up questions. This will encourage students to be more open about what they notice and wonder.

- If students would benefit from more information about Moore's law, you can share with them that Moore's law is an observation in the field of computer science that the number of transistors that manufacturers can fit on an integrated circuit doubles approximately every 2 years. The number of transistors on an integrated circuit is a good measure for computing speed, which means that every 2 years or so computers using the latest integrated circuit become "twice as fast." One reason why our computers (and cell phones) get smaller and become more powerful over time is that more transistors can fit on an integrated circuit.
- Next you can have students share what kind of function they might use to model the data. It is likely that students will suggest a linear function because the distribution of points on the scatterplot appears roughly linear.
- You can encourage some deeper analysis by asking students some probing questions like the following:
  - What do you notice about the values on the horizontal axis?

    The values on the horizontal axis are years starting in 1970. They are labeled in increments of two.
  - What do you notice about the values on the vertical axis?

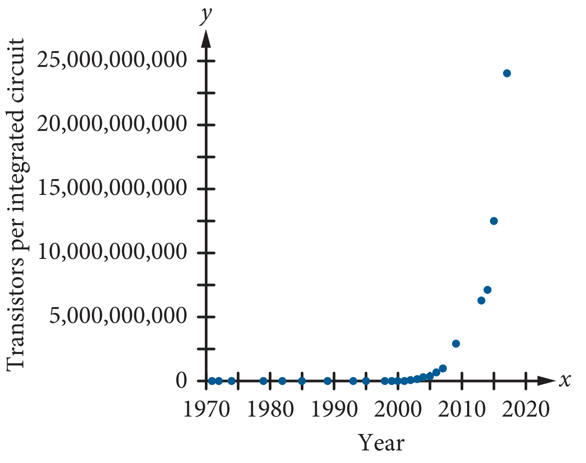
    The values on the vertical axis are the number of transistors, but their labels do not increase by a constant increment. They seem to increase by factors of 10 (e.g., 1,000, 10,000, 100,000, …) and include the values halfway between consecutive factors (e.g., 5,000 and 50,000).
  - What does this mean about the distance between the endpoints of the intervals indicated on the vertical axis?

    The distance between the endpoints of successive intervals (e.g., 1,000 to 10,000; 10,000 to 100,000) increases. That is, each interval is longer than the nonoverlapping interval immediately preceding it.
  - What kind of function have we used to model data where the output increases by a factor instead of a constant amount?

    We usually use exponential functions to model data where the output grows by equal factors over equal intervals.
  - Do you think an exponential function could be a good model for the data displayed on the scatterplot? Why or why not?

    Because the lengths of the intervals on the vertical axis are getting larger, an exponential function might be an appropriate model.

- Allow students some time to think about their answer to the last question. It is intended to let students ponder what kind of function might best model this data.
- After students have had a chance to discuss the possibility of modeling the data with an exponential function, you might want to show them another representation of the same data where the vertical axis is labeled in constant increments of 5 billion:



Guiding Student Thinking

Instead of showing students a graph in which both horizonal and vertical axes are shown with linear scales, you could have students take some time to construct a graph with these features using a graphing utility and values that they estimate from the Moore's law graph in the Student Task. This may help them internalize the difference between the different scales of the vertical axis.

- Once you show students the new graph, ask them again if they think an exponential function would be a good model for the data and why. Students may identify that because the number of transistors doubles approximately every two years, then an exponential function is likely an appropriate model.

Summarizing the Task

- The data associated with Moore's law provides students with an opportunity to examine the magnitude of large numbers and understand how challenging it can be to represent graphically a data set whose values increase dramatically relatively quickly.
- The Student Task had students analyze a graph where the vertical axis was scaled logarithmically rather than linearly. This kind of graph is called a *semi-log graph* because one axis uses powers of 10 rather than equal distances between the values.

- A semi-log graph can be a very effective way to express large numbers because it has the effect of presenting the data linearly rather than exponentially. This way of using logarithms to present data more manageably is one of the common uses of logarithms in modern science.
- You can let students know that in the next part of the lesson, they will explore how to use logarithms to transform an exponential function into a linear one.

Part 2: Linearizing Exponential Data

~30 MIN.

In this part of the lesson, students explore what happens when a data set that exhibits exponential behavior is linearized by taking a logarithm of the dependent variable values. This process is useful when the values are extremely large or extremely small and when there is an exponential relationship between the quantities. Mathematicians and scientists often use logarithms to make the range of values of one or both quantities in a data set smaller, which makes them easier to manipulate. At the end of this part of the lesson, students connect the parameters of a linear model they determined using the transformed dependent variable values with the parameters of the original exponential function. The goal of this part of the lesson is for students to develop an understanding of how logarithms are useful in scientific contexts, and not for them to develop mastery of linearizing a data set.

Student Task

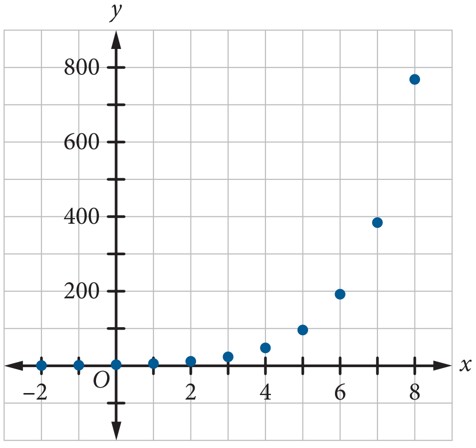
We want to explore how quickly the output values of an exponential function get too large to easily graph using constant scaling on both axes. Using the function you are assigned, create a table of values for integer inputs of –2 through 8. Then graph the function on graph paper.

x	$f(x)$
–2	
–1	
0	
1	
2	
3	
4	
5	
6	
7	
8	

Facilitating the Task

- Begin by assigning an exponential function to each pair of students. You may want to limit the number of different exponential functions you assign to the class so that you can quickly analyze any student's work as you circulate around the room. The solutions given in this part of the task are for the function $f(x) = 3(2^x)$. Some other reasonable functions you could assign include $f(x) = 5(2^x)$, $f(x) = 2(3^x)$, and $f(x) = 4(3^x)$. These functions have sufficiently large output values on the indicated interval of the domain to be interesting, but not so large that students will get lost.
- Allow students some time to construct a table of values for their assigned exponential function and create their graph. A sample table of values and graph are shown here:

x	$f(x)$
−2	0.75
−1	1.5
0	3
1	6
2	12
3	24
4	48
5	96
6	192
7	384
8	768



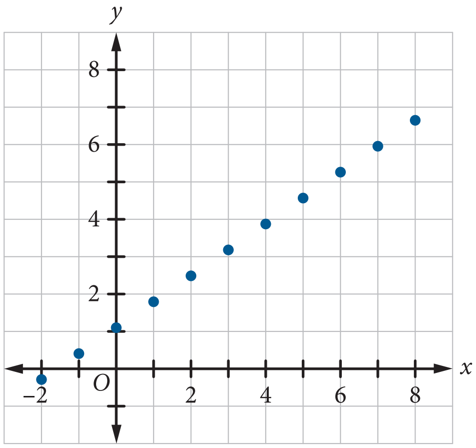
- As students finish their tables of values and plot the points they calculated, bring the class back together to have them share some observations they made while creating their graphs.
  - What do you notice about the output values of the function?

    The output values of the function are quickly getting very large.
  - How will this rapid increase in the output values affect the graph that you made?

    As the output values continue to get larger, they will soon be greater than our graph can accommodate.
- Let students know that they can use logarithms to “tame” the rapid growth they observed in their output values.

- Ask students to add a third column to their table of values. In this third column, they should take the natural logarithm of each output value. They can round these values to three decimal places. Once they have a table of values, they can plot the points on graph paper and make some observations about the graph. A sample table of values and graph follow:

x	$f(x)$	Natural log of $f(x)$
−2	0.75	−0.287
−1	1.5	0.405
0	3	1.099
1	6	1.792
2	12	2.485
3	24	3.178
4	48	3.871
5	96	4.564
6	192	5.257
7	384	5.951
8	768	6.644



Guiding Student Thinking

Students could use a logarithm of any base to calculate the values in the third column. If students are using a handheld calculator, then either the natural logarithm or the logarithm base 10 would be a good choice. If students use a logarithm with the same base as their function to calculate the values in the third column, the resulting graph will not be as interesting, because the exponential function and the logarithm function with the same base are inverses. In that case, the values in the third column would be the product of x and the coefficient a.

- As students complete the third column and graph their points, bring the class back together to have them share some observations they made while creating their graphs.
  - What do you notice about this graph? How is it different from the first graph?

    This graph appears to be linear. The first graph was exponential.

- Why do you think that taking the natural logarithm of the y-values transformed the appearance of the graph from exponential to linear?

  Exponential and logarithmic functions are inverses, which might explain why the exponential data appears linear after taking the natural logarithm of the y-values.

- Provide students with some time to think about the last question. Students may suggest that the inverse relationship between exponential and logarithmic functions has something to do with why the data set with the transformed output values appears linear when plotted.
- At this point, have students use a graphing utility to determine a linear regression equation for the x-values and the natural logarithm of the y-values. Encourage students to plot the residuals so that they can observe that the residual for every point is zero, meaning that the line perfectly fits the data. For $f(x) = 3 \cdot 2^x$, the regression equation is $y = 0.693x + 1.099$.
- Let students know that the laws of logarithms can be used to demonstrate that taking the logarithm of an exponential function yields a linear function.
- You may need to guide students a bit during this portion of the lesson. Start by having students express their transformed relationship algebraically. They should observe that taking the natural logarithm of $f(x)$ creates a new function that can be expressed as $g(x) = \ln\left(3 \cdot 2^x\right)$.
- Provide students some time to use the laws of logarithms to rewrite this new function g in an equivalent form. You can remind them that the graph of g looks linear, so they might be able to use the laws of logarithms to make the algebraic representation of g have the form of a linear function. Their work should look something like this:

$$\begin{aligned} \text{Original function:} \quad & g(x) = \ln\left(3 \cdot 2^x\right) \\ \text{Applying the Product rule:} \quad & g(x) = \ln(3) + \ln\left(2^x\right) \\ \text{Applying the Power rule:} \quad & g(x) = \ln(3) + x\ln(2) \end{aligned}$$

- At this point, you can suggest that students find the decimal approximation for each logarithmic expression in their function. This may help them see that they have written a linear function that is equivalent to the regression equation they determined. That is, $g(x) = \ln(3) + x\ln(2)$ can be expressed as $g(x) \approx 1.099 + 0.693x$.

Summarizing the Task

- The key property of logarithms is that they turn large or small numbers into more reasonable quantities that are easier to analyze. This transformative power of logarithms is often used in scientific contexts so that very large numbers can be expressed more concisely.
- There are several real-world examples of logarithmic scales with which students may be familiar, such as the decibel scale, the pH scale, and the Richter scale. The Richter scale is the focus of Part 3 of the lesson.
- The main takeaway from the first two parts of the lesson is that logarithms are a powerful tool for expressing large numbers as smaller, more manageable quantities. They are also used to turn small numbers, such as those expressed in scientific notation with negative exponents, into more reasonable quantities.

Part 3: Understanding the Richter Scale

~45 MIN.

In this part of the lesson, students examine the Richter scale as a real-world application of logarithms. The Richter scale quantifies the magnitude of an earthquake by comparing the relative intensity of a given earthquake to the smallest measurable earthquake. The Richter scale uses logarithms to express the wide-ranging intensities with smaller numbers. This example also provides student with a chance to see how the laws of logarithms can be used to solve applied mathematics problems.

Instructional Rationale

The Richter scale is not currently used by seismologists because the scale works best for describing certain categories of earthquakes, such as those that occur in Southern California, and therefore can give misleading information for types of earthquakes that occur elsewhere in the world. Regardless, the Richter scale is a good example of how logarithms can be used to express large numbers using smaller, more manageable numbers.

Student Task

The student task is provided on **Handout 3.4.A: Exploring the Richter Scale**. In this task, students learn how the Richter scale works. Students examine some facts about two earthquakes that occurred in Chile and California and respond to questions regarding these events. Solutions to the problems on the handout are provided in the Assess and Reflect section of the lesson.

Facilitating the Task

- Begin by providing students with Handout 3.4.A. The handout includes an introductory paragraph about seismologist Charles Richter's intent when he devised the Richter scale.
- You may find it helpful to show students the first 3 minutes and 24 seconds of the video, "How Does the Richter Scale Work?" which can be found at **www.youtube.com/watch?v=NaNw9LHq9dc**. After the video, allow students some time to share with the class what they learned about the Richter scale.
- Before having students answer the questions on the handout, be sure that they understand that the Richter scale quantifies the magnitude of an earthquake by taking the logarithm of the ratio of the intensity of a given earthquake to the intensity of the smallest measurable earthquake, which Richter called the standard earthquake. The magnitude of an earthquake measures the energy released at the earthquake's source. The intensity of an earthquake measures the strength of the shaking caused by the earthquake.
- Allow students time to work on the problems on Handout 3.4.A in small groups of four.
- As you circulate around the room while students work, be sure that they are using the properties of logarithms to answer the questions. You may need to provide some guidance to students who are unsure of how to answer the questions. Be attentive to students who may need support in understanding that the intensities of the earthquakes refer to how they compare to a benchmark standard earthquake rather than to an absolute measurement.
- As students complete the problems on the handout, it is important to allot time for groups to share their work and discuss their reasoning.

Classroom Ideas

Some students may not be familiar with how frequently small earthquakes occur. You can use the United States Geological Survey website, found at **earthquake.usgs.gov/**, to show students how many earthquakes occurred in the past 24 hours.

Summarizing the Task

- The goal of the Richter scale task is for students to see that some useful scales that scientists use to quantify intensities are logarithmic. That is, the magnitudes of earthquakes are reported using a linear scale, while their intensities span a much greater range. Logarithms are used to make very large values (or very small values, in the case of decibels) that represent the intensities more manageable.

- Be sure to work through problem 6 on Handout 3.4.A as a class. Students may employ a variety of different reasoning strategies that all result in the same solution. It is possible to use the product and quotient rules as well as the approach shown in the Assess and Reflect section of the lesson.
- For logarithmic scales like the Richter scale, the decibel scale, or the pH scale, a one-unit increase on the scale corresponds with a 10-fold increase in the intensity that the scale measures. Each one-unit increase on the Richter scale means an earthquake is 10 times more powerful; each one-unit increase on the decibel scale means a sound is 10 times more intense; each one-unit increase on the pH scale means that a solution is 10 times more basic.
- There is a short set of problems on **Handout 3.4.B: Applications of Logarithmic Functions**. This set uses the decibel scale, which quantifies the intensity of sound relative to the quietest sound that a human ear can hear.

UNIT 3

Assess and Reflect on the Lesson

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

In 2011, the Tohoku earthquake off the coast of Japan caused a tsunami (a massive wave of ocean water that impacts land) and the eventual meltdown of the Fukushima Daiichi nuclear power plant. This magnitude 9.1 earthquake was the largest ever recorded in Japan and the fourth largest in modern history. In 2015, there was a magnitude 7.8 earthquake near Nepal. Use the formula $M = \log_{10}\left(\frac{I}{S}\right)$, which relates the magnitude of an earthquake, M, to the intensity of the standard earthquake, S, and the intensity of the earthquake in question, I, to determine how many times more powerful the 2011 Tohoku earthquake was than the 2015 Nepal earthquake. Show your work and explain your answer.

One way to determine how many times more powerful one earthquake is than another involves converting the magnitudes to intensities and then determining the ratio of the intensities.

2011 Tohoku earthquake:

$$9.1 = \log_{10}\left(\frac{I}{S}\right)$$

$$10^{9.1} = \frac{I}{S}$$

$$10^{9.1} \cdot S = I$$

The Tohoku earthquake is $10^{9.1}$ times more powerful than the standard earthquake.

2015 Nepal earthquake:

$$7.8 = \log_{10}\left(\frac{I}{S}\right)$$

$$10^{7.8} = \frac{I}{S}$$

$$10^{7.8} \cdot S = I$$

The Nepal earthquake is $10^{7.8}$ times more powerful than the standard earthquake.

The ratio of the intensities, $\frac{10^{9.1} \cdot S}{10^{7.8} \cdot S} \approx 19.953$ suggests that the Tohoku earthquake was almost 20 times more powerful than the Nepal earthquake.

Guiding Student Thinking

An alternative way to approach the formative assessment goal involves the quotient rule for logarithms. Because the objective of the problem is to determine the ratio of the intensities, students may observe that it is possible to find the difference in magnitudes of the earthquakes, $9.1 - 7.8 = 1.3$, which quantifies its magnitude relative to the standard earthquake. The intensity of an earthquake with a magnitude of 1.3 is $10^{1.3}$ times more powerful than the standard earthquake. Because the standard earthquake has essentially no intensity, a difference in magnitude of 1.3 translates to an earthquake that is $10^{1.3} \approx 19.953$ times more powerful.

HANDOUTS

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 3.4.A: Exploring the Richter Scale

1. The magnitude of the standard earthquake can be determined by $M = \log_{10}\left(\frac{S}{S}\right) = \log_{10}(1) = 0$. Therefore, the magnitude of the standard earthquake is 0.
2. The magnitude of the Valdivia earthquake in 1960, which was 3.2 billion times more powerful than the standard earthquake, can be determined by $M = \log_{10}\left(\frac{3{,}200{,}000{,}000S}{S}\right) = \log_{10}(3{,}200{,}000{,}000) \approx 9.5$. Therefore, the magnitude of the Valdivia earthquake was about 9.5.
3. A magnitude 6 earthquake has an intensity of $10^6 \cdot S$, which means it is one million times more powerful than the standard earthquake. A magnitude 5 earthquake has an intensity of $10^5 \cdot S$, which means it is 100,000 times more powerful than the standard earthquake. The intensities of magnitude 6 and magnitude 5 are calculated as follows:

$$6 = \log_{10}\left(\frac{I_6}{S}\right) \qquad\qquad 5 = \log_{10}\left(\frac{I_5}{S}\right)$$

$$10^6 = \frac{I_6}{S} \qquad\qquad 10^5 = \frac{I_5}{S}$$

$$10^6 S = I_6 \qquad\qquad 10^5 S = I_5$$

This means that a magnitude 6 earthquake is 10 times more powerful than a magnitude 5 earthquake because $\frac{I_6}{I_5} = \frac{10^6 \cdot S}{10^5 \cdot S} = 10$.

4. A magnitude 8 earthquake has an intensity of $10^8 \cdot S$, which means it is 100 million times more powerful than the standard earthquake. A magnitude 5 earthquake has an intensity of $10^5 \cdot S$, which means it is 100,000 times more powerful than the standard earthquake. The intensities of magnitude 8 and magnitude 5 are calculated as follows:

$$8 = \log_{10}\left(\frac{I_8}{S}\right) \qquad\qquad 5 = \log_{10}\left(\frac{I_5}{S}\right)$$

$$10^8 = \frac{I_8}{S} \qquad\qquad 10^5 = \frac{I_5}{S}$$

$$10^8 S = I_8 \qquad\qquad 10^5 S = I_5$$

This means that a magnitude 8 earthquake is 1,000 times more powerful than a magnitude 5 earthquake because $\frac{I_8}{I_5} = \frac{10^8 \cdot S}{10^5 \cdot S} = 10^3 = 1{,}000$.

5. An increase in magnitude by 1 on the Richter scale translates to a 10-fold increase in intensity. This is because an earthquake with magnitude M has an intensity of $10^M \cdot S$ while a magnitude $M+1$ earthquake has an intensity of $10^{M+1} \cdot S$.

$$M = \log_{10}\left(\frac{I_M}{S}\right) \qquad\qquad M+1 = \log_{10}\left(\frac{I_{M+1}}{S}\right)$$

$$10^M = \frac{I_M}{S} \qquad\qquad 10^{M+1} = \frac{I_{M+1}}{S}$$

$$10^M S = I_M \qquad\qquad 10^{M+1} S = I_{M+1}$$

This means that a magnitude $M+1$ earthquake is 10 times more powerful than a magnitude M earthquake because $\frac{I_{M+1}}{I_M} = \frac{10^{M+1} \cdot S}{10^M \cdot S} = 10^1 = 10$.

6. We need to use the magnitude and intensity of the Valdivia earthquake to determine the magnitude and intensity of the San Francisco earthquake. We can call the intensity of the Valdivia earthquake I_V and the magnitude of the Valdivia earthquake M_V. Likewise, we can call the intensity of the San Francisco earthquake I_{SF} and the magnitude of the San Francisco earthquake M_{SF}. In problem 2, we found that $M_V = 9.5$, but we have to determine M_{SF} using the fact that $I_V = 50 I_{SF}$ or, alternatively, $I_{SF} = \frac{I_V}{50}$. This means that the intensity of the San Francisco earthquake

is $I_{SF} = \dfrac{3{,}200{,}000{,}000S}{50} = 64{,}000{,}000S$, or 64 million times more powerful than the standard earthquake. Therefore:

$$\begin{aligned} M_{SF} &= \log_{10}\left(\frac{I_{SF}}{S}\right) \\ M_{SF} &= \log_{10}\left(\frac{64{,}000{,}000S}{S}\right) \\ M_{SF} &= \log_{10}(64{,}000{,}000) \\ M_{SF} &\approx 7.8 \end{aligned}$$

Handout 3.4.B: Applications of Logarithmic Functions

1. The quietest sound the average human ear can hear is I_0. Therefore, the decibel level is $\text{dB} = 10\log_{10}\left(\dfrac{I_0}{I_0}\right) = 10\log_{10}(1) = 0$.
2. A sound that is one million times louder than the quietest sound the average human ear can hear measures $\text{dB} = 10\log_{10}\left(\dfrac{1{,}000{,}000I_0}{I_0}\right) = 10\log_{10}(1{,}000{,}000) = 60$ dB.
3. A sound that measures 85 dB is $10^{8.5} \approx 316{,}227{,}766$ times louder than I_0. The relationship between I, the intensity of a sound that measures 85 dB, and I_0, the quietest sound the average human ear can hear, is calculated as follows:

$$\begin{aligned} 85 &= 10\log_{10}\left(\frac{I}{I_0}\right) \\ 8.5 &= \log_{10}\left(\frac{I}{I_0}\right) \\ 10^{8.5} &= \frac{I}{I_0} \\ 10^{8.5} \cdot I_0 &= I \end{aligned}$$

4. A sound that measures 126 dB is $10^{12.6} \approx 4{,}000{,}000{,}000{,}000$ (four trillion) times louder than I_0. The relationship between I, the intensity of a sound that measures 126 dB, and I_0, the quietest sound the average human ear can hear, is calculated as follows:

$$\begin{aligned} 126 &= 10\log_{10}\left(\frac{I}{I_0}\right) \\ 12.6 &= \log_{10}\left(\frac{I}{I_0}\right) \\ 10^{12.6} &= \frac{I}{I_0} \\ 10^{12.6} \cdot I_0 &= I \end{aligned}$$

5. Since the decibel level of a sound is determined by the formula $\text{dB} = 10\log_{10}\left(\frac{I}{I_0}\right)$, the intensities can be substituted into the formula to determine the decibel level of a sound whose intensity measures $10^{-3}\frac{W}{m^2}$, as follows:

$$\text{dB} = 10\log_{10}\left(\frac{10^{-3}\frac{W}{m^2}}{10^{-12}\frac{W}{m^2}}\right)$$
$$\text{dB} = 10\log_{10}\left(10^9\right)$$
$$\text{dB} = 10(9)$$
$$\text{dB} = 90$$

Sounds with decibel levels greater than 85 dB are generally considered unsafe, so a sound with an intensity of $10^{-3}\frac{W}{m^2}$ would be considered unsafe.

PRACTICE PERFORMANCE TASK

Modeling the Relationship Between Pressure and Volume

LEARNING OBJECTIVES

1.1.1 Identify a function family that would appropriately model a data set or contextual scenario.

1.1.3 Construct a representation of a linear, quadratic, or exponential function both with and without technology.

3.1.3 Construct a representation of a logarithmic function.

3.1.4 Express a logarithmic function in an equivalent form to reveal properties of the graph and/or the contextual scenario.

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Connections Among Multiple Representations

SUGGESTED TIMING

~45 minutes

HANDOUT

- Unit 3 Practice Performance Task: Modeling the Relationship Between Pressure and Volume

PRACTICE PERFORMANCE TASK DESCRIPTION

In this practice performance task, students explore how logarithms can be used to “straighten” data so that a linear model can be fitted to the data. The goal of this practice performance task is for students to explore how transforming a data set using a logarithm yields a new data set that can be modeled with a linear function. Students then use their understanding of logarithms to express their functions in an equivalent form. The data set presented in this task is drawn from renowned chemist and physicist Robert Boyle’s original investigation of the relationship between the volume and pressure of a gas in a container. This relationship is most appropriately modeled by a simple rational function because volume and pressure are inversely proportional. Boyle’s original data used the height of a cylinder as a stand-in for the volume of the gas in the container.

AP Connections

This practice performance task supports AP preparation through alignment to the following AP Calculus Mathematical Practices:

- **1.E** Apply appropriate mathematical rules or procedures, with and without technology.

This practice performance task supports AP preparation through alignment to the following AP Statistics Course Skills:

- **4.B** Interpret statistical calculations and findings to assign meaning or assess a claim.

ELICITING PRIOR KNOWLEDGE

The goal of this task is for students to demonstrate their understanding of scatterplots, function families, lines of best fit, and logarithms. The type of problem in the task may be unfamiliar to students if they have not had prior experience exploring problems involving data straightening.

- You may find it helpful to prepare students to engage in the task by asking them to complete a few warm-up questions to elicit their prior knowledge, such as:
  - What do the parameters m and b represent in a linear function?

    In a linear function, m represents the slope of the line and the rate of change of the dependent variable with respect to the change in the independent variable. The parameter b is the vertical axis intercept, which represents the value of the dependent variable when the value of the independent variable is zero. It is sometimes called the initial value for the function.

  - How could you determine whether a data set is more appropriately modeled by a linear function or by an exponential function?

    A data set that exhibits a roughly constant rate of change would be appropriately modeled by a linear function. A data set whose associated scatterplot appears as a cluster of points that is roughly football-shaped is often appropriately modeled by a linear function. A data set that exhibits a roughly constant ratio of dependent variable values over equal intervals of independent variable values would be appropriately modeled by an exponential function. A data set whose associated scatterplot shows a cluster of points in a curved shape might be appropriately modeled by an exponential function.

  - Express these functions in an equivalent form using laws of exponents and laws of logarithms: $f(x)=e^{x+2}$, $g(x)=\frac{4}{e^x}$, $h(x)=\ln\left(5x^{-3}\right)$, and $k(x)=\frac{1}{2}\ln(x)-8$.

    An equivalent form for $f(x)=e^{x+2}$ is $f(x)=e^x \cdot e^2$.

    An equivalent form for $g(x)=\frac{4}{e^x}$ is $g(x)=4e^{-x}$.

    An equivalent form for $h(x)=\ln\left(5x^{-3}\right)$ is $h(x)=\ln(5)-3\ln(x)$.

    An equivalent form for $k(x)=\frac{1}{2}\ln(x)-8$ is $k(x)=\ln\left(\sqrt{x}\right)-\ln\left(e^8\right)=\ln\left(\frac{\sqrt{x}}{e^8}\right)$.

- If students struggle with the warm-up questions, it could indicate that they are not yet fully prepared to engage in the practice performance task. You may find it beneficial to provide a just-in-time review of the properties of logarithms. Lessons 3.3 and 3.4 offer several opportunities for students to practice using properties of logarithms to represent expressions in equivalent forms.

SUPPORTING STUDENTS

Here are a few possible implementation strategies you can use to help students engage with the task. You should rely on your knowledge of your students and your professional expertise to determine how to provide appropriate scaffolds while maintaining the cognitive demand of the task.

- **Previewing the Task:** To support students in identifying key features of the problem, you could display the introductory text and image of the problem to read and review with the entire class. This practice performance task has a great deal of information that students must read and analyze. To be historically authentic, the data set for the practice performance task involves fractions, which could be an obstacle for some students. Allow time for students to ask clarifying questions about anything they observe. This strategy may prevent students from working unproductively on the task.
- **Collaboration:** This practice performance task could be particularly challenging for students if they have not engaged in data analysis problems at regular intervals throughout the course. Many students would benefit from working with a partner so that they can discuss their approach to the task and help each other through the technological components of the task.
- **Using Technology:** If students use Desmos to complete the task but are somewhat unfamiliar with Desmos commands, then you may need to help them efficiently determine the natural logarithm of the height and pressure and plot these quantities.
  - Students should not manually determine the natural logarithm of each value in the data set. If students enter a table of values in Desmos, inputting the height data into the "x_1" column and the pressure data into the "y_1" column, then they could add a new column to the table by clicking to the right of the y_1 column and typing "$\ln(x_1)$" or "$\ln(y_1)$" into the header row. Desmos then calculates the corresponding values into that column.
  - By default, Desmos uses the first column as the independent variable and the other columns as the dependent variables for the purpose of plotting the points. Students can turn off the points plotted by default by clicking on the circle icon

in the header row. One way to plot the points for a data set is for students to use a command like "$(x_1, \ln(y_1))$", which directs Desmos to plot an ordered pair.

- It is possible to define a new list, such as x_2, to be the set of data that is the natural log of the data in list x_1. Students can use the command "$x_2 = \ln(x_1)$". This is advantageous because students can use the new variable in the regression command.

- **Debriefing the Task:** Part (e) of the task is intended to help students more deeply understand how properties of logarithms can be used to express logarithmic functions in multiple equivalent forms. In this case, if students perform a logarithmic transformation on both variables in the data set, they can derive the more familiar inversely proportional relationship between height and pressure. The symbolic manipulations will be familiar to students, but they may not be experienced with using properties of logarithms to condense an expression. The derivation shown below uses the natural logarithm, but a logarithm of any base would be equally appropriate. Note that the manipulation from line 4 to line 5 of the derivation that follows can be used to remind students of the inverse relationship between logarithmic and exponential functions.

Let h represent the height of the cylinder in inches. Then the natural log of the height, $\hat{h}$, can be expressed by the equation $\hat{h} = \ln(h)$. If p represents the pressure of a gas in inches of mercury, then the natural log of the pressure, $\hat{p}$, can be expressed by the equation $\hat{p} = \ln(p)$. The trend of data in a scatterplot of $(\hat{h}, \hat{p})$ can be appropriately modeled by a linear function, where $\hat{p} = m\hat{h} + b$. Using the properties of logarithms, this function can be expressed in an equivalent form:

$$\begin{aligned}
\hat{p} &= m\hat{h} + b \\
\hat{p} &= -1.00\hat{h} + 7.25 \\
\ln(p) &= -1.00\ln(h) + 7.25 \\
\ln(p) &= \ln\left(h^{-1.00}\right) + 7.25 \\
\ln(p) &= \ln\left(h^{-1.00}\right) + \ln\left(e^{7.25}\right), \text{ because } \ln\left(e^{7.25}\right) = 7.25 \\
\ln(p) &= \ln\left(h^{-1.00} \cdot e^{7.25}\right) \\
p &= e^{7.25} \cdot h^{-1.00} \\
p &\approx \frac{1{,}408}{h}
\end{aligned}$$

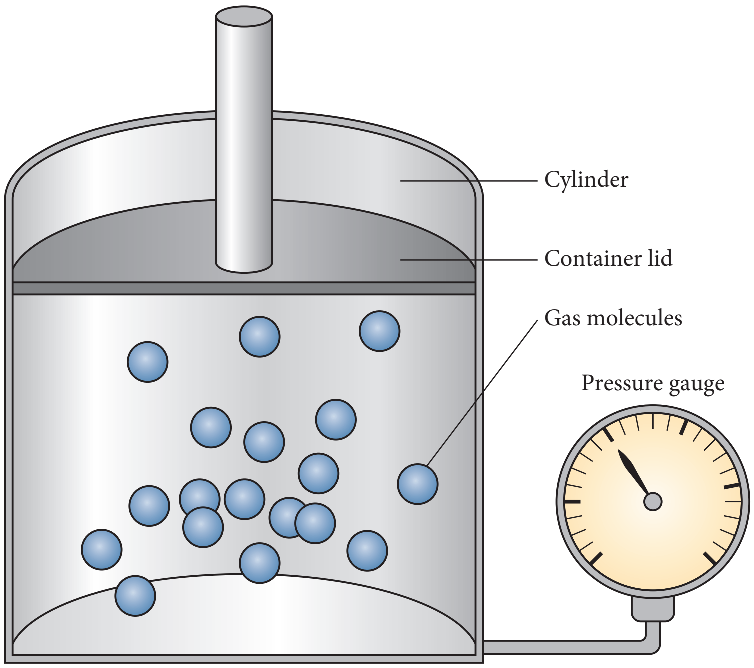
SCORING STUDENT WORK

Whether you decide to have students score their own solutions, have students score their classmates' solutions, or score the solutions yourself, you should use the results of the practice performance task to inform further instruction.

Modeling the Relationship Between Pressure and Volume

In 1662, chemist and physicist Robert Boyle published a conjecture about the relationship between the volume and pressure of a gas in a sealed container. After Boyle carried out extensive experimentation, his conjecture became known as Boyle's law. Essentially, he discovered that as the pressure of a gas decreases, its volume increases. You can see Boyle's law at work when a bicycle tire is inflated, an aerosol can sprays paint, or a syringe uses a vacuum to inject medication or withdraw samples of a fluid.

A modern version of Boyle's experiment can be demonstrated by a cylindrical container with a modified lid, whose vertical position within the container can change while maintaining its airtight seal. The movable lid effectively allows the height of the container to be changed. This apparatus allows an experimenter to measure the pressure of the gas, in inches of mercury, at different container heights, in inches (in.), to determine how the height of the cylinder and the pressure of the gas are related. Some of Boyle's original data follows the diagram. (Note: Boyle used fractions to record his data because decimals were not in common use in the late 1600s.)



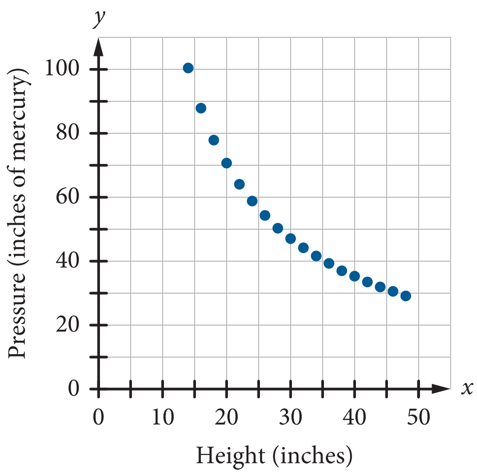
Height of container (inches)	Pressure of the Gas in the Container (inches of mercury)
48	$29\frac{2}{16}$
46	$30\frac{9}{16}$
44	$31\frac{15}{16}$
42	$33\frac{8}{16}$
40	$35\frac{5}{16}$
38	37
36	$39\frac{5}{16}$
34	$41\frac{10}{16}$
32	$44\frac{3}{16}$

Height of container (inches)	Pressure of the Gas in the Container (inches of mercury)
30	$47\frac{1}{16}$
28	$50\frac{5}{16}$
26	$54\frac{5}{16}$
24	$58\frac{13}{16}$
22	$64\frac{1}{16}$
20	$70\frac{11}{16}$
18	$77\frac{14}{16}$
16	$87\frac{14}{16}$
14	$100\frac{7}{16}$

There are 15 possible points for this practice performance task.

Student Stimulus and Part (a)

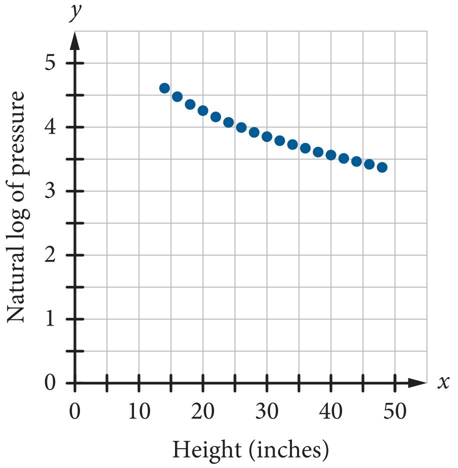
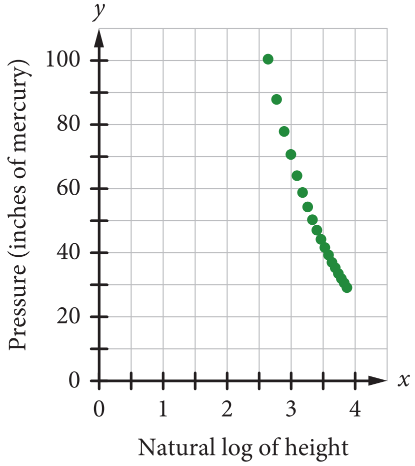
(a) Using a graphing tool, construct a scatterplot of the data. Think carefully about which set of measurements should be assigned as the *independent variable* and which should be assigned as the *dependent variable*. Given what you know about different function families, what type of function might serve as a good model for this data set? Explain your choice of function.

Sample Solution	Points Possible
A scatterplot of the data where the height of the cylinder is the independent variable and the pressure of the gas is the dependent variable looks like this:  Based on the scatterplot, it looks like an exponential function could be a good model for the data set. The values for height decrease by 2 in. for every row of the table and the pressure increases by a factor of about 1.04 for every row of the table. *Scoring note:* Students can choose either quantity to be the independent variable. This choice will affect all parts of the problem. Students should receive full credit if their work is correct and consistent throughout the problem.	**3 points maximum** 1 point for a correct scatterplot of height and pressure data, including labels 1 point for stating a type of function model that would be appropriate for this data set 1 point for a clear explanation that justifies the function model selected by the student *Scoring note:* Students can choose almost any function family for their model, but they must provide a clear and mathematically correct explanation.

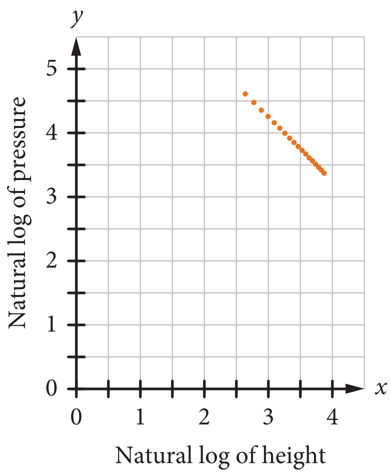
UNIT 3

Student Stimulus and Part (b)

(b) Construct three additional scatterplots that use the natural logarithm of one or both measurements: (i) *height* vs. $\ln(pressure)$, (ii) $\ln(height)$ vs. *pressure*, and (iii) $\ln(height)$ vs. $\ln(pressure)$. For each graph, describe how taking the logarithm of the data values affects the visualization of the data.

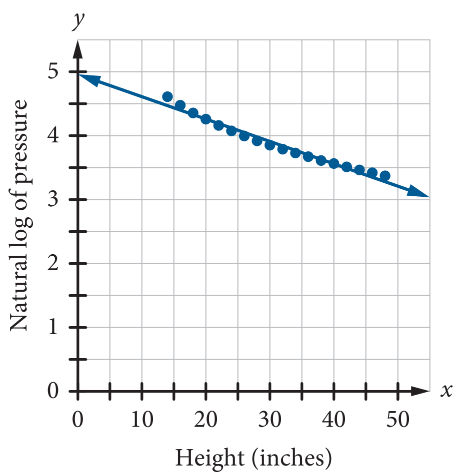
Sample Solution	Points Possible
The scatterplot for $(height, \ln(pressure))$ looks like:  The scatterplot for $(\ln(height), pressure)$ looks like: 	**4 points maximum** 1 point for a correct scatterplot of $(height, \ln(pressure))$, including axes labels 1 point for a correct scatterplot of $(\ln(height), pressure)$, including axes labels 1 point for a correct scatterplot of $(\ln(height), \ln(pressure))$, including axes labels 1 point for explaining that in each case, taking a logarithm of the data produces a graph that is straighter than the scatterplot for $(height, pressure)$

The scatterplot for $(\ln(height), \ln(pressure))$ looks like:



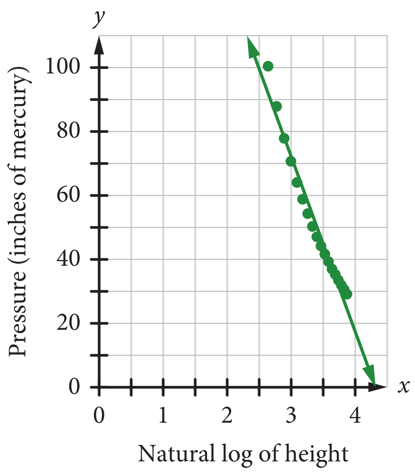
Student Stimulus and Part (c)

(c) Choose one of the scatterplots from part (b) that would be appropriately modeled by a linear function and fit a linear model to it. Explain the meaning of the parameters of your model.

Sample Solution	Points Possible
If the student used $(height, \ln(pressure))$, then the graph looks like: 	**3 points maximum** 1 point for fitting an appropriate linear model to the data 1 point for a correct explanation of the slope parameter for the linear model of the transformed data set selected by the student 1 point for a correct interpretation of the vertical axis intercept for the linear model of the transformed data set selected by the student

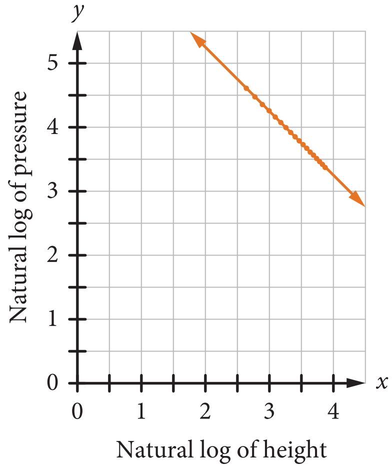
A reasonable linear model for this transformed data set is $\ln(p) = -0.035h + 4.960$. The slope of the line indicates that the natural log of pressure decreases by about 0.035 units for each 1-in. increase of the height of the cylinder. The vertical axis intercept of 4.960 indicates that when the height is 0 in., the natural log of pressure would be 4.960.

If the student used $\left(\ln(\textit{height}), \textit{pressure}\right)$, then the graph looks like:



A reasonable linear model for this transformed data set is $p = -54.770\ln(h) + 236.529$. The slope of the line indicates that the pressure decreases by about −54.770 inches of mercury for each one-unit increase of the natural log of the height. The vertical axis intercept of 236.529 indicates that when the natural log of the height is 0, the pressure would be 236.529 inches of mercury.

If the student used $(\ln(height), \ln(pressure))$, then the graph looks like:



A reasonable linear model for this transformed data set is $\ln(p) = -1.000\ln(h) + 7.252$. The slope of the line indicates that the natural log of pressure deceases by about 1 unit for each one-unit increase of the natural log of height. The vertical axis intercept of 7.252 indicates that when the natural log of height is 0, the natural log of pressure would be 7.252.

UNIT 3

Student Stimulus and Part (d)

(d) Using your linear model from part (c), predict the pressure of the gas in the container, in inches of mercury, if the height of the container was 12 in. Show the work leading to your answer. How close did your model predict Boyle's measured value of $117\frac{9}{16}$ inches of mercury?

Sample Solution	Points Possible
If the student used $\left(height, \ln\left(pressure\right)\right)$, then $\ln\left(p\right) = -0.035h + 4.960$. Substituting 12 in. for height yields: $\ln\left(p\right) = -0.035(12) + 4.960$ $\ln\left(p\right) = 4.54$ $p = e^{4.54}$ $p \approx 93.69$ The value of p (the predicted value of the pressure) calculated with this model differs from the actual value by -23.872 inches of mercury. If the student used $\left(\ln\left(height\right), pressure\right)$, then: $p = -54.770\ln\left(h\right) + 236.529$ Substituting 12 in. for height yields: $p = -54.770\ln\left(12\right) + 236.529$ $p \approx 100.430$ The value of p calculated with this model differs from the actual value by -17.132 inches of mercury. If the student used $\left(\ln\left(height\right), \ln\left(pressure\right)\right)$, then: $\ln\left(p\right) = -1.000\ln\left(h\right) + 7.252$ Substituting 12 in. for height yields: $\ln\left(p\right) = -1.000\ln\left(12\right) + 7.252$ $\ln\left(p\right) = 4.767$ $p = e^{4.767}$ $p \approx 117.58$ The value of p using this model differs from the actual value by -0.014 inches of mercury.	**2 points maximum** 1 point for using a linear model from part (c) to determine the pressure for a height of 12 in. 1 point for an analysis of how close the value predicted by the model is to the actual value

Student Stimulus and Part (e)

(e) Write a version of your function model from part (c) that includes a logarithm. Then use the properties of logarithms to express your function in an equivalent form. Explain why the properties of logarithms allow you to express the function model in this equivalent form.

Sample Solution	Points Possible
If the student used $\left(height, \ln(pressure)\right)$ data, then one possible way to write the equation would be $\ln(p) = -0.035h + 4.960$. An equivalent form of this equation is $p = e^{(-0.035h+4.960)}$. This form results from applying the inverse relationship between exponential and logarithmic functions. If the student used $\left(\ln(height), pressure\right)$ data, then one way to write the equation would be $p = -54.770\ln(h) + 236.529$. An equivalent form of this equation is $p = \ln\left(h^{-54.770}\right) + 236.529$. This form results from applying the power property of logarithms. If the student used $\left(\ln(height), \ln(pressure)\right)$ data, then one way to write the equation would be $\ln(p) = -1.000\ln(h) + 7.252$. An equivalent form of this equation is $\ln(p) = \ln\left(h^{-1.00}\right) + 7.252$ or $p = e^{\left(\ln\left(\frac{1}{h}\right)+7.252\right)}$. The first equation results from applying the power property of logarithms. The second equation results from applying the power property of logarithms and the inverse relationship between exponential and logarithmic functions. *Scoring note:* The student can receive full credit for this part of the task if they use an incorrect model from part (b), but they correctly use properties of logarithms to express the equation in an equivalent form.	**3 points maximum** 1 point for writing the model in a form that includes a logarithm 1 point for using properties of logarithms to express the model in an equivalent form 1 point for an explanation of the property of logarithms used and why it produces an equivalent form

PROVIDING FEEDBACK ON STUDENT WORK

Because this is a practice performance task, you could choose to share the scoring guidelines with students before you score their work. This would give students an opportunity to learn what a complete response looks like and allow them to self-assess the completeness and correctness of their answer. Be sure to identify trends in students' responses to inform further instruction. These trends should include topics that students consistently displayed mastery of, as well as conceptual errors that students commonly made. Possible trends and suggested guidance for each part of the task follow, although the patterns you observe in your classroom may differ.

(a) If students struggle to select a function family to model the trend in the data, it could be because they are not sure what characteristics of a data set to consider. Lessons 1.1 and 1.2 provide example problems to help students make sense of data sets.

Teacher Notes and Reflections

(b) If students struggle to construct these graphs, it could indicate that they are not familiar with how to transform data using logarithms and/or graph the transformed data set with a graphing tool. Students would benefit from additional opportunities to transform data sets with technology.

Teacher Notes and Reflections

(c) If students struggle to fit a linear model or to explain the meaning of its parameters, it could be because they have not developed sufficient mastery with using technology to create a linear model. Additional problems of this sort are provided in Lessons 1.3 and 1.4.

Teacher Notes and Reflections

(d) If students struggle with analyzing how close the value predicted by the model is to the actual value, it could indicate that they have not developed a sufficient understanding of how to use residuals to evaluate how well their model fits the data set. Additional problems involving residuals are provided in Lesson 1.5.

Teacher Notes and Reflections

(e) If students are unsure of how to use properties of logarithms to write equivalent forms of equations, it could be an indication that they could use additional practice in algebraic manipulations of more straightforward logarithmic expressions.

Teacher Notes and Reflections

Try to assure students that converting their score into a percentage does not provide an accurate measure of how they performed on the task. You can use the suggested score interpretations with students to discuss their performance.

Points Received	How Students Should Interpret Their Score
13 to 15 points	"I know all of these algebraic concepts really well. This is top-level work. (A)"
10 to 12 points	"I know all of these algebraic concepts well, but I made a few mistakes. This is above-average work. (B)"
7 to 9 points	"I know some of these algebraic concepts well, but not all of them. This is average-level work. (C)"
4 to 6 points	"I know only a little bit about these algebraic concepts. This is below-average work. (D)"
0 to 3 points	"I don't know much about these algebraic concepts at all. This is not passing work. (F)"

LESSON 3.5

A Field Guide to Polynomial Functions

LEARNING OBJECTIVES

3.2.1 Construct a representation of a polynomial function.

3.2.2 Express a polynomial function in an equivalent algebraic form to reveal properties of the function.

3.2.3 Identify key features of the graph of a polynomial function.

AREAS OF FOCUS

- Engagement in Mathematical Argumentation
- Connections Among Multiple Representations

SUGGESTED TIMING

~45 minutes

MATERIALS

- access to **Desmos.com**

HANDOUT

Lesson

- 3.5: A Field Guide to Polynomial Functions

LESSON OVERVIEW

CONTENT FOCUS

In this activity, students document a set of conceptual tools that they can use to identify the key features of any polynomial function. The end product of this activity is a field guide to polynomial functions that consists of a written list of observable features, with which students can identify the behavior of a polynomial function presented in factored, standard, or graphical form.

LESSON DESCRIPTION

Students engage in a digital card sort activity and then summarize the key characteristics of polynomial functions into a field guide that they can use to analyze polynomial functions presented in a variety of forms.

UNIT 3

FORMATIVE ASSESSMENT GOAL

This lesson prepares students to complete the following formative assessment activity.

You can identify several important features of a polynomial function, including its zeros, factors, degree, and leading coefficient by examining its algebraic form. You can identify other features of a polynomial function, including its intercepts, local extrema, and end behavior by examining its graph. Answer the following questions related to the key characteristics of a polynomial function:

(a) What feature of a polynomial function indicates the maximum number of real zeros that it could have?

(b) In what circumstances could a polynomial function have fewer than the maximum number of real zeros?

(c) How could a rough sketch of the graph of a polynomial function of degree n be constructed just by examining its fully factored form?

A Field Guide to Polynomial Functions

~45 MIN.

Students engage in a digital card sort activity and then summarize the key characteristics of polynomial functions into a field guide that they can use to analyze polynomial functions presented in a variety of forms.

Student Task

With a partner, complete the Polynomial Functions Card Sort Activity. While matching the cards of graphs with cards of functions, observe how features of each graph relate to the degree, the zeros, the maximum number of possible turning points, and the end behavior of the corresponding polynomial. Use your observations to make generalizations about these characteristics in **Handout 3.5: A Field Guide to Polynomial Functions**.

Facilitating the Task

- The card sort activity for this lesson can be found here: **preap.org/Desmos-Polynomials**. Be sure to assign it to your class and share the code with your students.
- Encourage students to try to match cards without using a graphing utility. Instead, have them focus on the connections between the algebraic form of the polynomial function and its graph.
- As students work on the task, you can ask them guiding questions such as:
  - What can you tell about each polynomial function based on its graph?
  - How can you determine the degree of a polynomial using its factored form?

Meeting Learners' Needs

If students struggle to get started, you could allow them to use a graphing utility to match one or two of the polynomial functions. Then encourage them to connect the features of each algebraic representation with its corresponding graphical representation. Challenge students to match a few polynomial function representations using only the categories of features they have identified rather than by using a graphing utility.

Summarizing the Task

- After students complete the card sort activity, give them some time to complete Handout 3.5.
- All students should complete the sections of the field guide for the degree, end behavior, and zeros of a polynomial. For the fourth key feature, you can let students choose a characteristic based on their prior work. Alternatively, you can assign different characteristics to different pairs of students to use for the fourth key feature. Other characteristics you might consider assigning include the y-intercept of the graph and the number of extrema.

- If you assign or students choose different characteristics, you can collect the different features from the class into a shared document, either physical or digital, so that all students in class have access to the same comprehensive polynomial field guide.
- The key takeaway for this activity is that polynomial functions have a variety of characteristics that manifest differently across their various representations. The field guide can help students identify which characteristics of a polynomial function can be determined from any representation they are given. Analyzing a polynomial function for zeros and extrema is an essential skill for students progressing to a calculus course.

Assess and Reflect on the Lesson

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

You can identify several important features of a polynomial function, including its zeros, factors, degree, and leading coefficient by examining its algebraic form. You can identify other features of a polynomial function, including its intercepts, local extrema, and end behavior by examining its graph. Answer the following questions related to the key characteristics of a polynomial function:

(a) What feature of a polynomial function indicates the maximum number of real zeros that it could have?

The degree of a polynomial function indicates the maximum number of real zeros it could have.

(b) In what circumstances could a polynomial function have fewer than the maximum number of real zeros?

A polynomial function could have fewer than the maximum number of real zeros if some of the zeros have a multiplicity of two or greater, or if the polynomial has nonreal zeros.

(c) How could a rough sketch of the graph of a polynomial function of degree n be constructed just by examining its fully factored form?

If a polynomial function is completely factored into linear factors, then it has at most n linear factors (including multiplicity). A rough sketch of the graph of the polynomial can be developed using the linear factors to identify its x-intercepts. The y-coordinate of the y-intercept of the graph of the polynomial is equal to the product of the constants in each linear factor, assuming that the leading coefficient of each linear factor is 1. If the degree of the polynomial is odd and the leading coefficient is positive, then as x increases without bound, the polynomial also increases without bound, and as x decreases without bound, the polynomial also decreases without bound. If the degree of the polynomial is odd and the leading coefficient is negative, then as x increases without bound, the polynomial decreases without bound, and as x decreases without bound, the polynomial increases without bound. If the degree of the polynomial is even and the leading coefficient is positive, then as x increases or decreases without bound, the polynomial increases without bound. If the degree of the polynomial is even and the leading coefficient is negative, then as x increases or decreases without bound, the polynomial decreases without bound.

HANDOUT

To supplement the information within the body of the lesson, additional answers and guidance on the handout are provided here.

Handout 3.5: A Field Guide to Polynomial Functions

1. The degree of a polynomial is the highest power of its terms. If a polynomial is in completely factored form, then the degree of the polynomial is equivalent to the number of linear factors, including multiplicity. If the polynomial is in standard form, the degree can be determined by inspection. For a polynomial function graph, the number of x-intercepts can provide a lower estimate for the degree.
2. The end behavior of a polynomial describes the trends in the values of the polynomial as the values of x increase or decrease without bound. If the degree of the polynomial is odd and the leading coefficient is positive, then as x increases without bound, the polynomial also increases without bound, and as x decreases without bound, the polynomial also decreases without bound. If the degree of the polynomial is odd and the leading coefficient is negative, then as x increases without bound, the polynomial decreases without bound, and as x decreases without bound, the polynomial increases without bound. If the degree of the polynomial is even and the leading coefficient is positive, then as x increases or decreases without bound the polynomial increases without bound. If the degree of the polynomial is even and the leading coefficient is negative, then as x increases or decreases without bound the polynomial decreases without bound.
3. The zeros of a polynomial, p, are the x-values that are solutions to the equation $p(x) = 0$. If the polynomial is in factored form, then the zeros of the polynomial can be determined by setting each linear factor equal to zero and solving the resulting equations. If the polynomial is in standard form, then the zeros can be determined by first expressing the polynomial in factored form and then following the method previously described, or by creating a graph of the polynomial and following the method that follows. For a polynomial function graph, the real zeros of the polynomial correspond to the x-coordinates of the x-intercepts of the graph of the polynomial.
4. Possible choices for this feature are the y-intercept of the graph or the number of extrema.
   - The y-intercept of a polynomial is the point at which the graph of the polynomial and the y-axis intersect. If the polynomial is in factored form, the y-coordinate of the y-intercept of the graph of the polynomial is equal to the product of the constants in each linear factor, assuming that the leading coefficient of each linear factor is 1. If the polynomial is in standard form, then the y-coordinate of the y-intercept can be found by evaluating the polynomial at $x = 0$. For a

polynomial function graph, the y-intercept can be determined or estimated by inspection.

- Extrema are local or global maximum or minimum values of a polynomial. A polynomial of degree n has at most $n-1$ extrema. If the polynomial is in factored form, then its degree can be determined from the number of linear factors, including multiplicity, and the number of extrema is one less than the degree. If the polynomial is in standard form, then its degree can be determined by inspection and the number of extrema is one less than the degree. For a polynomial function graph, the number of extrema can be determined by inspection.

Unit 3

Performance Task

PERFORMANCE TASK

Predicting the Number of Sections of a Circle

LEARNING OBJECTIVE

3.2.1 Construct a representation of a polynomial function.

PERFORMANCE TASK DESCRIPTION

This performance task provides students with an opportunity to use a polynomial function to model a sequential relationship presented in geometric terms. Specifically, students construct a function to model the maximum number of sections that can be created by randomly placing points on a circle and drawing all possible chords through them, such that the three chords do not intersect at the same point. Polynomial functions are appropriate models for sequences that have an *n*th-order finite difference. For the sequence given in this task, students determine that a fourth-degree polynomial is an appropriate function model. To enhance the mathematical modeling aspects of the task, students are expected to make a conjecture about a possible function model early in the problem. Then they collect additional data that they use to improve their model. This refinement process is a critically important aspect of mathematical modeling that students do not often get to experience.

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~45 minutes

MATERIALS

- access to **Desmos.com** or other graphing utility
- paper, compass, and straightedge

HANDOUT

- Unit 3 Performance Task: Predicting the Number of Sections of a Circle

AP Connections

This performance task supports AP preparation through alignment to the following AP Calculus Mathematical Practices:

- **3.F** Explain the meaning of mathematical solutions in context.

This performance task supports AP preparation through alignment to the following AP Statistics Course Skills:

- **4.B** Interpret statistical calculations and findings to assign meaning or assess a claim.

ELICITING PRIOR KNOWLEDGE

The goal of this task is for students to demonstrate an understanding of when a polynomial function is an appropriate model for a data set or contextual scenario.

- To begin, introduce students to the performance task.
- You may find it helpful to prepare students to engage in the task by posing a few short warm-up questions, such as the following:
  - Consider the sequence 2, 6, 18, 54, 162, 486, … . Can this sequence be modeled by an exponential function? Why or why not?

    This sequence can be modeled by an exponential function because there is a common ratio of 3 between consecutive terms.
  - Consider the sequence 1, 0, 1, 0, 1, 0, 1, 0, … . Can this sequence be modeled by a linear function? Why or why not?

    This sequence cannot be modeled by a linear function because it does not have a common difference (constant rate of change).
  - Consider the sequence $-2, -1, 3, 13, 32, 63, 109, \ldots$. Can this sequence be modeled by a polynomial function? Why or why not?

    This sequence can be modeled by a polynomial function because the third-order finite differences are constant. This suggests that a cubic polynomial would be an appropriate function model.
- If students struggle with the warm-up questions, it could indicate that they are not yet fully prepared to engage in the performance task. If students need help using technology to determine a regression equation to model a data set, there are examples in the model lessons for Key Concept 1.1. There are also some tips in the Supporting Students section below.

SUPPORTING STUDENTS

Here are a few possible implementation strategies you can use to help students engage with the task. You should rely on your knowledge of your students and your professional expertise to determine how to provide appropriate scaffolds while maintaining the cognitive demand of the task.

- **Collaboration:** Students could work in pairs to complete the task. Students may benefit from discussing the task with a partner because there are several different configurations of the points on the circle that students should consider in part (a). It would also be beneficial for students to compare their work with another student's to ensure that they do not make a data entry or counting error.

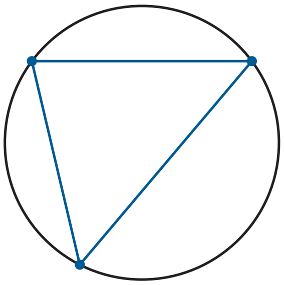
- **Chunking the Task:** You could chunk the task into two sections—parts (a) and (b), and then parts (c) and (d)—and have students complete one section at a time. They could check their solutions to each section with you or compare their work with that of another pair of students before moving on to the next section. They can focus on what changes, if any, they could make to their solution to craft a more complete response the next time they engage in a performance task.
- **Using Technology:** There are a few parts of the task where technology could be useful for students.
  - In part (b) of the task, students are asked to place six points on the circle and construct all chords such that no three chords intersect at the same point. Students may benefit from using a dynamic geometry tool, such as **desmos.com/geometry**, to create their sketches.
  - Note that if students use Desmos to determine a quartic polynomial model with the regression command, they should not use the letter e as one of the parameters in the polynomial. This is because Desmos interprets e as the natural base and not as a variable. Students can select a different letter to use as a parameter. As an alternative to using different letters, Desmos recognizes subscripts as a way to distinguish among parameters, so it is also possible to input the command, "$y_1 \sim a_1x_1^4 + a_2x_1^3 + a_3x_1^2 + a_4x_1 + a_5$" where x_1 represents the number of points placed on a circle and y_1 represents the number of distinct sections created when all chords are drawn and obtain the correct regression equation.

SCORING STUDENT WORK

Because this is a performance task and not a practice performance task, it is recommended that you score your students' work, rather than having each student score a peer's work. Be sure to use the results of the performance task to identify patterns and trends that can inform further instruction about modeling real-world scenarios with polynomial functions.

Predicting the Number of Sections of a Circle

You can randomly place points on a circle and draw all possible chords between them, such that no three chords intersect at the same interior point. The following figure shows three randomly placed points on the circle with all possible chords drawn between them.



Define a function, f, as the number of distinct sections of a circle created by chords associated with x, the number of points randomly placed on the circle. In the example above, $f(3) = 4$, because there are three points on the circle and four sections are created when all chords are drawn between them.

There are 10 possible points for this performance task.

Student Stimulus and Part (a)

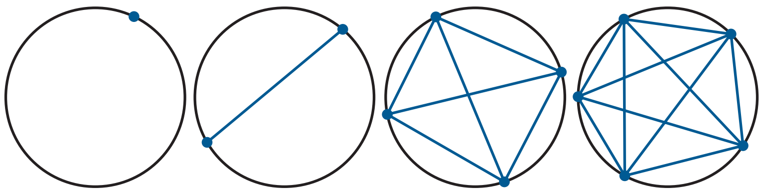
(a) Draw several examples of circles with randomly placed points. Draw all possible chords between these points such that no three chords intersect at an interior point. Then count the number of distinct sections of the circle created by these chords to complete the table of values below. Then use your completed table of values to make a conjecture about a function that models the data in the table. Make sure to provide an explanation to support your conjecture.

Number of Points, x	Number of Sections, $f(x)$
1	
2	
3	
4	
5	

Sample Solution	Points Possible

Sample Solution

Sample student drawings for $x = 1,\ 2,\ 4,$ and 5 are shown below.



A completed table of values is shown below.

Number of Points, x	Number of Sections, $f(x)$
1	1
2	2
3	4
4	8
5	16

The pattern in the table suggests that the function f could be defined by $f(x) = 2^{x-1}$.

Scoring note: If a student makes a conjecture based on a table that includes incorrect values, but their conjecture is consistent with those values, they can still receive the third point.

Points Possible

3 points maximum

1 point for drawing all chords between points on the circles, such that no three chords intersect at the same point

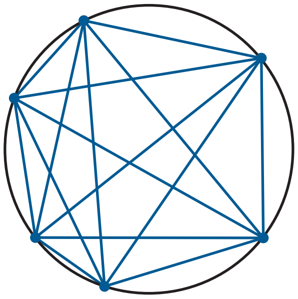
1 point for completing the table of values

1 point for providing a conjecture about the function

UNIT 3

Student Stimulus and Part (b)

(b) Consider the number of sections created when $x = 0$ and $x = 6$. Compare the number of sections your model predicts with the actual number of sections created. What does this suggest about your model?

Sample Solution	Points Possible
When there are 0 points placed, the circle has one section (the whole circle), so $f(0) = 1$. When there are 6 points placed, the circle is subdivided into 31 sections, so $f(6) = 31$.  However, the function model from part (a) suggests that when 0 points are placed, the circle should be subdivided into half of a section, and when 6 points are placed, the circle should be subdivided into 32 sections. This additional information suggests that the function model from part (a) is incorrect and requires revision.	**3 points maximum** 1 point for determining $f(0)$ and $f(6)$ 1 point for comparing the empirical evidence with the function model from part (a) 1 point for concluding that the function model in part (a) is not correct

Student Stimulus and Part (c)

(c) Use the values from parts (a) and (b) for the number of sections of a circle created by drawing all possible chords through randomly placed points on the circle for $x = 0$ through $x = 6$ to determine whether or not the data can be modeled by a polynomial function. If it can, what is the degree of the polynomial?

Sample Solution

The data suggest that we can model the scenario with a polynomial of degree 4 since it took four sets of common differences until a constant difference was observed.

x	*y*	1st common difference	2nd common difference	3rd common difference	4th common difference
0	1				
		0			
1	1		1		
		1		0	
2	2		1		1
		2		1	
3	4		2		1
		4		2	
4	8		4		1
		8		3	
5	16		7		
		15			
6	31				

Scoring note: If a student incorrectly calculates the finite differences so that no constant nth-order differences are observed and concludes that a polynomial model is inappropriate, they can still receive the second point. If a student incorrectly calculates the finite differences so that the constant nth-order difference occurs for $n \neq 4$, they can still receive the second point if they identify the degree of a polynomial model based on their value of n.

Points Possible

2 points maximum

1 point for calculating sets of finite differences until a constant difference is observed

1 point for concluding that a fourth-degree polynomial is an appropriate function model

UNIT 3

Student Stimulus and Part (d)

(d) Use a graphing utility to determine the best polynomial function model for the data you have collected. Then use your model to determine the number of unique sections that would be created if you randomly select 10 points on a circle and draw all possible chords between them, such that no three chords intersect at the same point.

Sample Solution	Points Possible
The finite differences from part (c) suggest that a fourth-degree polynomial would be an appropriate function model. A graphing utility yields the function $f(x)=\frac{1}{24}x^4-\frac{1}{4}x^3+\frac{23}{24}x^2-\frac{3}{4}x+1$. If 10 points are randomly placed on a circle and all chords between them are drawn, there will be $f(10)=256$ sections created.	**2 points maximum** 1 point for a finding a revised function model for the data 1 point for determining the value of $f(10)$ based on the revised function model

PROVIDING FEEDBACK ON STUDENT WORK

After scoring your students' work, it is important to identify trends in their responses to inform further instruction. These trends should include topics that students consistently displayed mastery of, as well as conceptual errors that students commonly made. Possible trends and suggested guidance for each part of the task follow, although the patterns you observe in your classroom may differ.

(a) If students struggle to make a conjecture about a function that could model the data they collect, it could be because they lack confidence in their understanding of modeling. You can reassure students that their conjectures only have to be consistent with their data and that any description they provide for a function that could feasibly model the data is acceptable. It may also help students who are reluctant to provide a conjecture to let them know that they will have opportunities to revise it over the course of the performance task.

Teacher Notes and Reflections

(b) It is important for students to recognize that a doubling pattern they might have conjectured is not consistent with the data collected after they add a sixth point. Encourage students to make multiple drawings and to attend to precision when counting the number of sections of the circle.

Teacher Notes and Reflections

(c) If students are unsure how to analyze their data for possible patterns, it could be because they have not had sufficient experience using polynomials to model sequences with *n*th-order constant differences. There are examples of sequences that can be modeled with quadratic functions in Lesson 1.6, and the warm-up questions in this performance task provide additional guidance for the type of data analysis that students should perform here.

Teacher Notes and Reflections

(d) If students have difficulty determining a function model for the data, it could be because they need more practice with using technology to determine a regression equation. There are examples of using regression technology in the model lessons for Key Concept 1.1 and Key Concept 3.1. Additionally, many technologies use different algorithms to calculate regression equations. This task provides opportunities to discuss attention to detail with regard to data entry as well as the variations in the processes used by technology to determine regression equations.

Teacher Notes and Reflections

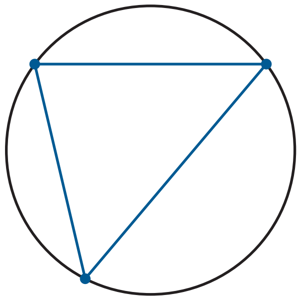
Try to assure students that converting their score into a percentage does not provide an accurate measure of how they performed on the task. You can use the suggested score interpretations with students to discuss their performance.

Points Received	How Students Should Interpret Their Score
8 to 10 points	"I know all of these algebraic concepts really well. This is top-level work. (A)"
6 or 7 points	"I know all of these algebraic concepts well, but I made a few mistakes. This is above-average work. (B)"
4 or 5 points	"I know some of these algebraic concepts well, but not all of them. This is average-level work. (C)"
2 or 3 points	"I know only a little bit about these algebraic concepts. This is below-average work. (D)"
0 or 1 point	"I don't know much about these algebraic concepts at all. This is not passing work. (F)"

Predicting the Number of Sections of a Circle

PERFORMANCE TASK

You can randomly place points on a circle and draw all possible chords between them, such that no three chords intersect at the same interior point. The following figure shows three randomly placed points on the circle with all possible chords drawn between them.



Define a function, f, as the number of distinct sections of a circle created by chords associated with x, the number of points randomly placed on the circle. In the example above, $f(3) = 4$, because there are three points on the circle and four sections are created when all chords are drawn between them.

(a) Draw several examples of circles with randomly placed points. Draw all possible chords between these points such that no three chords intersect at an interior point. Then count the number of distinct sections of the circle created by these chords to complete the table of values below. Then use your completed table of values to make a conjecture about a function that models the data in the table. Make sure to provide an explanation to support your conjecture.

Number of Points, x	Number of Sections, $f(x)$
1	
2	
3	
4	
5	

PERFORMANCE TASK

(b) Consider the number of sections created when $x = 0$ and $x = 6$. Compare the number of sections your model predicts with the actual number of sections created. What does this suggest about your model?

(c) Use the values from parts (a) and (b) for the number of sections of a circle created by drawing all possible chords through randomly placed points on the circle for $x = 0$ through $x = 6$ to determine whether or not the data can be modeled by a polynomial function. If it can, what is the degree of the polynomial?

(d) Use a graphing utility to determine the best polynomial function model for the data you have collected. Then use your model to determine the number of unique sections that would be created if you randomly select 10 points on a circle and draw all possible chords between them, such that no three chords intersect at the same point.

Unit 4T

Unit 4T
Trigonometric Functions

Overview

SUGGESTED TIMING: APPROXIMATELY 6 WEEKS

The final unit in this course provides an exploration of trigonometric functions. Trigonometry is the branch of mathematics that connects two fundamental geometric objects: triangles and circles. In Pre-AP Geometry with Statistics, students learned that the trigonometric ratios relate acute angle measures to ratios of side lengths in right triangles. Algebra 2 extends those relationships to include all real numbers. When the domains of trigonometric functions include angle measures greater than 90°, including greater than 360°, these functions are far more useful in modeling contextual scenarios that involve periodic phenomena, such as the rotation of a ceiling fan, the height of a Ferris wheel car, or the ebb and flow of tides. A sound understanding of trigonometric functions demystifies these common real-world contexts, putting the power of mathematical reasoning at students' command.

Beginning with the introduction of radians as units of angle measure, the unit continues with an investigation of the sine and cosine functions and their transformations, collectively referred to as sinusoidal functions. Students then use what they know about sinusoidal functions and properties of quotients of functions to understand the properties of the tangent function and the three reciprocal trigonometric functions. Finally, students use inverse trigonometric functions to solve problems related to circular and periodic motion. Please note that the Pre-AP three-year mathematics sequence includes trigonometry in Algebra 2 to create a more equitable pathway for students who take Algebra 1 in 9th grade to potentially enroll in AP Calculus AB in 12th grade. If a state's standards do not require trigonometry in Algebra 2, then completing the Pre-AP requirements for Unit 4T are optional for teachers in that state.

Technology Note: There are many different technologies available for students to use to construct scatterplots, determine regression equations, and create graphs of functions. Because facility with technology tools is an integral part of developing a deep conceptual understanding of the mathematical concepts in this course, you should select the technology tool with which you are most comfortable from the tools that are available to your students. The model lessons use Desmos, which is a free, online graphing tool available at **Desmos.com**. Different technology tools use different methods for calculating regression equations and for graphing functions. Where relevant, the model lessons alert you to possible discrepancies between the output displayed by Desmos and the output displayed by other common technology tools.

ENDURING UNDERSTANDINGS

This unit focuses on the following enduring understandings:

- Trigonometry connects the study of circles and the study of right triangles.
- Real-world contexts that exhibit periodic behavior or circular motion can be modeled by trigonometric functions.

KEY CONCEPTS

This unit focuses on the following key concepts:

- 4T.1: Radian Measure and Sinusoidal Functions
- 4T.2: The Tangent Function and Other Trigonometric Functions
- 4T.3: Inverting Trigonometric Functions

UNIT RESOURCES

The tables below outline the resources provided by Pre-AP for this unit.

Lessons for Key Concept 4T.1: Radian Measure and Sinusoidal Functions			
Lesson Title	**Learning Objectives Addressed**	**Suggested Timing**	**Areas of Focus**
4T.1: Measuring an Angle's Openness	4T.1.1	~165 minutes	Greater Authenticity of Applications and Modeling, Engagement in Mathematical Argumentation

Lessons for Key Concept 4T.1: Radian Measure and Sinusoidal Functions			
Lesson Title	**Learning Objectives Addressed**	**Suggested Timing**	**Areas of Focus**
4T.2: Determining Equivalent Angle Measures	4T.1.1, 4T.1.2	~45 minutes	Greater Authenticity of Applications and Modeling, Connections Among Multiple Representations
4T.3: Angles in the Coordinate Plane	4T.1.2	~45 minutes	Engagement in Mathematical Argumentation, Connections Among Multiple Representations
4T.4: A Model for Circular Motion	4T.1.3	~90 minutes	Greater Authenticity of Applications and Modeling, Connections Among Multiple Representations
4T.5: The Coordinates of Points on a Circle	4T.1.4	~60 minutes	Connections Among Multiple Representations, Engagement in Mathematical Argumentation
4T.6: Common Right Triangles in a Unit Circle	4T.1.4, 4T.1.7	~75 minutes	Connections Among Multiple Representations, Engagement in Mathematical Argumentation

UNIT 4T

Lessons for Key Concept 4T.1: Radian Measure and Sinusoidal Functions			
Lesson Title	**Learning Objectives Addressed**	**Suggested Timing**	**Areas of Focus**
4T.7: A Model for Periodic Phenomena	4T.1.5, 4T.1.6	~120 minutes	Connections Among Multiple Representations, Greater Authenticity of Applications and Modeling
	All learning objectives from this key concept are addressed with the provided materials.		

Performance Task for Unit 4T (~45 minutes)
This performance task assesses learning objectives from Key Concept 4T.1.

Learning Checkpoint 1: Key Concept 4T.1 (~45 minutes)
This learning checkpoint assesses learning objectives from Key Concept 4T.1. For sample items and learning checkpoint details, visit Pre-AP Classroom.

Lessons for Key Concept 4T.2: The Tangent Function and Other Trigonometric Functions			
Lesson Title	**Learning Objectives Addressed**	**Suggested Timing**	**Areas of Focus**
4T.8: The Tangent Function	4T.2.1, 4T.2.2	~120 minutes	Connections Among Multiple Representations, Engagement in Mathematical Argumentation
	All learning objectives from this key concept are addressed with the provided materials.		

Practice Performance Task for Unit 4T (~45 minutes)
This practice performance task assesses learning objectives from Key Concept 4T.2.

Lessons for Key Concept 4T.3: Inverting Trigonometric Functions	
There are no provided Pre-AP lessons for this key concept. As with all key concepts, this key concept is addressed in a learning checkpoint.	
	All Key Concept 4T.3 learning objectives should be addressed with teacher-developed materials.

Learning Checkpoint 2: Key Concepts 4T.2 and 4T.3 (~45 minutes)
This learning checkpoint assesses learning objectives from Key Concepts 4T.2 and 4T.3. For sample items and learning checkpoint details, visit Pre-AP Classroom.

UNIT 4T

LESSON 4T.1

Measuring an Angle's Openness

LEARNING OBJECTIVE

4T.1.1 Use the radian measure of an angle to relate the radius of a circle to the length of the arc subtended by that angle.

LESSON OVERVIEW

CONTENT FOCUS

In this lesson students build a conceptual understanding of radian measure to quantify the openness of an angle. Students explore a real-world context to develop a mathematical model that establishes the relationship among the measure of an angle, the radius length of a circle, and the measure of the associated arc length. They extend their understanding of radian measure by creating a full-circle radian protractor that allows them to determine angle measures that exceed 2π radians.

LESSON DESCRIPTION

Part 1: Comparing Attributes of Central Angles

In this part of the lesson, students make sense of a model of three satellites orbiting Earth in circular paths through the attribute of the central angle of a circle. This allows students to connect the measure of an angle's openness, the length of the radius of the circle, and the length of the arc cut off by the central angle's rays. These connections lead students to the definition of radian as another way to quantify the measure of an angle.

AREAS OF FOCUS

- Greater Authenticity of Application and Modeling
- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~165 minutes

LESSON SEQUENCE

- This lesson is part of a lesson sequence (~600 minutes total) of Lessons 4T.1 through 4T.7.

MATERIALS

- wax craft stick (preferred), pipe cleaner, or string
- ruler
- protractor
- compass
- scissors
- parchment paper or transparency sheets
- markers
- access to **Desmos.com**

HANDOUTS

Lesson

- 4T.1.A: Comparing Attributes of Central Angles
- 4T.1.B: Defining the Relationship Among Angle Measure, Radius Length, and Arc Length

Practice

- 4T.1.C: Practice with Radian Measure

Part 2: Creating a Radian Protractor

In this part of the lesson, students create a full-circle protractor to measure an angle's openness in radians. A full-circle protractor helps students connect the formula for the circumference of a circle to the radian measure of an angle that sweeps out a full circle.

Part 3: Defining the Relationship Among Angle Measure, Radius Length, and Arc Length

In this part of the lesson, students explore a central angle of a circle to relate the angle measure in radians, θ, the length of the radius, r, and the arc length, s, to develop the equivalent formulas $\frac{s}{r} = \theta$ and $s = \theta r$.

FORMATIVE ASSESSMENT GOAL

This lesson prepares students to complete the following formative assessment activity.

Ferris wheels are common tourist attractions that consist of a rotating upright wheel with multiple passenger cars. Two famous Ferris wheels are the Singapore Flyer and the London Eye.

The Singapore Flyer is a Ferris wheel with a height of 165 meters (m) and a diameter of 150 m.

The London Eye is a Ferris wheel with a height of 150 m and a diameter of 120 m.

Suppose that a rider travels 100 m along the curved path around each Ferris wheel.

(a) Without calculating exact measures, identify the Ferris wheel on which the rider sweeps out a larger angle of rotation.

(b) Verify your answer from part (a) by calculating the exact angle of rotation, in radians, the rider sweeps out on each Ferris wheel.

Part 1: Comparing Attributes of Central Angles

~75 MIN.

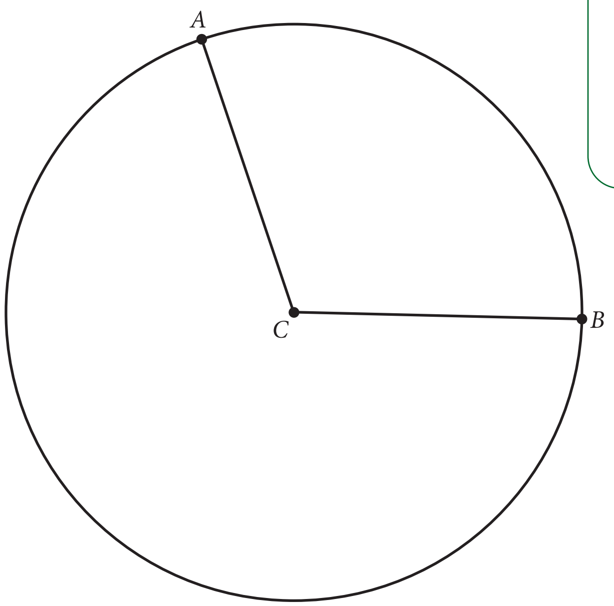
In this part of the lesson, students make sense of a model of three satellites orbiting Earth in circular paths through the attribute of the central angle of a circle. This allows students to connect the measure of an angle's openness, the length of the radius of the circle, and the length of the arc cut off by the central angle's rays. These connections lead students to the definition of radian as another way to quantify the measure of an angle.

Instructional Rationale

By design, there are very few references to π in this lesson. When radians are introduced as fractions of π, students often develop the misconception that all radian measures must be expressed using π. This lesson introduces students to radian measure as a way of quantifying the measure of an angle as the number of radius lengths necessary to span the subtended arc length. In Lesson 4T.2: Determining Equivalent Angle Measures, students explore commonly used angles and investigate how to convert measures between radians and degrees. Those explorations provide a natural setting for fractions of π to be included.

Warm-Up

- Display a central angle for students to see. One such central angle is shown in the following figure:



Classroom Ideas

To hold students accountable for their ideas during the warm-up, you can have them write their thoughts on their personal whiteboards, on sticky notes, or in a shared digital space, such as a Jamboard or Padlet.

- Have students work with a partner to identify all objects with attributes they can measure in the diagram.
- Once students have named several objects in the diagram that can be measured, have groups share their lists of objects and record students' ideas in a central location. Continue having groups share their lists of objects until all ideas have been captured. If a student suggests a feature of the diagram that cannot be measured, such as "center," you can remind students that they are looking only for attributes that can be measured.
- Make sure students have had the opportunity to review the definitions of angle, angle measure, central angle, radius, circumference, and arc length, but you do not have to limit the conversation to these features or attributes of the diagram.

Meeting Learners' Needs

Some students may not remember the correct mathematical terminology for some of the objects they observe. You can use this as an opportunity for a just-in-time review of key geometric vocabulary such as central angle, arc length, radius, and diameter. Encourage students to define these terms using their own language and, if necessary, to record the definitions in a central location accessible to all students.

Instructional Rationale

Unit 4T builds on students' understanding of right triangle trigonometry to develop the trigonometric functions in the context of circular motion. In this lesson, students focus on the dynamic nature of contexts such as an orbiting satellite, a rotating Ferris wheel, and the rising and falling of tides. The repeating motion in each of these contexts makes them interesting to study through the lens of trigonometric functions. Since many of the contexts involve circular motion, the lengths of radii are generally given or implied. In these contexts, a radian is a logical unit to measure an angle's openness because its measure in radians is directly related to the length of the radius. The lesson emphasizes a conceptual approach to radians so that students can both measure angles in radians and construct angles with a given radian measure.

Student Task

The student task for this part of the lesson is provided on **Handout 4T.1.A: Comparing Attributes of Central Angles**. In the task, students observe an image representing three satellites orbiting Earth in different circular paths. Students make predictions about which satellite travels the greatest distance in one day. The task ends with students investigating the measure of the angle of an arc of the satellites' orbits.

Facilitating the Task

- Begin by arranging small groups of three students or as many groups of three as possible. This group size is important because Part B of the task requires each student to investigate the orbit of one of the three satellites in the scenario.
- Next, provide students with Handout 4T.1.A and allow them time to closely observe the figure and ask clarifying questions. Then have students work in their small groups on Part A of the task.
- Once student groups have completed Part A, have them share their answers with the whole class. It is not necessary to discuss students' answers to Part A in detail, but it may be helpful to post their responses in a central location accessible to all students.
- Next, have each student in the group choose one of the three satellites to focus on. Have students use measuring tools, such as rulers, protractors, and wax craft sticks, to check the predictions they made in Part A. Have students record their measurements in the table on Handout 4T.1.A. The title of the last column of the table is intentionally left blank.
- As students take their measurements, it is important for them to try to make connections between the measure of an angle's openness (the angle measure), the measure of the arc length cut off by the central angle's rays, and the radius length of the circle.
- After students have had some time to record their measurements, bring the class together for a short discussion about the questions in Part A. You can guide the conversation using some questions like the following:
  - How did you determine the fraction of the orbit your satellite traveled?

    The fraction of the orbit traveled is the ratio of the arc length and the circumference.

Classroom Ideas

Wax craft sticks are the best tool for this task because they are pliable and slightly tacky, which allows them to hold a curved shape and not slide across the handout. If you cannot find wax craft sticks, you could have students use pipe cleaners or string. Pipe cleaners are better than string for this purpose because they hold their shape. String can be used, but students must be attentive to its positioning because its shape can be distorted by pulling it taut when measuring its length, producing inaccurate measurements of arc lengths.

- Suppose you know that two angles have the same measure. Which of the other quantities in the table will have the same value?

  If two angles have the same measure, then they open the same fraction of the circle and the ratio of the arc length to the radius will also be the same. The radius length, arc length, and circumference might all be different.

- What do you think would happen if we considered the ratio of arc length to radius length instead of arc length to circumference?

- To help students respond to the third question in the discussion, have students title the last column in the table $\frac{\text{arc length}}{\text{radius}}$ and have them complete the column.
- The second and third questions in the discussion are intended to get students thinking about different ratios they could form with the quantities they measured. They may not be able to respond meaningfully to the question or understand why that particular ratio is helpful.
- It is important for students to notice that the value of the ratio, 2.25, indicates that 2.25 radii would be needed to span the arc of each satellite's path. In other words, the measures in degrees of the angles for each satellite are equal, and the ratios of arc length to radius for each satellite's path are also equal.
- Make sure that the completed table is displayed where all students can see it so they can refer to it later in the lesson when you define *radian* as a unit of measure.
- At this point, it will be helpful to introduce students to the term *radian* and provide a working definition:

  One way to measure an angle's openness is with the ratio of the arc length to the radius length. This unit of measure is called a *radian*. In our satellite task, each satellite opened an angle that measures 2.25 radians.

- Let students know that their next task is to determine what it means for an angle to have a measure of 1 radian.
- Have the small groups return to their handouts and work collaboratively to construct an angle in which the arc cut off by the angle's rays has a length equal to the length of one radius. It will be helpful for students to mark the ends of the arc and use a straightedge to construct a central angle.

Summarizing the Task

- Engage students in a brief discussion about what they notice and what they wonder about the angle they constructed. You can use questions like the following:
  - What do you notice about your angles?
    The angles appear to be congruent.
  - For each of your angles, what is the ratio of the arc length to the radius? Why?
    The ratio of the arc length to the radius is 1 because we constructed an angle that cuts off an arc length of 1 radius.
  - What is the measure of the angle in both degrees and radians?
    The angle measures 1 radian or a little less than 60 degrees.

Meeting Learners' Needs

While comparing angles, students may observe that it is difficult to distinguish between the angles, since there is now more than one line on the circle. This motivates the need to mark the angle and leads students to understand why it is important to construct angles with the same starting position, or initial ray.

Guiding Student Thinking

Students might struggle to reconcile that the angles have congruent measures but different associated arc lengths. This could be because the radian measure of an angle is a ratio of the arc length and the radius of a circle, which differs from other units of measure that are more familiar to students such as inches or degrees. Remind students to think about the "openness" of a central angle in terms of the fraction of the circle it represents.

- In response to the first question, encourage students to hold their stack of papers up to the light to compare the angles. You can connect their observations to the understanding of congruence from geometry that two angles are congruent if one angle can be moved so it coincides exactly with the other without bending or stretching the angle.
- For the last question, some students might use a protractor to determine the angle measure in degrees or estimate that the angle measures about 60 degrees. The goal here is for students to see that they have just created an angle that is 1 radian in measure. This provides an opportunity to define what a radian means for an angle.
- At this point, you can show students a short animation to help students visualize wrapping a radius length around the circle: **preap.org/Desmos-Radian**.

- Now you can provide students with a formal definition of the term *radian*:

  One radian is the measure of the openness of the central angle that cuts off an arc whose length is equal to the length of the radius.

- You can have students create a vocabulary graphic organizer, like the one on Handout 1.1, to document what they know so far about radians. Then you can add this to your classroom's word wall.

Guiding Student Thinking

It may not be apparent to students that placing radius lengths along the circumference of the circle to measure the openness of the angle results in the same value as constructing the ratio of an angle's arc length to the radius length of the circle. To drive this critical point home, be sure to ask students some questions that forge the connection between the ratio of the arc length to the radius length and the process of placing our radius lengths.

- Before moving on to the next part of the lesson, ask students a question like the one here to help them see the relationship between the ratio of the arc length to the radius length and the process of placing the radius lengths along the circle.
  - What do you think will happen if you use the radius length of your satellite to measure the length of the path traveled by each satellite in 1 day?
- Have students return to their satellite diagrams and use radius lengths to measure the length of the path each satellite traveled in one day. They should determine that they need $2\frac{1}{4}$ radius lengths to span the opening of the angle, regardless of the length of the radius. This value, 2.25, is the same as the ratio of the arc length to radius length that they calculated earlier in the lesson.
- By this point in the lesson, students should start to make generalizations about the concept of a radian.

Part 2: Creating a Radian Protractor

~45 MIN.

In this part of the lesson, students create a full-circle protractor to measure an angle's openness in radians. A full-circle protractor helps students connect the formula for the circumference of a circle to the radian measure of an angle that sweeps out a full circle.

Student Task

How many radians do you think are in a full circle? We are going to create a protractor for a full circle with radian measures in 0.5 radian increments.

Facilitating the Task

- Before starting this task, make sure that students can describe how an angle measure in radians relates the radius length and the arc length. To elicit this type of reasoning you can ask questions like the following:
  - Imagine that one of your friends was absent for the first part of the lesson. How would you explain to them how to construct an angle that has a measure of 1 radian?

    Constructing an angle with a measure of 1 radian involves first constructing a circle and then marking an arc on the circle whose length is equal to the radius length. Then the central angle whose rays cut off that portion of the arc has a measure of 1 radian.
  - How would you explain to them how to construct an angle that has a measure of 3 radians?

    Constructing an angle with a measure of 3 radians involves first constructing a circle and then marking an arc on the circle whose length is 3 times the length of the radius. Then the central angle whose rays cut off that portion of the arc has a measure of 3 radians.
  - Does the size of the circle you construct change your approach for constructing these angles? Why or why not?

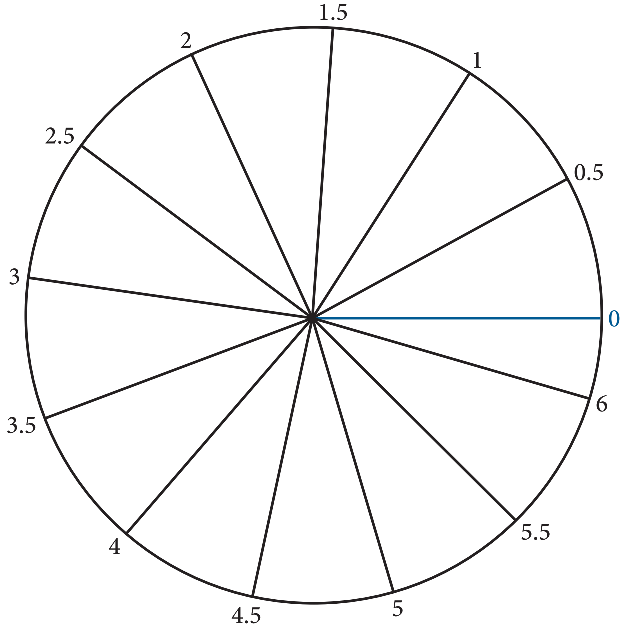
    No. The size of the circle does not change the approach for constructing these angles because radians indicate the ratio of the arc length to the radius length for any circle.
- Let students know they will create a measuring tool to help them create and measure angles in radians.
- Students will need a compass, a wax craft stick (or pipe cleaner or string), and a piece of parchment paper.
- Have students construct a circle that fits on their paper. Be sure students mark the center of the circle so that they can mark a radius and then cut their wax craft stick to the length of the radius.

Classroom Ideas

If you don't have parchment paper, you could use patty paper, tracing paper, or transparency paper and markers. You could also try using transparent sheet protectors and permanent markers. It is important to use something translucent or transparent so students can see angles through their protractors as they measure them.

UNIT 4T

- Have students work in groups of two or three to construct angles in increments of 0.5 radians, in a counterclockwise manner until they are unable to continue. A sample completed protractor is shown.



Instructional Rationale

Students may think they made a mistake because they will end with a little bit of the circle unmarked. This is intentional and will motivate the conversation in the next part of the lesson.

- If some students finish quickly, you can challenge them to work with a partner to use their radian ruler to measure angles. Have each partner draw a few angles on a piece of paper and then swap papers so that they can measure each other's angles. Students should estimate the measure of the angle to the nearest half of a radian.

Meeting Learners' Needs

To extend this activity for learners who would benefit from enrichment, you could have students mark every quarter radian or every tenth of a radian.

Summarizing the Task

- To debrief the task of creating a radian protractor, lead a conversation with students about what they noticed while working on the protractor. You can use questions like the following:
  - How many radians does an angle need to rotate to open a full circle?

    Just over 6 radians.

  - Why isn't the angle measure around a circle exactly 6 radians? How many radius lengths should span the entire circle?

    Because the circumference of a circle is equal to the product $2\pi r$, there are 2π radius lengths in the arc that opens a full circle. Therefore, an angle that opens a full circle has a measure of 2π radians.

  - What is a reasonable decimal approximation of 2π? How does this help to explain why an angle that measures 6 radians is not quite a full circle?

    A reasonable approximation of 2π is 6.28. Since 6.28 is greater than 6, it means that an angle that measures 6 radians does not sweep out a full circle because there is a little more than one-quarter of a radius length not opened by 6 radians.

  - At the beginning of the lesson, we saw that angles with equal measure sweep out equal fractions of a circle. What fraction of a circle does an angle with a measure of 1 radian sweep out?

    Since the fraction of the circle is the ratio of the arc length to the radius, if a circle has a radius of r, then an angle that has a measure of 1 radian sweeps out a fraction of $\frac{r}{2\pi r}$ or $\frac{1}{2\pi}$ of the circle.

- For the second question, you may need to help students make the connection that since the circumference of a circle is equal to $2\pi r$, there are 2π radius lengths in the arc that sweeps out a full circle. We can describe this in terms of angle measure: an angle that sweeps out a full circle has a measure of 2π radians.
- You can help students visualize the number of radius lengths that fit into a full circle using the same animation as earlier in the lesson, with the second slider: **preap.org/Desmos-Radian**.
- Before moving on, have students label an angle with a measure of 2π radians on their protractors. This helps students connect the task of making the radian protractor with their thinking about the openness of an angle that sweeps out a full circle. This also sets students up for Lesson 4T.2: Determining Equivalent Angle Measures, in which they partition the full circle, 2π radians, to create angles with equal measures.

Part 3: Defining the Relationship Among Angle Measure, Radius Length, and Arc Length

~45 MIN.

In this part of the lesson, students explore a central angle of a circle to relate the angle measure in radians, θ, the length of the radius, r, and the arc length, s, to develop the equivalent formulas $\frac{s}{r} = \theta$ and $s = \theta r$.

Student Task

The student task for this part of the lesson is provided on **Handout 4T.1.B: Defining the Relationship Among Angle Measure, Radius Length, and Arc Length**. In the handout, students compare the arc lengths, angles of rotation, and radii for satellites presented in three different scenarios.

> **Meeting Learners' Needs**
> Some students may still have difficulty conceptualizing the term *orbit*. Before starting the task, it might be helpful to elicit a definition of the term from the class so that you can create a shared understanding of what an orbit is. In this context, an orbit is a curved path around a planet, and one full trip along this path is called a *revolution*.

Facilitating the Task

- Begin by providing students with Handout 4T.1.B and allowing pairs of students some time to closely observe the task and ask clarifying questions.
- As students are working, circulate around the room and ask students probing questions such as the following:
  - If you know the radius length and the angle measure in radians, how can you determine the arc length that a satellite travels?

    I know that the angle measure in radians is the ratio of the arc length to the radius length, so I can set up the proportion angle measure = $\frac{\text{arc length}}{\text{radius length}}$.
  - How is this process related to how you created your radian protractor?

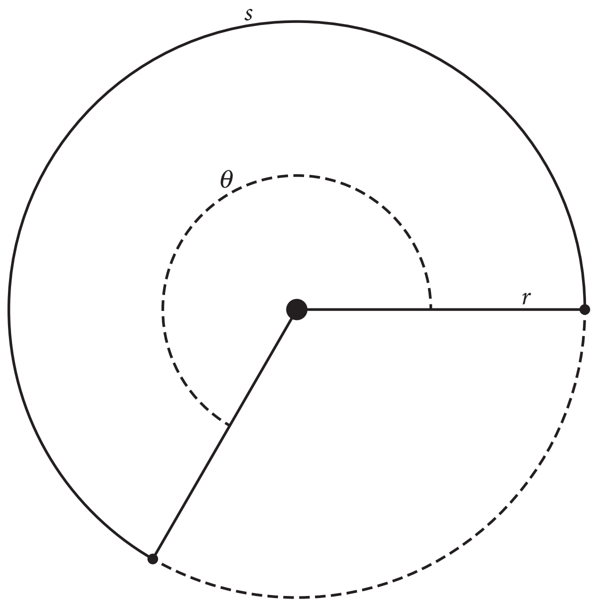
    Instead of starting with the radius length and marking out arc lengths, I can work backward.
  - Would your method work if the angle measure was given in degrees? Why or why not?

    No, because the degree measure of an angle is not related to the arc length or the radius of the circle.
- Answers for the three problems on Handout 4T.1.B are given in the Assess and Reflect section of the lesson.

> **Meeting Learners' Needs**
> Students might benefit from drawing diagrams to represent the situation. They can use the scale 1 mm = 1 km and their radian ruler from Part 3 to help them draw scaled diagrams.

Summarizing the Task

- By the end of this lesson, it is important that students sufficiently understand the relationship among the angle measure in radians, the arc length, and the length of the radius to meaningfully interpret each algebraic form of this relationship. For example, they should be able to describe that the formula $\frac{s}{r} = \theta$ conveys that the angle measure in radians indicates the number of radius lengths that fit into the arc, and equivalently $s = \theta r$ conveys that the arc length is the result of placing θ copies of the radius length around the circle.
- As students complete Handout 4T.1.B, have two pairs of students compare their observations and engage in a brief conversation about what their observations have in common and where they differ.
- Next you can ask students some questions to generalize the relationships among the three quantities: radius length, r, arc length, s, and the angle measure, θ. It might be helpful to display a generalized diagram like the following:



Guiding Student Thinking

Greek letters, such as theta (θ) and alpha (α), are often used to represent angle measures because of the Ancient Greek influence on the development of mathematical ideas. It is important in this course to challenge students to incorporate Greek letters into their work appropriately. Students might also recognize that π is also a Greek letter, used to represent the ratio of the circumference to the diameter of any circle.

- Suppose there is another satellite with a circular orbit that has a radius of r km. If the satellite travels s km along its orbit, how can you determine the angle measure the satellite has swept out in radians? Explain how you know.

  I am trying to find the number of radius lengths that span the arc length, so I can determine the angle measure by dividing the arc length, s, by the radius length, r.

- Suppose you know the radius of a satellite's orbit is r km and the angle measure the satellite sweeps out is θ radians. How can you determine the distance that the satellite travels along its orbit? Explain how you know.

  The distance the satellite traveled along its orbit is the product of the radius and the angle measure in radians, so I can determine this distance by multiplying the radius length, r, by the angle measure, θ.

- Suppose you know that a satellite sweeps out θ radians and the distance that the satellite traveled along its orbit is s km. How can you determine the radius of the satellite's orbit? Explain how you know.

  Because we determined the angle measure to be the ratio of the arc length and the radius, I can find the radius of the satellite's orbit with the ratio of the arc length, s, to the angle measure, θ.

- To help students understand that the three methods they developed to answer the questions on Handout 4T.1.B are different forms of the same relationship, you can write down algebraic representations of the three methods: $\frac{s}{r} = \theta$, $s = \theta r$, and $\frac{s}{\theta} = r$.
- While all three formulas express the same relationship, the version $\frac{s}{r} = \theta$ represents the process of determining the number of radius lengths that span the arc length, and this number of radius lengths is the measure of the angle in radians.
- Help students see that the angle measure in radians tells them how many times they need to place radius lengths along the circle. This iterative process should remind students of multiplication and making θ copies of r. This way of thinking will help students make sense of the formula $s = \theta r$.
- There are several practice problems related to radian measure and arc length on **Handout 4T.1.C: Practice with Radian Measure**. You could have students work on these problems to help them develop fluency with the relationship among the measure of a central angle, the associated arc length, and the length of the radius of a circle.

Assess and Reflect on the Lesson

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

Ferris wheels are common tourist attractions that consist of a rotating upright wheel with multiple passenger cars. Two famous Ferris wheels are the Singapore Flyer and the London Eye.

The Singapore Flyer is a Ferris wheel with a height of 165 meters (m) and a diameter of 150 m.

The London Eye is a Ferris wheel with a height of 150 m and a diameter of 120 m.

Suppose that a rider travels 100 m along the curved path around each Ferris wheel.

(a) Without calculating exact measures, identify the Ferris wheel on which the rider sweeps out a larger angle of rotation.

The rider would sweep out a larger angle of rotation on the London Eye. Since the radius of the London Eye is smaller than the radius of the Singapore Flyer, 100 m is a greater portion of the circumference of the London Eye.

(b) Verify your answer from part (a) by calculating the exact angle of rotation, in radians, the rider sweeps out on each Ferris wheel.

For the Singapore Flyer, a 100-meter arc length sweeps out an angle that measures $\frac{100}{75} \approx 1.333$ radians. For the London Eye, a 100-meter arc length sweeps out an angle that measures $\frac{100}{60} \approx 1.667$ radians.

HANDOUTS

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 4T.1.A: Comparing Attributes of Central Angles

See lesson for guidance about the handout.

Handout 4T.1.B: Defining the Relationship Among Angle Measure, Radius Length, and Arc Length

1. The distance EarthSat traveled can be thought of as 3 radii, each of length 175 km, placed along the circumference of its orbit. This means that EarthSat traveled a distance of $3 \cdot 175 = 525$ km. Similarly, the distance MoonSat traveled can be thought of as 4 radii, each of length 160 km, placed along the circumference of its orbit. This means that MoonSat traveled a distance of $4 \cdot 160 = 640$ km. Therefore, MoonSat traveled a longer distance in the same span of time.
2. Satellite Yin's angle of rotation can be determined by calculating the number of radius lengths of 180 km that span the total distance traveled, 936 km. This means Satellite Yin swept out an angle that measures $\frac{936}{180} = 5.2$ radians. Similarly, Satellite Yang's angle of rotation can be determined by calculating the number of radius lengths of 195 km that span the total distance traveled, 916.5 km. This means Satellite Yang swept out an angle that measures $\frac{916.5}{195} = 4.7$ radians. Therefore, Satellite Yin swept out a greater angle of rotation.
3. Because the arc length is the product of the radian measure of the angle and the radius, the radius of Satellite Bravo's orbit can be determined by solving the equation $3.2r = 640$ for r. This means that Satellite Bravo's orbit has a radius of $\frac{640}{3.2} = 200$ km. Similarly, the radius of Satellite Zulu's orbit can be determined by solving the equation $4.8 \cdot r = 816$ for r. This means that Satellite Zulu's orbit has a radius of $\frac{816}{4.8} = 170$ km. Therefore, Satellite Bravo's orbit has the greater radius.

UNIT 4T

Handout 4T.1.C: Practice with Radian Measure

1. Because $\frac{s}{r} = \theta$, or equivalently $s = \theta r$, we can determine the missing values:

	θ	r	s
(a)	$\frac{\pi}{5}$	2	$\frac{2}{5}\pi$
(b)	$\frac{4}{3}$	$\frac{3}{2}$	2
(c)	2	$\frac{3}{2}$	3

2. The measure of a central angle whose arc length is equal to one radius length is 1 radian. This is the definition of a radian.

3. The measure of a central angle whose arc length is equal to two radius lengths is 2 radians. This is an extension of the definition of a radian.

4. (a) The measure of the central angle of a sector whose arc length is 8 feet and whose radius measures 24 feet is $\frac{1}{3}$ radians because $\frac{s}{r} = \frac{8}{24} = \frac{1}{3}$.

   (b) The measure of the central angle of a sector whose arc length is 8 feet and whose radius measures 8 feet is 1 radian because $\frac{s}{r} = \frac{8}{8} = 1$.

   (c) The measure of the central angle of a sector whose arc length is 8 feet and whose radius measures 10 feet is $\frac{4}{5}$ radian because $\frac{s}{r} = \frac{8}{10} = \frac{4}{5}$.

5. The arc length of a sector of a circle whose central angle measures π radians and whose radius length measures 10 centimeters is 10π centimeters because $s = \theta r = 10\pi$.

6. The radius of a circle that has a sector whose central angle measures $\frac{\pi}{2}$ radians and whose arc length is 2π feet has a length of $r = \frac{s}{\theta} = \frac{2\pi}{\frac{\pi}{2}} = 4$ feet.

7. (a) The central angle for each slice measures $\frac{2\pi}{8} = \frac{\pi}{4}$ radians.

   (b) The arc length for each slice of pizza is $s = \theta r = \frac{\pi}{4}(6) \approx 4.71$ inches.

UNIT 4T

LESSON 4T.2

Determining Equivalent Angle Measures

LEARNING OBJECTIVES

4T.1.1 Use the radian measure of an angle to relate the radius of a circle to the length of the arc subtended by that angle.

4T.1.2 Determine when two angles in the coordinate plane are coterminal.

LESSON OVERVIEW

CONTENT FOCUS

In this lesson, students extend their understanding of radian measures by constructing commonly used angles and determining their measures in both degrees and radians. The lesson engages students in constructing a conceptual model that involves reasoning with proportions to determine the measures of these angles. By the end of the lesson, students develop formulas that allow them to convert angle measures between degrees and radians.

LESSON DESCRIPTION

Part 1: Partitioning a Circle

In this part of the lesson, students construct commonly referenced angles by first partitioning a circle into a specified number of equally sized pieces. Then they use proportional reasoning to determine the measures of the resulting angles in both radians and degrees. Students return to these concepts later in Lesson 4T.5: The Coordinates of Points on a Circle to determine the coordinates of the intersection point of the terminal ray of an angle and a circle centered at the origin.

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Connections Among Multiple Representation

SUGGESTED TIMING

~45 minutes

LESSON SEQUENCE

- This lesson is part of a lesson sequence (~600 minutes total) of Lessons 4T.1 through 4T.7.

MATERIALS

- paper plate or large printed circle
- access to **Desmos.com**

HANDOUTS

Lesson

- 4T.2.A: How Many Radians?
- 4T.2.B: Converting Between Units of Angle Measure

Practice

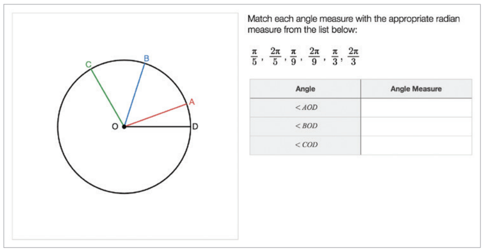
- 4T.2.C: Practice with Radians and Degrees

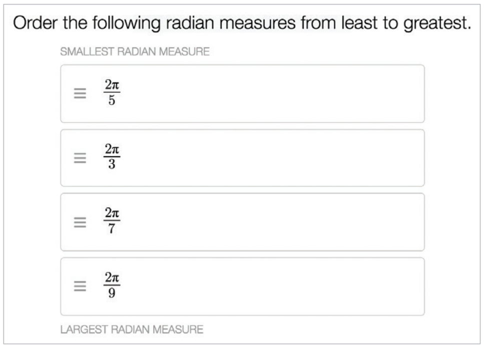
Part 2: Converting Between Units of Angle Measure
In this part of the lesson, students leverage their proportional reasoning from Part 1 to develop a way of thinking that allows them to move between radian and degree measures. Finally, students apply their thinking to develop a formula to convert the units of angle measures between radians and degrees.

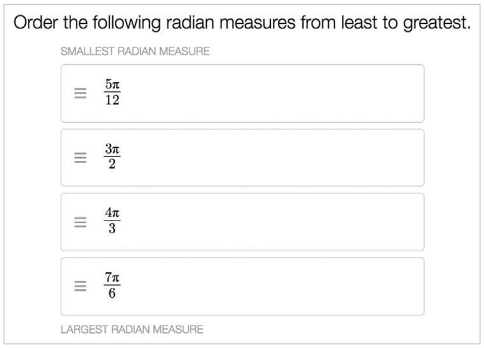
FORMATIVE ASSESSMENT GOAL

This lesson prepares students to complete the following formative assessment activity.

The Formative Assessment Goal for this lesson is a Desmos activity located at **preap.org/Desmos-RadiansAndDegrees**. In this activity, students must match several central angles to their corresponding radian measures and put sets of radian measures in order from least to greatest. You must assign the activity to your class before they are able to complete it.







UNIT 4T

Part 1: Partitioning a Circle

~30 MIN.

In this part of the lesson, students construct commonly referenced angles by first partitioning a circle into a specified number of equally sized pieces. Then they use proportional reasoning to determine the measures of the resulting angles in both radians and degrees. Students return to these concepts in Lesson 4T.5: The Coordinates of Points on a Circle to determine the coordinates of the intersection point of the terminal ray of an angle and a circle centered at the origin.

Warm-Up

Suppose that a satellite circles the Earth with an orbit that has a radius of 160 kilometers (km).

- What is the length of the satellite's orbit around the Earth? How do you know?

  The circumference of the orbit is $2 \cdot \pi \cdot 160 = 320\pi$. The length of the satellite's orbit around the Earth is 320π km.

- What is the measure of the angle, in radians, that the satellite sweeps out in one full orbit around the Earth? How do you know?

  The measure of the angle that the satellite sweeps out in one full orbit around the Earth can be determined by dividing the circumference by the radius: $\frac{320\pi}{160} = 2\pi$ radians.

Guiding Student Thinking

Students may need help understanding why expressing their answers in terms of π is the only way to represent the exact value of certain measurements. Students may incorrectly believe that π is equal to exactly 3.14, and not understand that 3.14 is an approximate value for π. If needed, it would be beneficial for students' mathematical understanding to spend a little time exploring this concept during the warm-up. Later in the lesson, it is critical for students to express angle measures as fractions of π.

Student Task

The student task is given on **Handout 4T.2.A: How Many Radians?** In this task, students construct a model to express as fractions different angles of revolution of the satellite around the Earth along its orbit. Students determine the location of the satellite and the angle of revolution from its initial location based on the fractional distance of the one revolution traveled. The measure of each central angle is recorded in both radians and degrees.

Facilitating the Task

- Begin by distributing Handout 4T.2.A and allowing small groups of students some time to closely observe the task and ask clarifying questions.
- Give each student a circular paper plate or a large, printed circle that they can use to mark up and fold. Explain to students that the plate (or circle) represents the orbit of the satellite. Have students mark the initial location of the satellite at the 3 o'clock position. In Lesson 4T.3: Angles in the Coordinate Plane, students learn that an angle placed with its initial ray on the 3 o'clock position is said to be in *standard position*.
- Let students know that this task is related to the satellite task in Lesson 4T.1. In the previous lesson, they were given either the distance the satellite traveled along the orbit or its angle of revolution. In this task, students are given a fractional part of the orbit that the satellite completes and are asked to determine the location of the satellite and the angle of rotation from the initial location.
- Have students work on problem 1 on the handout. Encourage students to first mark their plates to represent the satellite's location after completing half of its orbit. Students might estimate the location or fold the plate in half. Then ask students to determine and label the measure of the angle the satellite sweeps out in half of its orbit in both radians and degrees. Having students label the measures of common central angles in both degrees and radians helps them recognize that radians and degrees are both units used to quantify an angle's openness. If students are tempted to estimate the angle measure as "about 3 radians," rather than express an exact value of the angle measure as π radians or 180 degrees, you can encourage students to use their radian protractors from Lesson 4T.1 to help them determine the openness of the angle.

Meeting Learners' Needs
It is best for students to partition their circles themselves. If you have students with accessibility needs such that partitioning the circle on their own would be a barrier to completing the task, then you can provide students with a circle that is already partitioned so that they can focus on determining the angle measures.

Meeting Learners' Needs
Students may struggle with folding their circle into 12 equal parts. To partition the circle into 12 equal sectors, they should fold their circle into quarters, and then fold each quarter into 3 equal parts. If done correctly, each circle should display 3 equal parts in each quadrant.

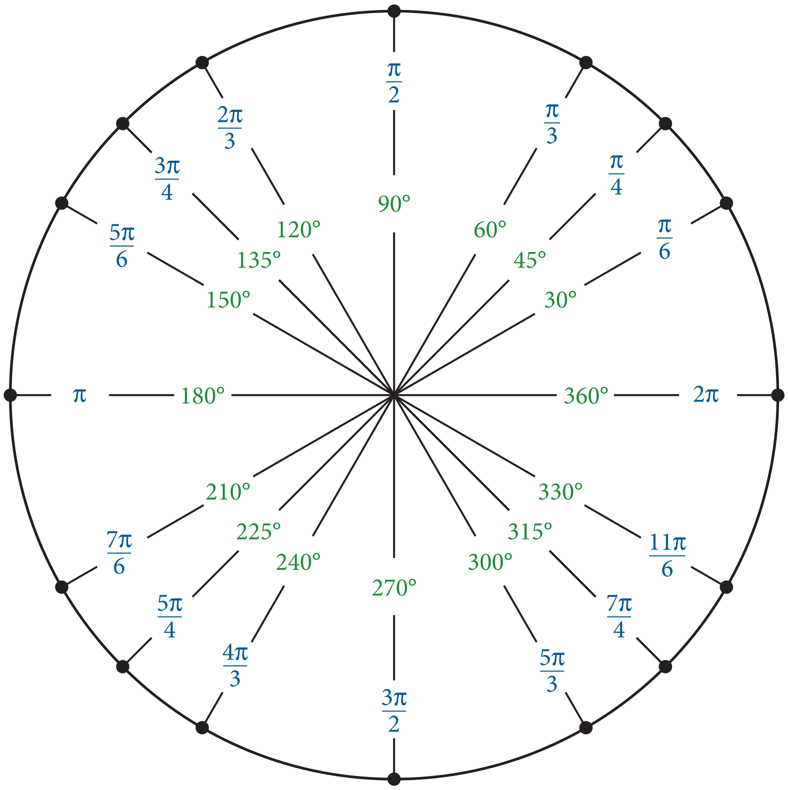
Guiding Student Thinking

Students might attempt to determine the angle measure in radians by measuring the number of radius lengths that span the arc along the circle, an approach consistent with the thinking students used in Lesson 4T.1. In this activity, however, that approach will provide an approximation for each angle measure, not an exact value. Instead, encourage students to use proportional reasoning to determine the exact value for each angle measure. For example, if the satellite sweeps out half of a full orbit, and a full orbit corresponds to an angle measure of 2π radians, then sweeping out half the orbit corresponds to an angle measure of $\frac{1}{2} \cdot 2\pi = \pi$ radians. This thinking about a portion of an orbit as a fraction of 2π supports students in determining equivalent angle measures, which they do in Part 2. Similarly, students should focus on proportional reasoning instead of using tools such as a protractor to measure angles in degrees. Therefore, because a full circle measures 360 degrees, half a circle would be $\frac{1}{2} \cdot 360 = 180$ degrees.

- Have students complete problems 2 through 4. As students work, help them find the location of the satellite for each description by folding their circles into the specified number of equal parts and then determining the corresponding angle measures using proportional reasoning. For example, to complete $\frac{3}{4}$ of the orbit, the satellite needs to sweep out an angle with a measure of $\frac{3}{4} \cdot 2\pi = \frac{3\pi}{2}$ radians or $\frac{3}{4} \cdot 360 = 270$ degrees.

Summarizing the Task

- By the end of the activity, students should have a labeled circle similar to the following figure.



- Once students have marked up their circle, have them answer problem 5 on the handout. Then debrief problem 5 with some questions like the ones that follow.
  - How do you know that an angle that measures 225° is congruent to an angle that measures $\frac{5\pi}{4}$ radians?

    An angle that measures 225° rotates $\frac{225°}{360°} = \frac{5}{8}$ of the full circle. An angle that measures $\frac{5\pi}{4}$ radians represents $\frac{\frac{5\pi}{4}}{2\pi} = \frac{5}{8}$ of a full circle. Because these two angles rotate the same fraction of a full circle, they must be congruent.

- Help students articulate some of the relationships they identified in this part of the lesson, such as the following:
  - To complete one full revolution, an angle sweeps out 2π radians.
  - The circumference of a circle is spanned by exactly 2π radius lengths.
  - Angle measures in radians are often reported as fractions of 2π.

Part 2: Converting Between Units of Angle Measure

~20 MIN.

In this part of the lesson, students leverage their proportional reasoning from Part 1 to develop a way of thinking that allows them to move between radian and degree measures. Finally, students apply their thinking to develop a formula to convert the units of angle measures between radians and degrees.

Student Task

The student task is given on **Handout 4T.2.B: Converting Between Units of Angle Measure**. In this task, students work toward fluently converting angle measures between radians and degrees. By the end of this task, students generate formulas to support converting angle measures between radians and degrees. Answers for Handout 4T.2.B are given in the Assess and Reflect section of the lesson.

Facilitating the Task

- Begin by distributing Handout 4T.2.B and allowing small groups of students time to closely observe the task and ask clarifying questions.
- As you circulate around the room, observe how students make sense of converting between radians and degrees. For students who need additional direction, you may suggest that they use the given information (in degrees or radians) to determine the fraction of the circle that an angle sweeps out, and then use that information to determine the angle measure in the other units of measure (radians or degrees).
- For problem 4, students generalize their work from problems 1 through 3 to generate a formula for converting an angle measure from degrees to radians.
- Similarly, for problem 7, students generalize their work from problems 1, 5, and 6 to generate a formula for converting an angle measure from radians to degrees.

Summarizing the Task

- Once students have completed the handout and made some generalizations about converting between degrees and radians, engage them in some questions like the ones that follow to help them make connections between the two formulas they developed in problems 4 and 7.
    - What is the formula you developed in problem 4 that would help you convert from degrees to radians?

Meeting Learners' Needs

You may find that students need more than two or three concrete examples of how to convert angle measures between degrees and radians. It can be challenging for some students to generalize a technique from only a few examples. You can provide students with additional degree or radian measures to convert to the other unit to help them develop a conversion method.

- What is the formula you developed in problem 7 that would help you convert from radians to degrees?
- How are these formulas similar and how are they different?
- Do the two formulas you developed in problems 4 and 7 express the same relationship between the angle measure in degrees and the angle measure in radians? How do you know?

Guiding Student Thinking

Students often memorize formulas without understanding the relationships they express. Rather than have students memorize two different formulas, it is better to have them understand the proportional relationship between an angle measure in degrees and the equivalent angle measure in radians. In other words, students should understand that there is a direct variation between the angle measure in degrees and the angle measure in radians, where the fraction $\frac{2\pi}{360} = \frac{\pi}{180}$ is the constant of proportionality. This understanding allows students to flexibly use proportional reasoning to convert between degrees and radians without having to remember two different formulas.

- Try to have students verbalize that the two formulas, $\frac{\pi}{180}(\text{angle measure in degrees}) = \text{angle measure in radians}$ and $\frac{180}{\pi}(\text{angle measure in radians}) = \text{angle measure in degrees}$, express the same proportional relationship between the angle measure in degrees and the angle measure in radians. That is, both formulas can be summarized with the proportion $\frac{\pi}{180} = \frac{\text{angle measure in radians}}{\text{angle measure in degrees}}$.
- This is a good opportunity to have students update the vocabulary graphic organizer about radians that they began Lesson 4T.1 so they can include the relationship between radians and degrees.
- At this point, you could use the problems on **Handout 4T.2.C: Practice with Radians and Degrees** to provide students with additional practice opportunities.

Meeting Learners' Needs

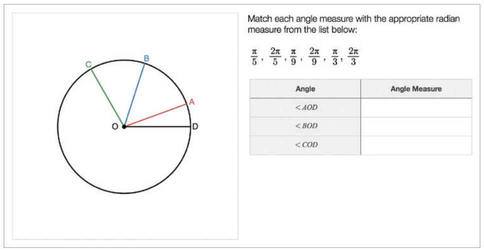
You may find that some students are unsure of where the fraction $\frac{\pi}{180}$ comes from. Encourage these students to work with the equivalent proportion $\frac{2\pi}{360} = \frac{\text{angle measure in radians}}{\text{angle measure in degrees}}$. This may help them understand the meaning of the proportion and why it is a true statement.

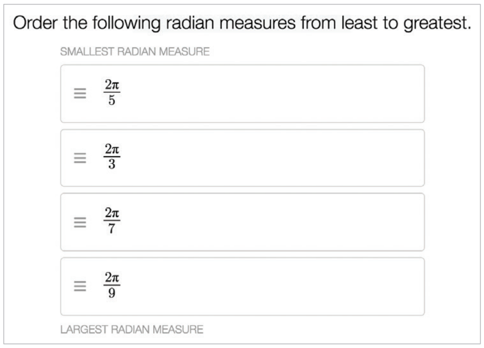
Assess and Reflect on the Lesson

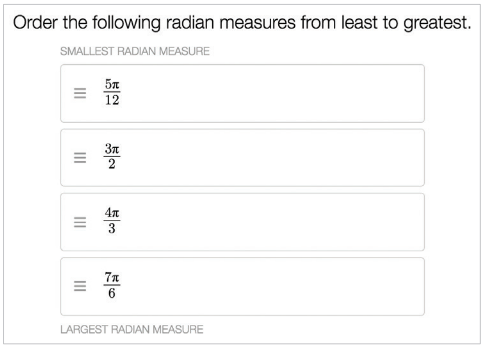
FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

The Formative Assessment Goal for this lesson is a Desmos activity located at **preap.org/Desmos-RadiansAndDegrees**. In this activity, students must match several central angles to their corresponding radian measures and put sets of radian measures in order from least to greatest. You must assign the activity to your class before they are able to complete it.







Screen 1: Angle AOD measures $\frac{\pi}{9}$ radians, angle BOD measures $\frac{2\pi}{5}$, and angle COD measures $\frac{2\pi}{3}$.

Screen 2: The correct order of radian measures from least to greatest is: $\frac{2\pi}{9}, \frac{2\pi}{7}, \frac{2\pi}{5}, \frac{2\pi}{3}$.

Screen 3: The correct order of radian measures from least to greatest is: $\frac{5\pi}{12}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}$.

Guiding Student Thinking

Students may be tempted to convert the radian measures in Screens 2 and 3 to decimal approximations and then order the decimals. Encourage them to use reasoning to compare and order the fractions without first converting them to decimal forms. For the radian measures on Screen 2, students should observe that because the numerators are equal, they can order the fractions by comparing their denominators. They should reason that the fraction with the greatest denominator has the least value because it partitions the numerator into the most equal parts. That is, $\frac{2\pi}{9}$ partitions 2π into 9 equal parts, and so on. For the radian measures on Screen 3, encourage students to express the four fractions into equivalent forms with a common denominator. Then they can order the fractions by comparing their numerators. Students should reason that the fraction with the greatest numerator is the greatest fraction because it represents the greatest number of pieces of the same size. That is, $\frac{3\pi}{2}$ can be expressed as $\frac{18\pi}{12}$, which can be thought of as 18π pieces of size $\frac{1}{12}$, or 18 pieces of size $\frac{\pi}{12}$, and so on.

HANDOUTS

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 4T.2.A: How Many Radians?

See lesson for answers to the handout.

Handout 4T.2.B: Converting Between Units of Angle Measure

1. An angle that sweeps out $\frac{1}{5}$ of a circle has a measure of $\frac{1}{5}(360°) = 72°$. An angle that sweeps out $\frac{1}{5}$ of a circle has a measure of $\frac{1}{5}(2\pi) = \frac{2\pi}{5}$ radians.

2. An angle that has a measure of 120° represents $\frac{120°}{360°} = \frac{1}{3}$ of the circle. This angle has a measure of $\frac{1}{3}(2\pi) = \frac{2\pi}{3}$ radians.

3. An angle that has a measure of 315° represents $\frac{315°}{360°} = \frac{7}{8}$ of the circle. This angle has a measure of $\frac{7}{8}(2\pi) = \frac{14\pi}{8} = \frac{7\pi}{4}$ radians.

4. An angle that has a measure of x degrees represents $\frac{x}{360}$ of the circle. This angle has a measure of $\frac{x}{360}(2\pi) = \frac{2\pi x}{360} = \frac{\pi}{180}x$ radians.

5. An angle that has a measure of $\frac{\pi}{5}$ radians represents $\frac{\frac{\pi}{5}}{2\pi} = \frac{1}{10}$ of the circle. This angle has a measure of $\frac{1}{10}(360) = 36$ degrees.

6. An angle that has a measure of $\frac{9\pi}{8}$ radians represents $\frac{\frac{9\pi}{8}}{2\pi} = \frac{9}{16}$ of the circle. This angle has a measure of $\frac{9}{16}(360) = 202.5$ degrees.

7. An angle that has a measure of x radians represents $\frac{x}{2\pi}$ of the circle. This angle measures $\frac{x}{2\pi}(360) = \frac{360x}{2\pi} = \frac{180}{\pi}x$ degrees.

Handout 4T.2.C: Practice with Radians and Degrees

1. (a) 315° is equivalent to $\frac{7}{4}\pi$ radians because $\frac{\pi}{180} = \frac{\frac{7}{4}\pi}{315}$.

   (b) 90° is equivalent to $\frac{\pi}{2}$ radians because $\frac{\pi}{180} = \frac{\frac{\pi}{2}}{90}$.

   (c) 36° is equivalent to $\frac{\pi}{5}$ radians because $\frac{\pi}{180} = \frac{\frac{\pi}{5}}{36}$.

2. (a) $\frac{3}{8}\pi$ radians is equivalent to 67.5 degrees because $\frac{\pi}{180} = \frac{\frac{3\pi}{8}}{67.5}$.

   (b) $\frac{3\pi}{2}$ radians is equivalent to 270 degrees because $\frac{\pi}{180} = \frac{\frac{3\pi}{2}}{270}$.

   (c) $\frac{7\pi}{5}$ radians is equivalent to 252 degrees because $\frac{\pi}{180} = \frac{\frac{7\pi}{5}}{252}$.

3. (a) An angle that measures 1.5 radians is acute because it sweeps out less than a quarter of a circle.

   (b) An angle that measures π radians is straight because it sweeps out half a circle.

   (c) An angle that measures 0.5 radians is acute because it sweeps out less than a quarter of a circle.

   (d) An angle that measures $\frac{2\pi}{3}$ radians is obtuse because it sweeps out more than a quarter of a circle but less than half a circle.

   (e) An angle that measures $\frac{\pi}{2}$ radians is a right angle because it sweeps out a quarter of a circle.

4. The central angle of a circle with a radius of 12 inches and an arc length of 18 inches has a measure of $\frac{18}{12}=\frac{3}{2}$ radians or equivalently $\frac{270}{\pi}\approx 85.944°$ because $\frac{\pi}{180}=\frac{\frac{3}{2}}{\frac{270}{\pi}}$.
   The area of the sector formed by the central angle measures $\left(\pi \cdot 12^2\right)\frac{\frac{3}{2}}{2\pi}=108$ square inches.

5. The central angle formed by the hands of an analog clock at 4 o'clock measures 120° or equivalently $\frac{4\pi}{6}=\frac{2\pi}{3}$ radians. A clock face is divided into 12 equal parts where the central angle of one sector measures 30° or $\frac{\pi}{6}$ radians. Since 4 o'clock would have four equal sectors, the central angle measures $4 \cdot 30 = 120$ degrees or $4 \cdot \frac{\pi}{6}=\frac{2\pi}{3}$ radians.

6. If a circular gear is revolving at 40 revolutions per minute, then we can determine that the speed in degrees per minute is $40 \cdot 360° = 14{,}400$ degrees per minute and the speed in radians per minute is $40 \cdot 2\pi = 80\pi \,\frac{\text{rad}}{\text{min}}$.

UNIT 4T

LESSON 4T.3

Angles in the Coordinate Plane

LEARNING OBJECTIVE

4T.1.2 Determine when two angles in the coordinate plane are coterminal.

LESSON OVERVIEW

CONTENT FOCUS

In this lesson, students make sense of how two angles can share the same initial ray and the same terminal ray but have different measures. Such angles are called *coterminal angles*. By the end of the lesson, students should understand three important features of an angle measure: the magnitude of an angle's measure quantifies its amount of rotation from its initial ray; an angle's measure can exceed 360°, or 2π radians; and the sign of an angle's measure indicates its direction of rotation from its initial ray.

LESSON DESCRIPTION

Part 1: Rotating a Crane
In this part of the lesson, students use a contextual scenario involving a stationary crane to make sense of how to use positive and negative values to indicate the direction of rotation of an angle in standard position.

Part 2: Creating Coterminal Angles
In this part of the lesson, students use the crane scenario to define and identify coterminal angles. Through this task, they learn the relationship between the measures of coterminal angles whose terminal rays rotate counterclockwise and clockwise.

AREAS OF FOCUS

- Engagement in Mathematical Argumentation
- Connections Among Multiple Representations

SUGGESTED TIMING

~45 minutes

LESSON SEQUENCE

This lesson is part of a lesson sequence (~600 minutes total) that includes Lessons 4T.1 through 4T.7.

MATERIALS

- compass
- colored pencils/crayons (optional)
- access to **Desmos.com**

HANDOUTS

Lesson

- 4T.3.A: Rotating a Crane

Practice

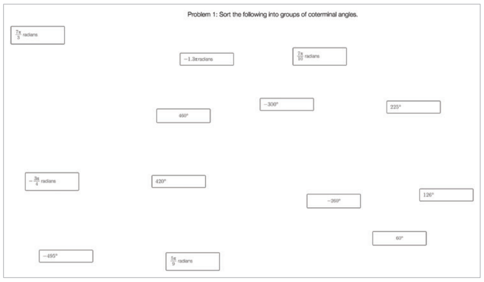
- 4T.3.B: Practice with Coterminal Angles

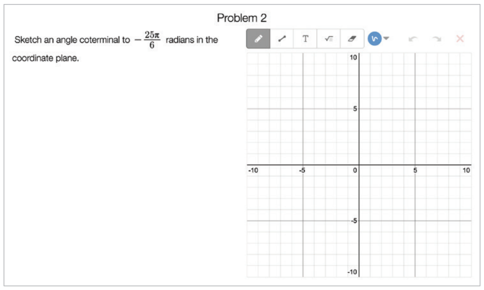
UNIT 4T

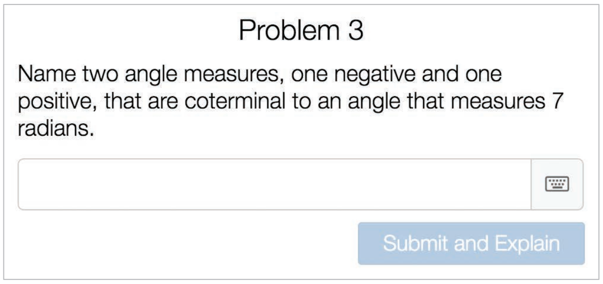
FORMATIVE ASSESSMENT GOAL

This lesson prepares students to complete the following formative assessment activity.

The Formative Assessment Goal for this lesson is a Desmos activity located at **preap.org\Desmos-CoterminalAngles**. In this short activity, students use what they know about coterminal angles to complete a card sort related to coterminal angles and identify groups of coterminal angles. You must assign the activity to your class before they are able to complete it.







UNIT 4T

Part 1: Rotating a Crane

~20 MIN.

In this part of the lesson, students use a contextual scenario involving a stationary crane to make sense of how to use positive and negative values to indicate the direction of rotation of an angle in standard position.

Student Task

The student task is provided on **Handout 4T.3.A: Rotating a Crane**. In Part A of the handout, students use a coordinate grid to construct a model of a building and the four regions that surround it. Then they use their model to answer questions about the scenario.

Facilitating the Task

- Begin by providing Handout 4T.3.A to each student and at least one compass to each pair of students. Allow them to ask clarifying questions and share things they notice about the scenario described in the handout.
- Let students know they will use a coordinate plane to help them organize their thinking and represent the crane's movements. As indicated in the task, the crane is located on a tower at the center of the building, which should coincide with the origin of the coordinate plane used to represent the crane and the surrounding regions.

Meeting Learners' Needs

If students ask about what a construction tower crane looks like or how it operates, you may want to find a short video or website to show to help them better understand the scenario. One possible resource can be found at **science.howstuffworks.com/transport/engines-equipment/tower-crane.htm.**

- Have students complete problem 1 by constructing a diagram of the building on the coordinate grid provided on the handout. Students should account for the scale of the drawing since the distance between each grid line represents 3 meters. A sample student diagram is shown in the Assess and Reflect section of the lesson.
- Once students have completed their scale drawings, have them work on problem 2. This problem helps orient them to the coordinate plane and the four regions designated for different purposes. You may find that students naturally want to refer to the quadrants of the graph. That is, they may refer to the retail area as quadrant I, the fitness center as quadrant II, etc.).

- Have students use a compass to draw the circular path the tip of the crane makes. To help students think about the meaning of this circular path, you can ask students if the crane can reach all points within the building.
- You may have to clarify for students that the crane always picks up its load from the loading dock and then rotates a specified angle measure before dropping the load in its final location.

Meeting Learners' Needs
You could ask students who would benefit from an enrichment activity to determine the equation of the circular path the tip of the crane makes. Based on the crane's position on the coordinate grid, the equation is $x^2 + y^2 = 30^2$.

- Tell students that we say the crane's angle of rotation is in *standard position* because its vertex is at the origin and one of its rays lies along the positive x-axis. The ray that lies along the positive x-axis is called the *initial ray*. The other ray is called the *terminal ray*.
- Ask students some questions to get them to reflect on the advantages and disadvantages of constructing angles on a coordinate grid in standard position, such as the following:
  - Why might it be helpful to place angles in standard position?

    If you know that an angle is in standard position, then you know the location of the vertex and the initial ray. This means that the location of the terminal ray would be easy to determine because the degree of rotation begins from the positive x-axis.
  - What might be a disadvantage of placing angles in standard position?

    In a problem with multiple angles in standard position, their shared initial ray might make it hard to tell the angles apart.

Meeting Learners' Needs
Some students may benefit from constructing a vocabulary graphic organizer for *an angle in standard position*, with the terms *initial ray* and *terminal ray*. A blank vocabulary graphic organizer can be found on Handout 1.1. Then you can add the organizer to your classroom's word wall.

Instructional Rationale

In their geometry course, students likely dealt with angles as static objects, so there was no need to indicate the direction of rotation from the initial ray to the terminal ray. In this unit, we work with dynamic contexts where the direction of rotation should be specified. Problem 3 on Handout 4T.3.A is designed to have students consider how the magnitude of an angle measure without a direction could lead to different drop-off locations for the crane.

- Have students work with a partner to complete problem 3. It is reasonable for students to assert that if the crane rotates an angle of 1 radian, then the package of bricks could be dropped in either the retail area or the parking garage. Once each pair of students has answered problem 3, have two pairs compare their answers and discuss the similarities and differences. Then you can ask students some questions to get them to confront the ambiguous direction of rotation, such as the following:
  - Did the crane rotate clockwise or counterclockwise to drop off the package of bricks?

    It is not clear what direction the crane rotated. Therefore, it could have rotated either clockwise or counterclockwise.
  - How might we differentiate between rotating 1 radian counterclockwise and 1 radian clockwise?

    We might be able to indicate the direction of rotation with the sign of the angle measure.

Summarizing the Task

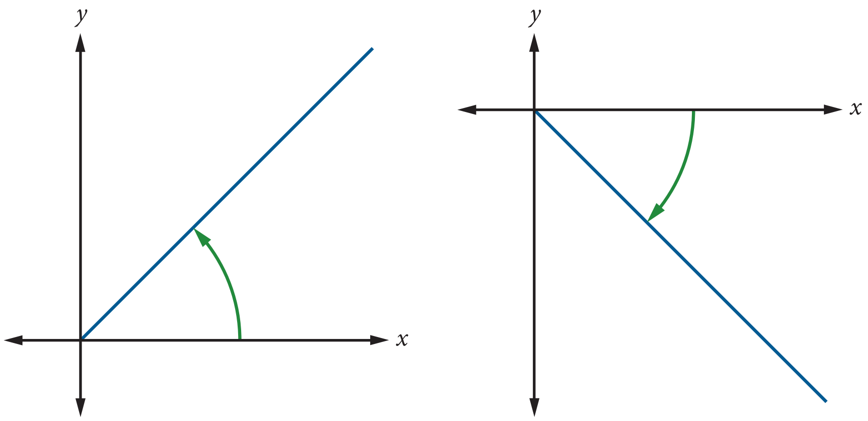
- At this point, you can explain to students that mathematicians developed the convention that positive values of angles indicate counterclockwise rotation from the initial ray and negative values of angles indicate clockwise rotation from the initial ray.

Meeting Learners' Needs

Some students may not know or readily use the terms *clockwise* and *counterclockwise*. You can help make sense of the direction angles in standard position sweep out by indicating that the terminal rays of angles with positive measures rotate toward the positive y-axis and the terminal rays of angles with negative measures rotate toward the negative y-axis.

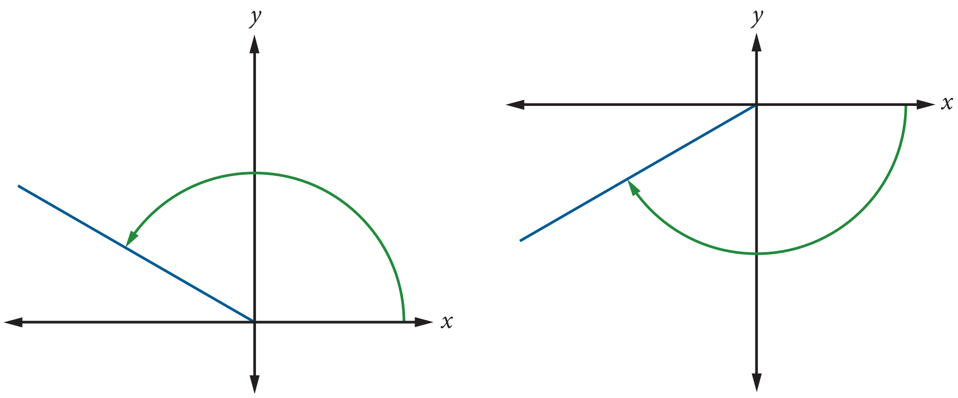
- You may find that students need more practice reasoning with positive and negative angle measures. You can give them additional opportunities to work with positive and negative angle measures by asking questions such as the following:
  - Draw two angles in standard position: one that has a measure of $\frac{\pi}{4}$ radians and one that has a measure of $-\frac{\pi}{4}$ radians. How are these two angles similar? How are they different?

    An angle that measures $\frac{\pi}{4}$ radians sweeps out in the counterclockwise direction. An angle that measures $-\frac{\pi}{4}$ radians sweeps out in the clockwise direction. The terminal rays of the angles have swept out the same amount of rotation, but their locations are different.



  - Draw two angles in standard position: one that has a measure of 150° and one that has a measure of −150°. How are these two angles similar? How are they different?

    An angle that measures 150° sweeps out in the counterclockwise direction. An angle that sweeps out −150° rotates in the clockwise direction. The angles have the same amount of rotation, but the locations of their terminal rays are different.



- Is it possible for two different angles to open the same amount in different directions but have the same terminal ray? Why or why not?

  If both angles are odd integer multiples of 180° or both angles are even integer multiples of 180° (or equivalently are odd integer multiples of π radians or even integer multiples of π radians), then the angles would open by the same amount, and though the direction of rotation would be different, they would have the same terminal ray.

- By the end of Part 1, it is essential for students to understand that signed angle measures are used to specify the direction of rotation of the terminal ray, that positive angle measures indicate counterclockwise rotation, and that negative angle measures indicate clockwise rotation.

Part 2: Creating Coterminal Angles

~25 MIN.

In this part of the lesson, students use the crane scenario to define and identify coterminal angles. Through this task, they learn the relationship between the measures of coterminal angles whose terminal rays rotate counterclockwise and clockwise.

Student Task

The student task is provided on Handout 4T.3.A. In Part B, students use their understanding of clockwise and counterclockwise angles of rotation to investigate coterminal angles in the crane context.

Facilitating the Task

- Have students continue to Part B of Handout 4T.3.A and complete problems 4 and 5. Encourage students to mark the locations on their diagrams where the crane needs to deposit the packages of bricks. Students can determine the angles of rotation using their radian protractor from Lesson 4T.1 or a standard degree protractor. Have the pairs of students check their reasoning with another pair's before moving on to the next question. As you circulate around the room, ask students probing questions such as the following:
  - How much more would the crane need to rotate to return to the intended drop-off location?

    If the crane rotates another 360°, or 2π radians, it will be back at the intended drop-off location. The total angle of rotation of the crane will have a measure of 460°. Since 100° is equivalent to $\frac{5\pi}{9}$ radians and the crane rotates an additional 2π radians, the total angle of rotation of the crane will have a measure of $\frac{23\pi}{9}$ radians.

- What could the crane operator do if the cement truck was still at the drop-off location on the crane's second rotation?

  The crane could rotate another 360° or 2π radians. That would mean the crane would have rotated a total of 820°, or $\frac{41\pi}{9}$ radians, from when the crane initially left the loading dock.

Guiding Student Thinking

Since the angle measure in problem 4 is given in degrees, students will likely continue to reason in degrees. Throughout the handout, encourage students to reason in both degrees and radians. This additional practice during class will help them better understand the relationship between radians and degrees. It will also allow you to identify students who have not yet mastered converting angle measures between degrees and radians.

- Encourage students to develop a definition for angles whose terminal rays coincide. Let students use their own language rather than listen to and repeat a formal definition you provide at this time.
- Have students work with their partner to complete problem 6. This problem gives students an opportunity to convert between angle measures in degrees and radians.
- Have students work with a partner to complete problem 7. As you circulate around the room, ask students probing questions like the following:
  - How do we indicate a clockwise rotation of the crane?

    We indicate a clockwise rotation of the crane with a negative angle measure.
  - What is the relationship between the angle measure rotated counterclockwise and smallest possible angle measure rotated clockwise to reach the same drop-off location?

    The angle measures differ by 360°.
  - What is the relationship between the angle measure rotated counterclockwise and the smallest possible angle measure rotated clockwise to the same drop-off location if the angle measures are given in radians?

    The angle measures differ by 2π radians.

Summarizing the Task

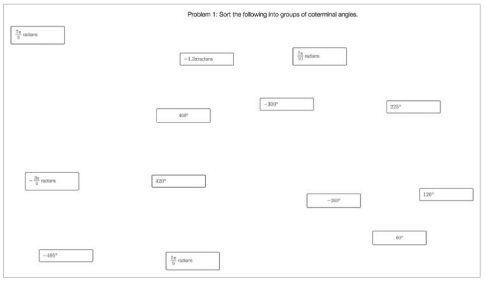
- Have students return to their definition of *coterminal angles* to expand it, if necessary, to include negative angle measures and angle measures in different units. You may need to guide the discussion so that all students have a clear definition of *coterminal angles*: Two angles are called *coterminal* if their initial rays and their terminal rays coincide. The angles may have different measures. The angle measures will differ by an integer multiple of 360° or 2π radians.
- If you have not had students create a vocabulary graphic organizer for the terms in this lesson, this is a good opportunity to do so. Students can make one organizer for the term *standard position* that includes the terms *initial ray* and *terminal ray*. They can create a second organizer for the term *coterminal angles*. Then you can add these terms to your classroom's word wall. A blank vocabulary graphic organizer is provided on Handout 1.1.
- By the end of the lesson students should be able to articulate the relationship between the measures of coterminal angles for angle measures in degrees or radians. For additional practice opportunities, students could complete problems on **Handout 4T.3.B: Practice with Coterminal Angles**.

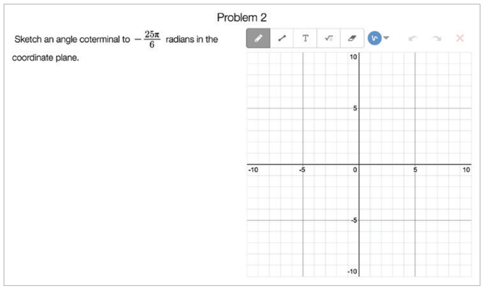
Assess and Reflect on the Lesson

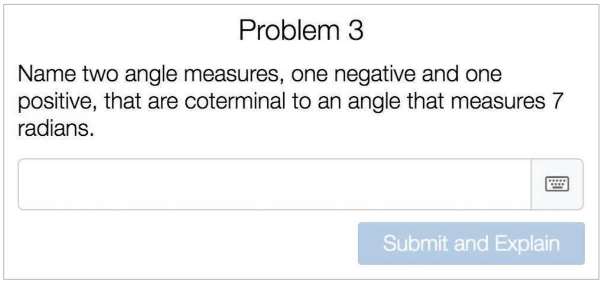
FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

Complete the Desmos activity located at **preap.org\Desmos-CoterminalAngles**. In this short activity, students use what they know about coterminal angles to complete a card sort related to coterminal angles and identify groups of coterminal angles.







Screen 1: Card Sort

Group 1: $60°, -300°, 420°, \frac{7\pi}{3}$ radians

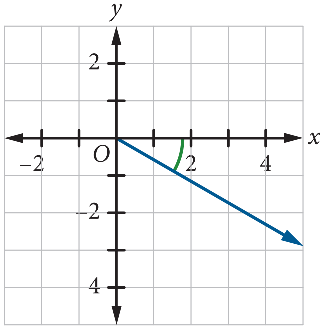
Group 2: $225°, -495°, -\frac{3\pi}{4}$ radians

Group 3: $\frac{5\pi}{9}$ radians, $-260°, 460°$

Group 4: $126°, \frac{7\pi}{10}$ radians, -1.3π radians

UNIT 4T

Screen 2: An angle that is coterminal to $-\frac{25\pi}{6}$ in the coordinate plane will be coterminal to the following:



Screen 3: One example of an angle with a positive angle measure that is coterminal to an angle that measures 7 radians is $7 + 2\pi$ radians. An example of an angle with a negative angle measure that is coterminal to an angle that measures 7 radians is $7 - 4\pi$ radians.

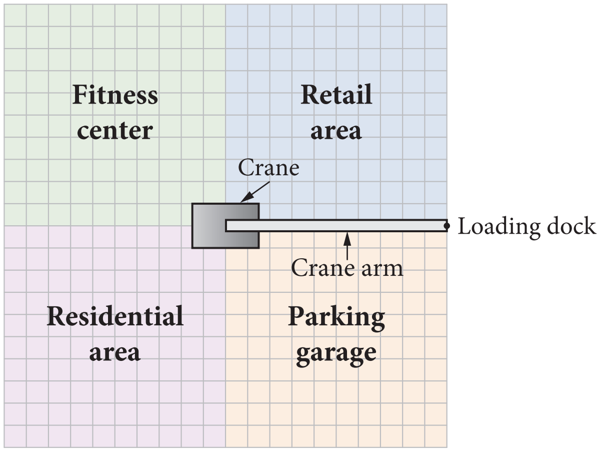
HANDOUTS

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 4T.3.A: Rotating A Crane

Part A

1.



2. (a) parking garage
   (b) residential area
   (c) This point is outside the building.
   (d) Any ordered pair indicating a location inside the building where the x-coordinate is negative and the y-coordinate is positive is in the fitness center.
   (e) Any ordered pair indicating a location inside the building where the x-coordinate is positive and the y-coordinate is positive is in the retail area.
   (f) (30, 0)
3. Either the retail area or the parking garage is an acceptable answer.

Part B

4. The operator would need to rotate the crane another 360° or 2π radians.
5. The operator can fix this error by rotating another 355°, or $\frac{71\pi}{36}$ radians, to get back to the intended drop-off point.
6. The afternoon operator should interpret the angle measure to be positive $\frac{5\pi}{4}$ radians.
7. The crane can rotate clockwise by an angle of –300°, or $-\frac{5\pi}{3}$ radians, to reach the drop-off point for the second box.

Handout 4T.3.B: Practice with Coterminal Angles

1. Two angles that are between 360° and –360° that are coterminal with 500° are 140° and –220°. An angle that measures 500° is 140° more than one full rotation; therefore 500° and 140° differ by an integer multiple of 360°. Because 140° and –220° differ by an integer multiple of 360°, they must be coterminal angles.
2. Two angles that are coterminal with $-\frac{\pi}{6}$ are $\frac{11}{6}\pi$ and $-\frac{13}{6}\pi$. Because $-\frac{\pi}{6}$ and $\frac{11}{6}\pi$ differ by an integer multiple of 2π, they must be coterminal angles. Likewise, because $-\frac{\pi}{6}$ and $-\frac{13}{6}\pi$ differ by an integer multiple of 2π, they must be coterminal angles.
3. Coterminal angles measured in degrees differ by an integer multiple of 360°. Therefore, all angles that are coterminal with 27° can be determined with the expression $(27+360n)°$.
4. Coterminal angles measured in radians differ by an integer multiple of 2π. Therefore, all angles that are coterminal with 27° can be determined with the expression $\frac{3\pi}{8}+2\pi \cdot n$.

5. The coterminal angles are 45° and 405°, 270° and –90°, 390° and 30°, 90° and 450°, 850° and 130°, and –240° and 120°.

6. The coterminal angles are $\frac{\pi}{2}$ and $\frac{5\pi}{2}$, 3π and π, $\frac{2\pi}{3}$ and $\frac{8\pi}{3}$, $-\frac{\pi}{2}$ and $\frac{3\pi}{2}$, $\frac{\pi}{3}$ and $-\frac{5\pi}{3}$, and 0 and 2π.

7. (a) 80° and 200° are not coterminal. An angle coterminal with 80° would have a measure of $(80+360n)°$ and an angle coterminal with 200° would have a measure of $(200+360n)°$ for integer values of n.

(b) 6π and -2π are coterminal.

(c) 60° and $\frac{\pi}{3}$ are coterminal.

(d) $\frac{3\pi}{4}$ and 270° are not coterminal. An angle coterminal with $\frac{3\pi}{4}$ radians would have a measure of $\left(\frac{3\pi}{4}+2\pi n\right)$ radians and an angle coterminal with 270° would have a measure of $(270+360n)°$ for integer values of n.

LESSON 4T.4

A Model for Circular Motion

LEARNING OBJECTIVE

4T.1.3 Construct a representation of a sinusoidal function.

LESSON OVERVIEW

CONTENT FOCUS

This lesson introduces students to sinusoidal functions as models for circular motion. Sinusoidal functions relate the measure of an angle in standard position to the location of the intersection point of the terminal ray of the angle and a circle whose center is the origin. Students observe that the angle measure and the height of the endpoint above a horizontal axis are related by a sine function. Similarly, they observe that the angle measure and the distance of the endpoint from the vertical axis are related by a cosine function. Through the investigation of real-world contexts, students develop a practical and theoretical understanding of sine and cosine functions.

LESSON DESCRIPTION

Part 1: Creating a Model of a Ferris Wheel Car

In this part of the lesson, students begin their exploration of sinusoidal functions through the context of a Ferris wheel, in which they relate the angle that the wheel has turned from its starting position to the height of a car from the horizontal axis. Students first construct a circle with a specified radius and partition the circle into 12 arcs of equal length. Then students measure the height of a Ferris wheel car above the horizontal axis and relate this measurement to the angle made by the horizontal axis and

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Connections Among Multiple Representations

SUGGESTED TIMING

~90 minutes

LESSON SEQUENCE

This lesson is part of a lesson sequence (~600 minutes total) that includes Lessons 4T.1 through 4T.7.

MATERIALS

- blank paper or graph paper
- compass
- protractor
- ruler
- wax craft sticks
- string
- index cards or card stock
- access to **Desmos.com**

HANDOUT

Practice

- 4T.4: Practice Using Sine and Cosine

the radius that includes the car. Students then create a graph with their data to reveal the shape of a new function family.

Part 2: Defining the Sine Function
In this part of the lesson, students measure the height of the Ferris wheel car from the loading platform again, this time using a ruler whose units are based on the radius of their circles. After plotting this new data, students observe that their graphs are the same regardless of the radius length of their individual circles. This new function relates the angle measure in radians to the height of the point on the circle measured in radius lengths. Students define this relationship as the sine function.

Part 3: Defining the Cosine Function
In this part of the lesson, students explore the cosine function through the context of a modern wind turbine. Students measure the distance of each blade tip from the vertical axis when the wind turbine is in motion. As in Part 1, they construct a table of values that relates the angle measure in radians that a single blade makes with the horizontal axis to the distance of the blade tip to the vertical axis. Then they define this relationship as the cosine function.

FORMATIVE ASSESSMENT GOAL

This lesson prepares students to complete the following formative assessment activity.

Use a ruler and the graph of the sine and cosine functions to determine the approximate value of each of the following expressions:

(a) $\sin(1)$

(b) $\sin(4)$

(c) $\cos(1)$

(d) $\cos(4)$

UNIT 4T

Part 1: Creating a Model of a Ferris Wheel Car

~90 MIN.

In this part of the lesson, students begin their exploration of sinusoidal functions through the context of a Ferris wheel, in which they relate the angle that the wheel has turned from a starting position to the height of a car from the horizontal axis. Students first construct a circle with a specified radius and partition the circle into 12 arcs of equal length. Then students measure the height of a Ferris wheel car above the horizontal axis and relate this measurement to the angle made by the horizontal axis and the radius that includes the car. Students then create a graph with their data to reveal the shape of a new function family.

Warm-Up

- Because this lesson requires students to work with angle measures in radians, you may find it helpful to start the lesson with some brief questions to activate students' prior knowledge about radians and degrees, such as the following:
  - What does it mean for an angle to measure 1 radian? What is the approximate degree measure of an angle that measures 1 radian?

    An angle that measures 1 radian sweeps out an arc whose length is equal to the radius of the circle for which the angle is a central angle. An angle that measures 1 radian also measures a little less than 60°.
  - What does it mean for an angle to measure 2π radians? What is the degree measure of an angle that measures 2π radians?

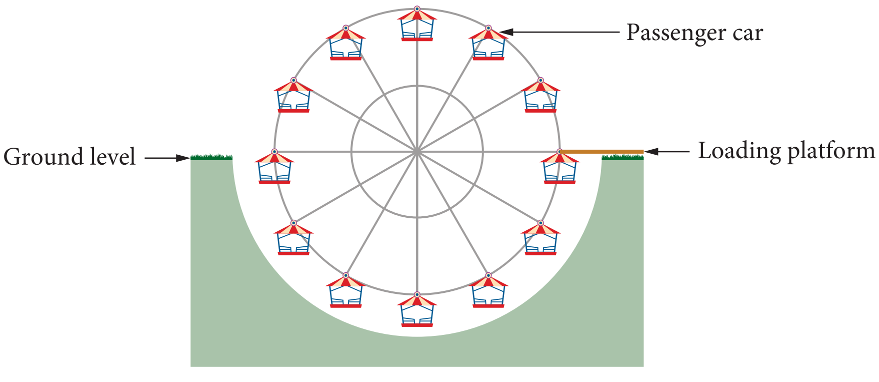
    An angle that measures 2π radians sweeps out an arc whose length is equal to 2π times the radius of the circle for which the angle is a central angle. Since $2\pi r$ is the circumference of a circle, an angle that measures 2π radians sweeps out an arc whose length is the full circle. An angle that measures 2π radians also measures 360°.
  - Describe the relationship between the radian measure of an angle and the degree measure of an angle.

    The degree measure of an angle is directly proportional to the radian measure of an angle. The constant of proportionality is $\frac{2\pi}{360}$ or $\frac{\pi}{180}$.
  - Use the relationship between radian and degree measures to determine the degree measure of an angle that measures $\frac{2\pi}{5}$ radians and the radian measure of an angle that measures 135°. Explain how you know.

The degree measure of an angle that measures $\frac{2\pi}{5}$ radians is 72° because $\frac{\pi}{180}(72) = \frac{2\pi}{5}$. The radian measure of an angle that measures 135° is $\frac{3\pi}{4}$ radians because $\frac{\pi}{180}(135) = \frac{3\pi}{4}$.

Student Task

Fun-Time Amusement Park is building Ferris wheels in an unusual configuration: The lower half of each Ferris wheel will be underground, while the upper half will be above ground. Each Ferris wheel has 12 passenger cars equally spaced around its outer edge. The amusement park has decided to build two such Ferris wheels at the park, with the smaller one for children and the larger one for teens and adults.



The loading platform for the Ferris wheel will be at ground level. This engineer's model for the Ferris wheel is not complete. You will draw a model of one of these Ferris wheels. Your drawing should include the height of each car when the Ferris wheel is in a stationary position as shown in the figure.

Facilitating the Task

- Allow students some time to read the scenario and ask clarifying questions. This would be a good opportunity for students to make a list of what they notice and what they wonder about the scenario. If you have students generate lists of their ideas and observations, provide time for them to share their thoughts with a partner and/or the whole class.

Meeting Learners' Needs

Having students engage in the full activity is the most valuable way for them to experience mathematical modeling. Some students may need scaffolding to participate meaningfully in this activity. Consider providing them with pre-drawn circles or giving them circles already partitioned into 12 equal parts. This would allow students to focus on the heights of the nonorigin endpoints and the corresponding angle measures.

- Have students complete the task in small groups of 3 or 4 students. You can assign each group a different radius length to use in their Ferris wheel model to encourage meaningful peer-to-peer conversation during the jigsaw portion of the lesson. For a circle to fit on a standard piece of paper, the radius needs to be 10 cm or less.
- Try to avoid assigning a radius of 1 cm or 1 inch to any group; it is a little too small for students to measure with precision. Later in the lesson, students will create a nonstandard ruler whose unit increment is the length of their radius. A group whose circle is already 1 unit may not take away the intended meaning of the exercise when using a radius length as a unit of measure.
- As students create their circles, it could be helpful to remind them that the circles are their models of the Ferris wheel. They may also need to be reminded that half the Ferris wheel will be underground, and half the Ferris wheel will be above ground. Students could use a compass or a digital tool to create their circles. If they are using compasses, they will also need a ruler to set the compass opening to the correct width.
- Students who want to partition their circles by equal arc lengths will need a thin, flexible object, such as wax craft sticks, a pipe cleaner, or a length of string that they can wrap around the circle. Students who want to partition the circle by equal angle measures will need a protractor. Students who want to partition the circle by equal radian measures will need their radian protractor from Lesson 4T.1. It is best if they directly measure either the arc lengths or the angles so that they continue to develop the essential skill of using measuring devices in their work.

Classroom Ideas

You could provide students with graph paper to help them construct their circular models by hand. If you choose to assign radius lengths in centimeters, then use centimeter graph paper. If you choose to assign radius lengths in inches, then use $\frac{1}{4}$-in. or $\frac{1}{8}$-in. graph paper.

Instructional Rationale

It is important for students to be specific about the reference point and direction of the angle measure because it reinforces the meaning of negative values in height and angle measures. The Ferris wheel in the student task is intentionally constructed so that half the Ferris wheel is above ground. This unorthodox design ensures that the heights and angle measures swept out counterclockwise from the loading platform are initially positive. Because the lower half of the Ferris wheel is below ground, the heights and angle measures swept out clockwise from the loading platform are initially negative. The design of the problem helps students make sense of the signed measurements of a unit circle.

- Have students measure the angle that each car makes with the horizontal axis, the location of the loading platform, relative to the center of the Ferris wheel. They can generate a table of values like the following sample table, which displays measurements for a circle with a radius of 3 cm. Students will need rulers with markings of sufficiently small intervals to measure the heights with some precision. For example, millimeter markings for a centimeter ruler or $\frac{1}{16}$-in. markings for an inch ruler will work.
- As you circulate around the room, ask students some probing questions such as:
  - What are the quantities you are investigating?

    The quantities are the angle measures of the Ferris wheel car and the corresponding heights of the car relative to the ground.
  - How are you measuring your angles?

    The angle measure of the Ferris wheel car should be measured in radians counterclockwise from the loading platform, and the height of the car relative to the ground should be measured in centimeters.
  - What are the maximum and minimum heights of the Ferris wheel cars from the loading platform?

    Since the circular model has a radius of r cm, the maximum height of a Ferris wheel car is r cm and the minimum height of the car is $-r$ cm.

Angle Measure (CCW in radians)	Height Above Ground (cm)
0	0
$\frac{\pi}{6}$	1.5
$\frac{\pi}{3}$	2.6
$\frac{\pi}{2}$	3
$\frac{2\pi}{3}$	2.6
$\frac{5\pi}{6}$	1.5
π	0

Angle Measure (CCW in radians)	Height Above Ground (cm)
$\frac{7\pi}{6}$	-1.5
$\frac{4\pi}{3}$	-2.6
$\frac{3\pi}{2}$	-3
$\frac{5\pi}{3}$	-2.6
$\frac{11\pi}{6}$	-1.5
2π	0

- Once students complete their tables of values, they can plot the points on a coordinate plane. Students must decide which quantity should be assigned as the independent variable and which should be assigned as the dependent variable. Encourage them to consider how their assignments of independent and dependent variables allow them to model the relationship between the quantities with a function.

Guiding Student Thinking

If students need help understanding how their assignments of variables affect their ability to model the scenario with a related function, you could focus their attention on the height column in their table of values. Prompt them to reason through how using height as the independent variable would not be a good choice since several values for height are each associated with different angle measures.

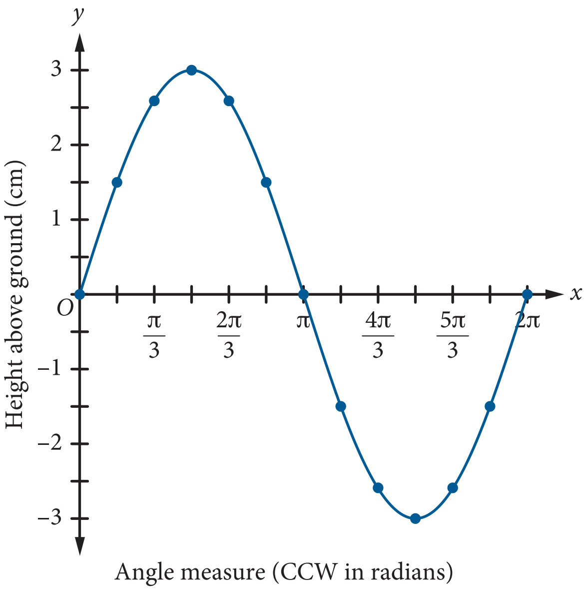
Summarizing the Task

- After students plot the discrete points, they can draw a continuous curve through the points. A sample graph is shown in the figure. Ask students some questions such as:
  - Does it make sense that we can draw a curve through all the points? Why or why not?

    It makes sense to draw a curve through the points because the Ferris wheel car achieves every angle measure from 0 to 2π radians (or from 0° to 360°) as it turns. The heights of the Ferris wheel car would follow a predictable pattern along the curve.
  - What do the points that are on the curve, but not in your table of values, indicate?

    Any point on the curve represents two related values: a measure of an angle formed by a radius whose nonorigin endpoint is a Ferris wheel car and the platform (ground level), as well as its height from the ground.
  - How similar is your graph to another group's graph?

    All groups have curves with the same general shape, but with different heights for corresponding angles.
- Students can now define this function as the relationship between the two quantities: the angle measure of the Ferris wheel car in radians and the height of the car from the loading platform in centimeters.
- Let students give the function a name. Some students may want to call it h for height.



- To make sure that students understand the inputs and outputs of this new function, you can ask them questions such as:
  - What are the inputs and outputs of the function h?

    The inputs of h are angle measures in radians and the outputs are the heights of the Ferris wheel cars for the model in centimeters.

  - What is the meaning of $h(2)$? What do you know about the value of $h(2)$? What is the approximate value of $h(2)$ for your model?

    $h(2)$ is the height of the Ferris wheel car when the angle of the car is 2 radians. The value of $h(2)$ is positive because an angle measure of 2 radians sweeps out an arc length of 2 radii. A 2-radii arc length has a height that is above ground. Examining the graph suggests that a value for $h(2)$ is approximately 2.7 cm for a circular model with a radius of 3 cm.

- Engage students in a brief thought experiment about the periodicity of the sinusoidal function:
  - Our table of values shows the heights of the car when it is at various angles within one full rotation of the Ferris wheel. Suppose that we extended the table of values for angle measures beyond one full rotation of the wheel. How would the heights of the car in the second rotation of the wheel compare to the heights of the car in the first rotation of the wheel? Explain your reasoning.

    The heights of the car in the second rotation would be the same as the heights of the car in the first rotation for coterminal angles. This is because the terminal rays of coterminal angles coincide, which means the heights of the car are the same.

Part 2: Defining the Sine Function

~30 MIN.

In this part of the lesson, students measure the height of the Ferris wheel car from the loading platform again, this time using a ruler whose units are based on the radius of their circles. After plotting this new data, students observe that their graphs are the same regardless of the radius length of their individual circles. This new function relates the angle measure in radians to the height of the point on the circle measured in radius lengths. Students define this relationship as the sine function.

Student Task

Suppose that we measured the height of the Ferris wheel car in a nonstandard unit of measure, like the radius of the circle, instead of in centimeters. Complete each of the following:

(a) Create your own custom ruler whose length is 2 radii, and then partition each radius length into at least 8 equal parts.

(b) Use your new ruler to measure the height of the Ferris wheel car at the 12 angle measurements from your original table of values.

(c) Plot the new data and connect the data points with a curve.

(d) Compare your new graph to your old graph. How are they the same? How are they different?

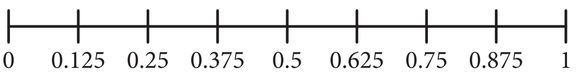
Facilitating the Task

- Before having students begin the task, provide a few moments for them to predict how taking measurements using this new unit of measure will affect the graphs they will construct from this data. You can ask students the following questions, although it is not necessary for them to get the correct answers yet. Engaging with the questions will help them identify what to watch for as they work through the task.
  - Why do you think we are making a new ruler? What is the advantage of using the radius length of your circle as the unit of measure?
  - How will the new height measurements affect the graph you made in Part 1?

Classroom Ideas

Instead of having students create a new table of values, you could suggest that they add a third column to their original table. This will save some time during class and may also provide an opportunity for students to observe that the values they are measuring with their radius rulers are the measurements they determined with the standard ruler scaled by a factor of $\frac{1}{r}$.

- As you circulate around the room, be sure that students are creating a ruler whose 1-unit length is the length of their radius. They can use the edge of an index card or a piece of card stock as the edge of their new ruler.
- It is critical that students partition their ruler into at least 8 equal parts. If the radius lengths are in centimeters, then they may find that partitioning their ruler into 10 equal parts is more convenient. A sample nonstandard ruler is shown in the following figure. Students can choose to use decimals or fractions to mark their personal rulers.

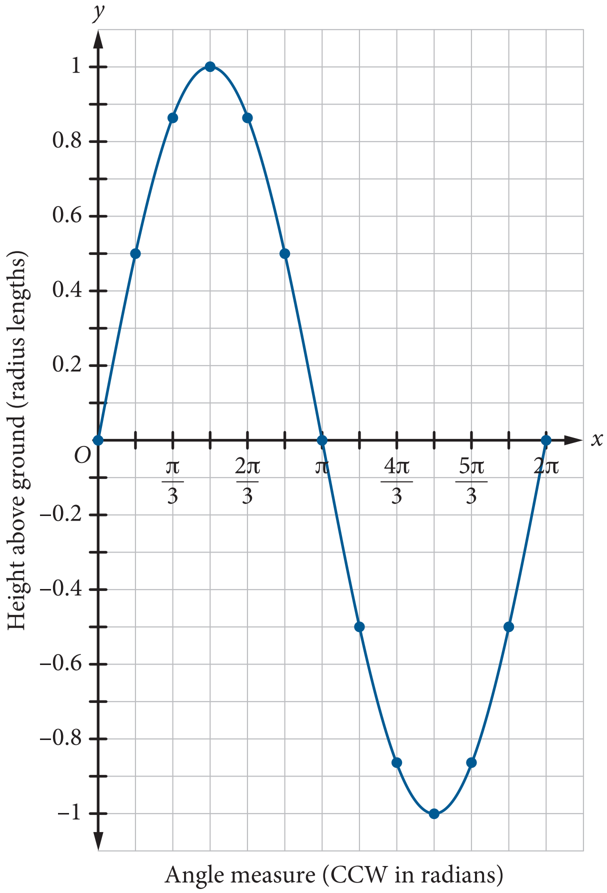


- Once students complete their nonstandard ruler, they can continue the task by measuring the height of the Ferris wheel car in radius lengths and then graph the data. All groups should produce the same table of values and the same graph. A table of values and the associated graph follow.

Angle Measure (CCW in radians)	Height Above Ground (radius lengths)
0	0
$\frac{\pi}{6}$	0.5
$\frac{\pi}{3}$	0.87
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	0.87
$\frac{5\pi}{6}$	0.5
π	0

Angle Measure (CCW in radians)	Height Above Ground (radius lengths)
$\frac{7\pi}{6}$	−0.5
$\frac{4\pi}{3}$	−0.87
$\frac{3\pi}{2}$	−1
$\frac{5\pi}{3}$	−0.87
$\frac{11\pi}{6}$	−0.5
2π	0

- As groups start to complete their tables and graphs, encourage them to compare their work with that of other groups. They should start to observe that the tables and graphs produced by all groups are the same.



Summarizing the Task

- Have one or two groups whose original circles had radii of different lengths share their tables of values and graphs with the whole class. Provide students with some time to make observations and pose questions about why the tables of values and graphs are identical despite the radii being different lengths.
- At this point, you can talk students through their ideas and observations about this new function. You can structure this discussion in any way that engages students. The key takeaways for students that need to be discussed in the class debrief include the following:
  - The function in this part of the lesson is different than the function from Part 1. Even though the general behavior of the Part 2 function is the same as the Part 1

function, h, the outputs associated with each input are different. That means they are distinct functions.

- There is something unique about the Part 2 function because it produces the same input-output pairs regardless of the length of the radius of the circle.

- Let students know that mathematicians wanted to make this important function easy to reference, so they assigned it a specific name: the *sine function*. We can use function notation to write it as $f(\theta) = \sin(\theta)$. The input of the sine function is an angle measure in radians, θ, and the output is a height measured in radius units. The function students explored in Part 1 is called a *sinusoidal function* because the output was scaled by the length of the radius. In the sine function, the maximum output value is 1 and the minimum output value is -1.
- To help students understand that the sine function relates the two quantities, angle measure and height above the horizontal axis, and does not represent the actual motion of the Ferris wheel, you could show students this animation: **preap.org/Desmos-SineTrace**.
- At this time, you could have students begin a vocabulary graphic organizer for the *sine function*. A blank organizer is available on Handout 1.1.

Instructional Rationale

Some students might want to understand how this use of sine compares to what they studied in their geometry class. This is an important question, but it should not be addressed at this time. In Lesson 4T.5: The Coordinates of Points on a Circle, students learn how the sine and cosine functions they explored through circular motion are connected to the study of right triangles.

Part 3: Defining the Cosine Function

~30 MIN.

In this part of the lesson, students explore the cosine function through the context of a modern wind turbine. Students measure the distance of each blade tip from the vertical axis when the wind turbine is in motion. As in Part 1, they construct a table of values that relates the angle measure in radians that a single blade makes with the horizontal axis to the distance of the blade tip to the vertical axis. Then they define this relationship as the cosine function.

Student Task

Wind power is a source of renewable energy. The principle of how a wind turbine works is straightforward: Wind turns a mechanism that looks like an airplane propeller. The blades of the mechanism are connected to a rotor, which is simply a machine that rotates. The rotor spins an electric generator, which creates electricity. A modem wind turbine usually has three long blades equally spaced around the rotor, as shown in the following figure:

Credit: **Rafa Irusta / Adobe Stock**

Construct a table of values that relates the angle that a single blade makes with the horizonal axis of the rotor to the distance of the blade tip to the vertical axis. Then use your table to make a graph. Your table of values and graph should be constructed so that they can apply to blades of any length.

(a) Using your Ferris wheel as a model for the wind turbine and your custom ruler, measure the distance of each blade tip to the vertical axis at the 12 angle measurements used in the table in Part 2.

(b) Plot the new data and connect the data points with a curve.

(c) Compare your graph to the graph in Part 2. How are they the same? How are they different?

Facilitating the Task

- Begin by providing small groups of 3 or 4 students with some time to closely observe the task and ask clarifying questions.
- Let students know that they will use the ideas they developed in the Ferris wheel problem to define a new sinusoidal function.
- While circulating around the room, encourage students to use their nonstandard ruler to measure the distances between the endpoint of the blade and the vertical axis in radius lengths.

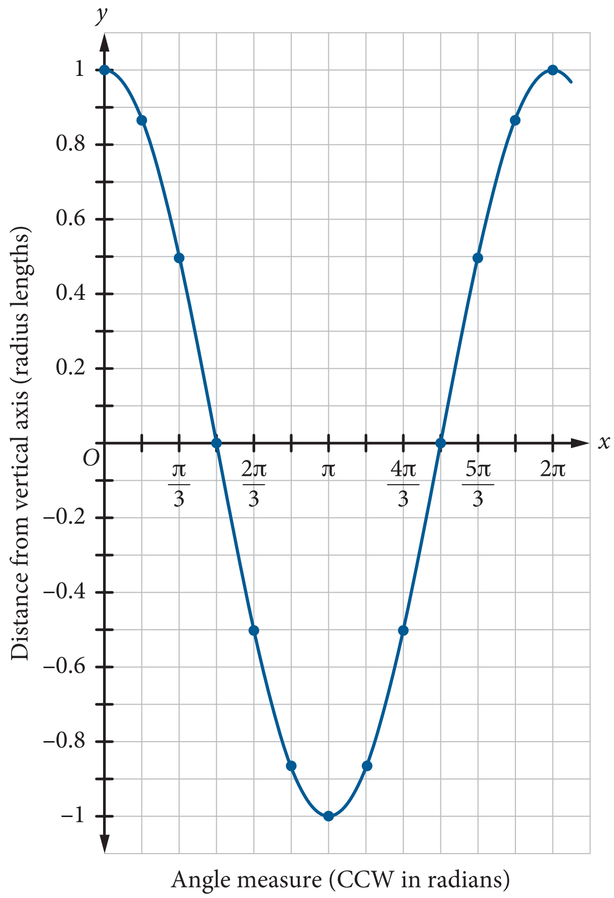
Meeting Learners' Needs

If you sense that some students are unsure why they should use radius lengths to measure the distances, you can allow them to use a ruler with standard units of measure, such as inches or centimeters, like they did in Part 1 of the lesson. If they draw a model of the wind turbine that has a radius that is not equal to 1, then they will produce a graph that looks like the graph of the cosine function, but it will be scaled by the length of the radius.

- For part (b) of the task, students should generate a table of values like the one that follows. Once students complete their tables of values, they can construct a graph of this new sinusoidal function using the rows of the table of values as coordinate pairs.
- As groups of students complete their tables and graphs, encourage them to compare their work with each other. Students should compare and contrast the graph produced in this part of the lesson to the graph in Part 2 and be prepared to articulate their observations.

Angle Measure (CCW in radians)	Distance from Vertical Axis (radius lengths)
0	1
$\frac{\pi}{6}$	0.87
$\frac{\pi}{3}$	0.5
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	–0.5
$\frac{5\pi}{6}$	–0.87
π	–1

Angle Measure (CCW in radians)	Distance from Vertical Axis (radius lengths)
$\frac{7\pi}{6}$	–0.87
$\frac{4\pi}{3}$	–0.5
$\frac{3\pi}{2}$	0
$\frac{5\pi}{3}$	0.5
$\frac{11\pi}{6}$	0.87
2π	0



Summarizing the Task

- Have one or two groups of students share their observations of the similarities and differences between their graphs of the cosine and sine functions. The important mathematical insights that students should discuss in the class debrief include the following:
    - The general behavior of the new graph is similar to the sine function graph because they are both wavelike.
    - The sine function graph intersects the y-axis at $(0, 0)$, while this new function graph intersects the y-axis at $(0, 1)$.
    - Like the graph of the sine function, the new graph produces the same input-output pairs regardless of the length of the radius of the circle used to model the wind turbine.

- Let students know that they have developed another sinusoidal function, called the *cosine function*. We can use function notation to write it as $g(\theta) = \cos(\theta)$. The input of the cosine function is an angle measure in radians, θ, and the output is the distance between the endpoint on the circle and the vertical axis. The maximum output value is 1 and the minimum output value is -1.
- To help students understand that the cosine function relates the two quantities, angle measure and the distance of the end of the fan blade from the vertical axis, and does not represent the actual motion of the fan blade, you could show students this animation: **preap.org/Desmos-CosineTrace**.
- At this time, you could have students begin a vocabulary graphic organizer for the cosine function. A blank organizer is available on Handout 1.1.
- There are practice problems involving the sine and cosine functions available on **Handout 4T.4: Practice Using Sine and Cosine**.

UNIT 4T

Assess and Reflect on the Lesson

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

Use a ruler and the graph of the sine and cosine functions to determine the approximate value of each of the following expressions:

(a) $\sin(1)$
$\sin(1) \approx 0.84$

(b) $\sin(4)$
$\sin(4) \approx -0.76$

(c) $\cos(1)$
$\cos(1) \approx 0.54$

(d) $\cos(4)$
$\cos(4) \approx -0.65$

Guiding Student Thinking

It is important to ask students to determine outputs for angle measures that do not include π to reinforce their understanding that not all angle measures must be expressed in terms of π. Also, having students determine approximate values for the sine of an angle using their ruler and the graph will demystify how tech tools determine the values for sine.

HANDOUTS

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 4T.4: Practice Using Sine and Cosine

1. (a) The car has swept out an angle of $\frac{20}{70}$ radians.

(b) The car is at a height of $\sin\left(\frac{20}{70}\right)$ meters relative to the horizontal diameter. Because $\frac{20}{70} < \pi$, the angle swept out less than a half-circle. Therefore, it is above the horizontal diameter. The height of the car in meters is approximately 0.28 meters.

(c) The car is at a distance of $\cos\left(\frac{20}{70}\right)$ meters from the vertical diameter. Because $\frac{20}{70} < \frac{\pi}{2}$, the angle swept out less than a quarter circle. Therefore, it is to the right of the vertical diameter. The car is approximately 0.96 meters from the vertical diameter.

2. (a) $\sin(2.5) > 0$ because an angle of 2.5 radians spans less than a half circle, counterclockwise. The radius is above the horizontal axis, so the value of $\sin(2.5)$ is positive.

(b) $\sin(-3) < 0$ because an angle of -3 radians spans less than a half circle, clockwise. The radius is below the horizontal axis, so the value of $\sin(-3)$ is negative.

(c) $\cos(6) > 0$ because an angle of 6 radians spans just less than a full circle, counterclockwise. The radius is to the right of the horizontal axis, so the value of $\cos(6)$ is positive.

(d) $\cos(-4) < 0$ because an angle of -4 radians spans more than a half circle, clockwise. The radius is to the left of the vertical axis, so the value of $\cos(-4)$ is negative.

3. The angle swept out to form an arc length of 10, with a radius of 5, has a measure of 2 radians. This means that $u = \cos(2)$ and $v = \sin(2)$.

4. (a) $\sin(-a) = -\sin(a)$ because an angle that sweeps out in a clockwise direction is on the opposite side of the horizontal axis from, and therefore has the opposite sign of, an angle that sweeps out the same magnitude in a counterclockwise direction.

(b) $\cos(-a) = \cos(a)$ because an angle that sweeps out in a clockwise direction is on the same side of, and has the same distance from, the vertical axis as an angle that sweeps out the same magnitude in a counterclockwise direction.

(c) $\sin(2\pi + a) = \sin(a)$ because two angles whose measures differ by 2π are coterminal and the angles they sweep out relative to the horizontal axis are equal.

(d) $\cos(\pi + a) = -\cos(a)$ because two angle measures that differ by π have nonorigin endpoints of their radii that are endpoints of a diameter. Therefore, they will be on opposite sides of, but have the same distance from, the vertical axis.

5. (a) The angle that Car 1 makes with the horizontal axis measures $\frac{\pi}{6}$ radians. Therefore, the height of the car is $4 + 20 + 20\sin\left(\frac{\pi}{6}\right) \approx 34$ feet.

(b) The angle that Car 2 makes with the horizontal axis measures $\frac{\pi}{3}$ radians. Therefore, the height of the car is $4 + 20 + 20\sin\left(\frac{\pi}{3}\right) \approx 41.4$ feet.

(c) The angle that Car 8 makes with the horizontal axis measures $\frac{8\pi}{6} = \frac{4\pi}{3}$ radians. Therefore, the height of the car is $4 + 20 + 20\sin\left(\frac{4\pi}{3}\right) \approx 6.6$ feet.

(d) The angle that Car 1 makes with the vertical axis measures $\frac{\pi}{6}$ radians. Therefore, the distance of the car from the vertical axis is $20\cos\left(\frac{\pi}{6}\right) \approx 17.4$ feet.

(e) The angle that Car 2 makes with the vertical axis of the Ferris wheel measures $\frac{\pi}{3}$ radians. The angle that Car 4 makes with the vertical axis of the Ferris wheel measures $\frac{2\pi}{3}$ radians. Therefore, Car 2 is $20\cos\left(\frac{\pi}{3}\right) = 10$ feet from the vertical axis and Car 4 is $20\cos\left(\frac{2\pi}{3}\right) = -10$ feet from the vertical axis. That means the cars are 20 feet apart.

UNIT 4T

LESSON 4T.5

The Coordinates of Points on a Circle

LEARNING OBJECTIVE

4T.1.4 Determine the exact coordinates of any point on a circle centered at the origin.

LESSON OVERVIEW

CONTENT FOCUS

Trigonometry is the branch of mathematics that links two fundamental geometric objects: circles and triangles. This lesson connects students' understanding of the values of the sine and cosine ratios of right triangle trigonometry to values of the sine and cosine functions. Students use reference triangles to show that the terminal ray of an angle in standard position intersects a circle centered at the origin at the point $(r\cos(\theta), r\sin(\theta))$, where θ is the measure of an angle in standard position and r is the length of the radius of the circle.

LESSON DESCRIPTION

Part 1: Determining the Coordinates of Points on the Unit Circle

In this part of the lesson, students develop the relationship between the coordinates on a unit circle and the outputs of the sine and cosine functions. Students represent the signed vertical distance of a point from the horizontal diameter of a unit circle as $\sin(\theta)$ and represent the signed horizontal distance of a point from the vertical diameter of a unit circle as $\cos(\theta)$. These signed distances give the coordinates of a point on the unit circle where $(x, y) = (\cos(\theta), \sin(\theta))$.

AREAS OF FOCUS

- Connections Among Multiple Representations
- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~60 minutes

LESSON SEQUENCE

- This lesson is part of a lesson sequence (~600 minutes total) that includes Lessons 4T.1 through 4T.7.

MATERIALS

- patty paper or blank sheet of paper
- straightedge
- compass
- wax craft stick

HANDOUTS

Lesson

- 4T.5.A: Determining the Coordinates of Points on the Unit Circle

Practice

- 4T.5.B: Exploring Reference Triangles

Part 2: Constructing Congruent Reference Triangles
In this part of the lesson, students use reference triangles to explore the relationship between the values of sine and cosine for two angle measures whose terminal rays are reflections of each other across the *y*- or *x*-axis.

FORMATIVE ASSESSMENT GOAL

This lesson prepares students to complete the following formative assessment activity.

Suppose a circle centered at the origin has a radius of 5 units. Determine the coordinates of the point on the circle where the terminal ray of each of the following angle measures intersects the circle.

(a) $\frac{\pi}{2}$ radians

(b) 4 radians

(c) $-\frac{11\pi}{6}$ radians

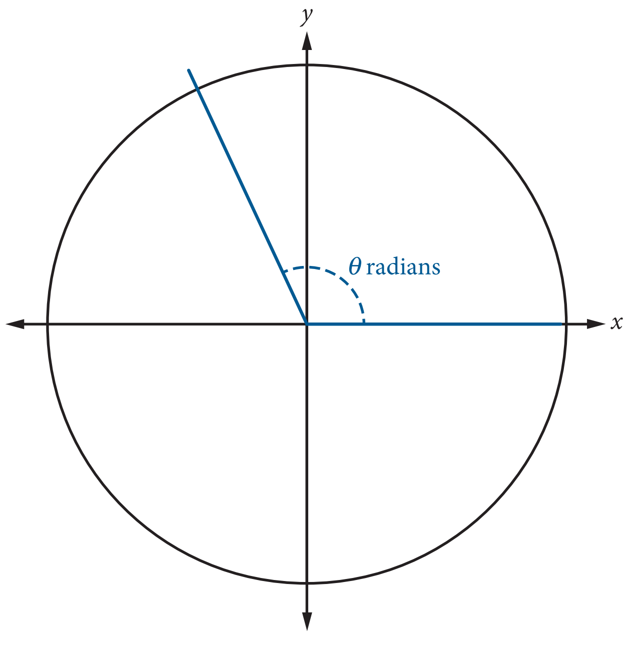
Part 1: Determining the Coordinates of Points on the Unit Circle

~30 MIN.

In this part of the lesson, students develop the relationship between the coordinates on a unit circle and the outputs of the sine and cosine functions. Students represent the signed vertical distance of a point from the horizontal diameter of a unit circle as $\sin(\theta)$ and represent the signed horizontal distance of a point from the vertical diameter of a unit circle as $\cos(\theta)$. These signed distances give the coordinates of a point on the unit circle where $(x, y) = (\cos(\theta), \sin(\theta))$.

Warm-Up

- You may find it helpful to start the lesson with some brief questions to activate students' prior knowledge about measuring the length of an arc with a nonstandard unit like the radius length. Have students draw a diagram similar to the one shown and have them answer the questions that follow. It is not necessary for all diagrams to be identical because this warm-up is less about arriving at a correct answer and more about understanding how to measure with a nonstandard unit.



- Estimate the measure of the angle in radians.

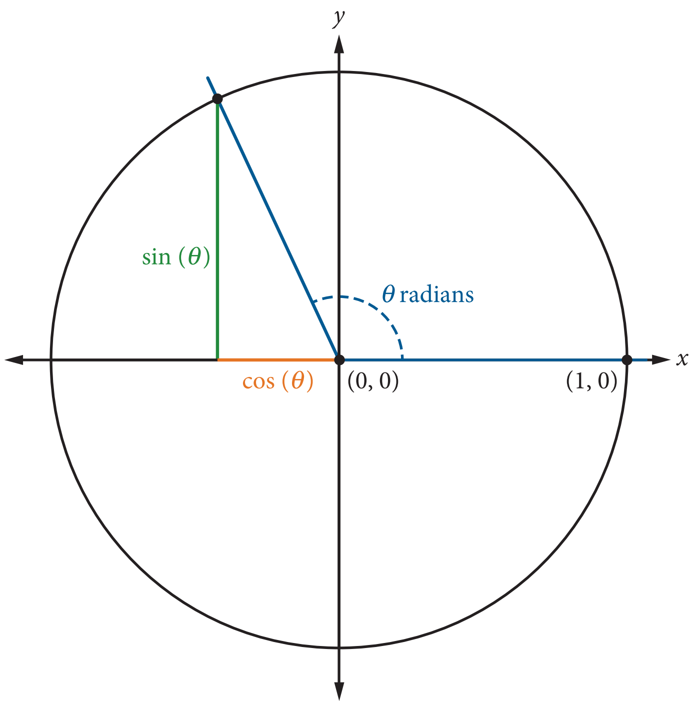
  This angle has a measure of about 2 radians because it takes two copies of the radius to span the arc cut off by the angle's rays.

Meeting Learners' Needs

You may find it helpful to provide students with wax craft sticks to support them with measuring the arc length using the radius to estimate the angle measure.

- Draw line segments that represent the values of $\sin(\theta)$ and $\cos(\theta)$. Explain your answer.

  The segments are shown in the following figure. The length of the vertical segment is equal to $\sin(\theta)$ because the sine of the angle is the signed vertical distance that a point is from the horizontal axis measured in radius units. The length of the horizontal segment is equal to $\cos(\theta)$ because the cosine of the angle is the signed horizontal distance that a point is from the vertical axis.



- Estimate the value of $\sin(\theta)$. Explain how you determined your answer.

  The value of $\sin(\theta)$ is between 0.5 and 1. The maximum value of $\sin(\theta)$ is 1 at $\theta = \frac{\pi}{2}$, so $\sin(\theta)$ must be less than 1. Since the height seems to be more than $\frac{1}{2}$ of the radius of the circle, then $0.5 < \sin(\theta) < 1$.

- Estimate the value of $\cos(\theta)$. Explain how you determined your answer.

  The value of $\cos(\theta)$ is between 0 and -0.5. The minimum value of $\cos(\theta)$ is -1 at $\theta = \pi$, which means that the indicated value must be greater than -1. Since the distance seems to be less than $\frac{1}{2}$ of the radius of the circle, then $-0.5 < \cos(\theta) < 0$.

Student Task

The student task is provided on **Handout 4T.5.A: Determining the Coordinates of Points on the Unit Circle**. In this task, a unit circle is used to model an amusement park ride that swings the riders over another type of ride called the Lazy River. Students use their model to determine specific coordinates at which the riders are directly above the Lazy River. For this part of the lesson, students should answer only problems 1 and 2 on the handout. Solutions for the handout are provided in the Assess and Reflect section of the lesson.

Instructional Rationale

For the scenario on Handout 4T.5.A, using the uncommon angle measure of 0.5 radians allows students to focus on how to represent the values of $\cos(\theta)$ and $\sin(\theta)$ in the coordinate plane rather than on determining the exact values for each cosine and sine expression. In Part 2 of this lesson, students explore reference triangles associated with commonly used angle measures.

Facilitating the Task

- Provide each student with a copy of Handout 4T.5.A and have them read the introductory paragraph. Allow students to ask clarifying questions and share things they notice about the situation.
- Have students work with a partner to complete Part 1. This problem orients students to the model of the swing ride and the lazy river it crosses.

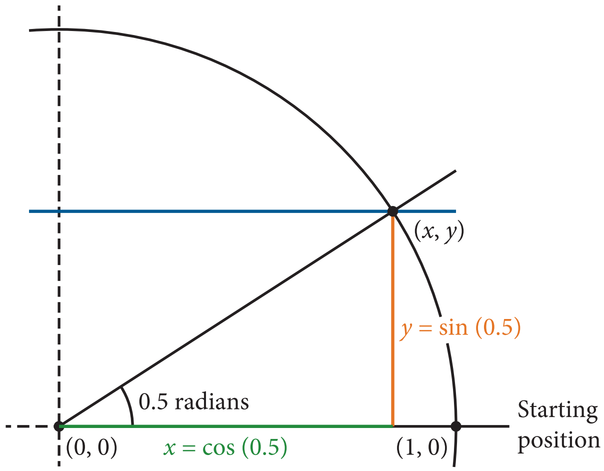
Classroom Ideas

If students are unfamiliar with the amusement park rides presented in this scenario, consider showing them online videos of a swing ride and a lazy river.

- While you circulate around the room, be sure to support students in making sense of how to measure distances in this circle. Just as students used a radius ruler of length "one radius" to measure distances in Lesson 4T.4, they should consider the radius of the swing ride to have a length of "1." This understanding helps students answer problem 1(b).
- In problem 1(c), students are expected to explain what negative x- and y-coordinates mean for the scenario. Students may describe negative x-coordinates as referring to points to the left of the vertical axis and negative y-coordinates as referring to points below the horizontal axis. These descriptions are correct but do not fully answer the question. Encourage students to think in terms of geographical

directions to describe what the negative coordinates mean about the location of the swing along its circular path.

- Once you are confident that students understand Part 1 of the task, they can move on to Part 2. As students start to answer problem 2, you may notice that they draw a right triangle on the model of the ride to determine the coordinates where the swing is directly over the lazy river. You may need to help them see that the (x, y) coordinates of a point on the circular path indicate a distance east (to the right) of the center of the ride and a distance north of (above) the center of the ride, respectively. A sample figure is shown:



- Students should leverage their understanding from Lesson 4T.4 that $\cos(\theta)$ gives the horizontal distance from the center of the circle and $\sin(\theta)$ gives the vertical distance from the center of the circle. Have students label the legs of their triangles with $x = \cos(0.5)$ and $y = \sin(0.5)$.
- Some students might remember the trigonometric ratios from their geometry class and wonder if their work in this lesson is related to that. Allow students some time to explore whether or not using the right triangle approach yields the same results. Students should find that for the sine ratio, $\sin(\theta) = \dfrac{\text{length of opposite leg}}{\text{length of hypotenuse}}$, the length of the leg opposite θ is y and the length of the hypotenuse is 1. Likewise for the cosine ratio, $\cos(\theta) = \dfrac{\text{length of adjacent leg}}{\text{length of hypotenuse}}$, the length of the leg adjacent to θ is x and the length of the hypotenuse is 1. When the lengths of the sides of the triangle are measured in radius lengths, the trigonometric ratio yields the same

solution as applying the understanding that $\cos(\theta)$ is the signed horizontal distance of a point from the origin and $\sin(\theta)$ is the signed vertical distance of a point from the origin.

- Have students use a calculator to determine approximate values for $\cos(0.5)$ and $\sin(0.5)$. Make sure students' calculators are in radian mode rather than degree mode. You may need to support students in understanding that the exact coordinate of the point where the swing ride is directly over the lazy river is $(\cos(0.5), \sin(0.5))$, but for practical purposes, the approximate coordinates $(0.876, 0.479)$ are often preferred.
- Have each pair of students compare their solution method with that of another pair of students. Finally, let each group of four students share their solution and method with the class. Be sure to emphasize that there are two equivalent methods that they can use: the circle method and the triangle method. In the circle method, they can use the coordinates of a circle with a radius of 1 unit $(\cos(0.5), \sin(0.5))$ and scale up to a circle whose radius is 30 feet by multiplying each coordinate by 30. This can be expressed exactly as $(30\cos(0.5), 30\sin(0.5))$. In the triangle method, students can set up the ratios $\sin(0.5) = \frac{\text{length of opposite leg}}{30}$ and $\cos(0.5) = \frac{\text{length of adjacent leg}}{30}$ and solve for the unknown lengths, which would yield length of opposite leg $= 30\sin(0.5)$ and length of adjacent leg $= 30\cos(0.5)$.

Guiding Student Thinking

Some students may be uncomfortable with leaving the location of the camera in the form $(30\cos(0.5), 30\sin(0.5))$. You can allow them to get a decimal approximation of the coordinate, $(26.327, 14.328)$, but be sure to emphasize that this is only an approximation and not the exact location of the point on the circle. For practical purposes, the decimal approximation is both useful and necessary. In a purely mathematical problem, the exact value would be the preferred way to report the solution.

Summarizing the Task

- Through this part of the lesson, students should come to understand that the coordinates of a point on a circle with a radius of 1 unit can be determined with the formula $(x, y) = (\cos(\theta), \sin(\theta))$, and that the coordinates of a point on a circle with a radius of r can be determined with the formula $(x, y) = (r\cos(\theta), r\sin(\theta))$.
- Explain to students that mathematicians often use one radius length to construct and measure distances in a circle. Such a circle is called a *unit circle* because its

radius is 1 unit. You can use this as an opportunity to add the term *unit circle* to your classroom's word wall if you have one.

- At this point in the lesson, you can ask students some questions in which they practice determining the coordinates of points on the unit circle. Some example questions follow:
  - Suppose the swing ride stops turning after it rotates 4.1 radians. What are the coordinates of the point at which the rider is located when the ride stops turning, in terms of radius lengths? How do you know?

    The coordinates of the location of the rider are $(\cos(4.1), \sin(4.1))$. In terms of the radius length, the horizontal distance of the rider from the vertical diameter is $\cos(4.1)$ and the vertical distance of the rider from the horizontal diameter is $\sin(4.1)$.

  - Suppose the ride stops turning after it rotates θ radians. What are the coordinates of the point at which the rider is located when the ride stops turning? How do you know?

    The coordinates of the location of the rider are $(\cos(\theta), \sin(\theta))$. In terms of the radius length, the horizontal distance of the rider from the vertical diameter is $\cos(\alpha)$ and the vertical distance of the rider from the horizontal diameter is $\sin(\alpha)$.

  - Another amusement park has a swing ride with a radius of 45 feet. The ride stops after the rider has rotated 2.5 radians from the starting position. What are the coordinates of this point measured in radius lengths and in feet?

    The coordinates of the location of the rider in terms of radius lengths are $(\cos(2.5), \sin(2.5))$. The coordinates of the location of the rider in feet are $(45\cos(2.5), 45\sin(2.5))$.

  - Another amusement park has a swing ride with a radius of r feet. The ride stops after the rider has rotated θ radians from the starting position. What are the coordinates of this point measured in radius lengths and in feet?

    The coordinates of the location of the rider in terms of radius lengths are $(\cos(\theta), \sin(\theta))$. The coordinates of the location of the rider in feet are $(r\cos(\theta), r\sin(\theta))$.

- Before moving on to the next part of the lesson, students should be able to articulate that the coordinates of a point on a unit circle can be determined with the formula $(x, y) = (\cos(\theta), \sin(\theta))$, and that the coordinates of a point on a circle with a radius of r can be determined with the formula $(x, y) = (r\cos(\theta), r\sin(\theta))$.

UNIT 4T

Part 2: Constructing Congruent Reference Triangles

~30 MIN.

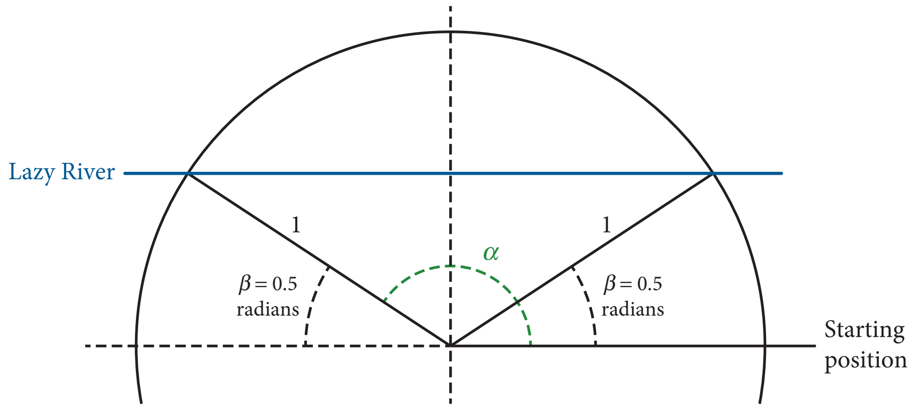
In this part of the lesson, students use reference triangles to explore the relationship between the values of sine and cosine for two angle measures whose terminal rays are reflections of each other across the y- or x-axis.

Student Task

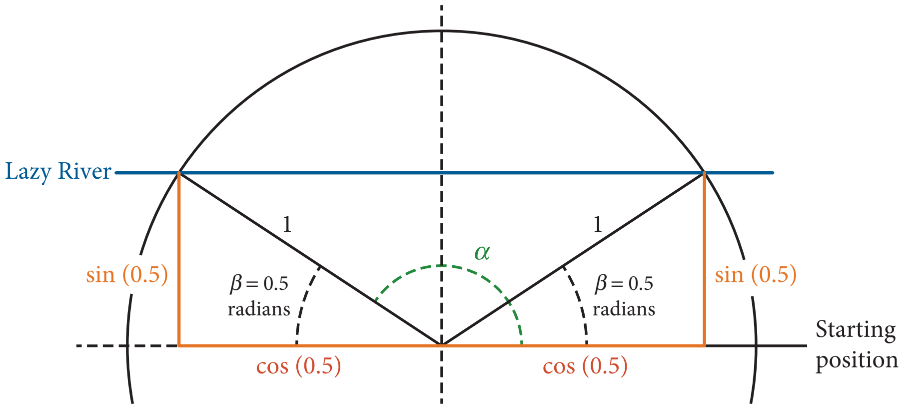
The student task for this part of the lesson is problem 3 on Handout 4T.5.A. Solutions for the handout are provided in the Assess and Reflect section of the lesson.

Facilitating the Task

- Have students read problem 3 on Handout 4T.5.A and discuss with their partner how they would construct an angle with a measure of radians that represents the next point at which the rider would be directly over the lazy river. Students might be eager to determine the measure of this angle, but for right now, help students focus on precisely constructing the angle.
- To support students in constructing the angle, have each student pair write out how the new angle is similar to and different from the original angle. You can have each student pair share their similarities and differences with another pair, and then have several of these small groups of four share their ideas with the whole class.
- As students share their ideas, be sure to bring attention to the idea that the terminal ray of the angle that represents the second location where the rider passes directly over the lazy river is a reflection across the y-axis of the original angle's terminal ray. Encourage students to label the measure of the reflected angle from the initial ray as α on their diagram, and the measure of the terminal ray of the reflected angle to the negative x-axis as β. Their figure should look like the following:



- Once students have constructed a correct diagram, give them time to work with their partner to determine the value of α. Encourage them to use what they remember from their geometry course to help them identify the relationship between angle α and angle β: Because they share a ray, and their nonshared rays form a straight angle, angle α and angle β are supplementary.
- You may need to help students reason through the measures of the supplementary angles when these measures are given in radians. Since supplementary angles sweep out half a circle, the sum of their measures is π radians.
- Encourage students to construct two triangles to help them determine the relationship between $\sin(0.5)$ and $\sin(\alpha)$, and between $\cos(0.5)$ and $\cos(\alpha)$. Give students time to discuss with their partner what they notice and what they wonder about these two triangles.
- You can let students know that the right triangle formed by the terminal ray, the vertical line segment with length $\sin(0.5)$, and the horizontal line segment with length $\cos(0.5)$ on the left side of the graph is a *reference triangle* and that 0.5 radians is called the *reference angle* for these reference triangles.



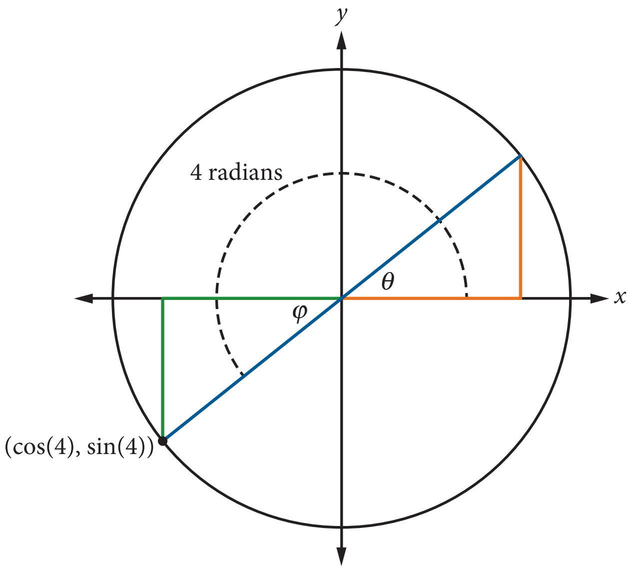
- Finally, have students determine the coordinates of the second point at which the rider passes over the lazy river. They should reason that the coordinates of this point in terms of radius lengths are $(\cos(\alpha), \sin(\alpha)) = (\cos(\pi - 0.5), \sin\pi(\pi - 0.5))$ and in terms of feet are $(30\cos(\alpha), 30\sin(\alpha)) = (30\cos(\pi - 0.5), 30\sin\pi(\pi - 0.5))$.

Summarizing the Task

- In this task, students developed the concept of a reference triangle, which is a right triangle with a horizontal leg that lies on the x-axis, a vertical leg that is perpendicular to the x-axis, and a hypotenuse formed by a terminal ray. This right triangle refers to a congruent right triangle in the first quadrant. Reference triangles are useful for solving problems that involve angles in quadrants II, III, and IV because a congruent right triangle in the first quadrant can be used to determine the relevant lengths, and the quadrant where the terminal ray lies can be used to determine the sign of the values.
- Engage the class in a conversation about how they can use reference triangles to explain the relationship between the values $\sin(0.5)$ and $\sin(\alpha)$, and between $\cos(0.5)$ and $\cos(\alpha)$.
    - How is the value of $\sin(\alpha)$ related to the value of $\sin(0.5)$? Explain how you know.

      The values of $\sin(\alpha)$ and $\sin(0.5)$ are equivalent. The first quadrant right triangle was formed using an angle of 0.5 radians with the positive x-axis and the second quadrant right triangle was formed using an angle of 0.5 radians with the negative x-axis. Furthermore, the radii (the hypotenuses of the right triangles) are congruent, which means the two right triangles are congruent and the vertical distance from the x-axis must be equal.
    - How is the value of $\cos(\alpha)$ related to the value of $\cos(0.5)$? Explain how you know.

      The values of $\cos(\alpha)$ and $\cos(0.5)$ have opposite signs. The first quadrant right triangle was formed using an angle of 0.5 radians with the positive x-axis and the second quadrant right triangle was formed using an angle of 0.5 radians with the negative x-axis. Furthermore, the radii are congruent, which means the two right triangles are congruent. This means that the horizontal legs have the same length, but the second quadrant right triangle extends to the left. We indicate this by making the value of the $\cos(\alpha)$ negative.
- To make sure that students can construct a reference triangle for any given angle, have them sketch an angle that measures 4 radians. Then ask them to construct a reference triangle in the first quadrant that they could use to determine the sign of the outputs of the sine and cosine functions for this angle. An example diagram follows.



- Ask students some questions like the ones that follow to make sure they understand the relationship between the angles in the first and third quadrants and the values of the sines and cosines of these angles.
  - Look at the angle formed by the negative x-axis and the terminal ray of the angle measuring 4 radians. Let's call this angle φ. What is the relationship between the measure of angle φ and 4 radians? Explain how you know.

    The measure of angle φ is $4 - \pi$ radians. This is because angle φ is the difference between an angle that sweeps out 4 radians and an angle that sweeps out half a circle.

  - How is the value of $\sin(\varphi)$ related to the value of $\sin(4)$? Explain how you know.

    The values of $\sin(\varphi)$ and $\sin(4)$ are equivalent. This is because the hypotenuse of the triangle in the third quadrant coincides with the terminal ray of the angle that sweeps out 4 radians.

  - How is the value of $\cos(\varphi)$ related to the value of $\cos(4)$? Explain how you know.

    The values of $\cos(\varphi)$ and $\cos(4)$ are equivalent. This is because the hypotenuse of the reference triangle in the third quadrant coincides with the terminal ray of the angle that sweeps out 4 radians.

- Now look at the reference triangle in the first quadrant. Let's call the angle at the origin θ. What is the relationship between the measure of angle θ and 4 radians? Explain how you know.

  The measure of angle θ is equivalent to the measure of φ because they are vertical angles. This means the measure of θ is $4-\pi$ radians.

- How is the value of $\sin(\theta)$ related to the value of $\sin(4)$? Explain how you know.

  The values of $\sin(\theta)$ and $\sin(4)$ have the same absolute value but opposite signs. This is because the hypotenuse of the reference triangle in the first quadrant is above the x-axis and the hypotenuse of the reference triangle in the third quadrant is below the x-axis.

- How is the value of $\cos(\theta)$ related to the value of $\cos(4)$? Explain how you know.

  The values of $\cos(\theta)$ and $\cos(4)$ have the same absolute value but opposite signs. This is because the hypotenuse of the reference triangle in the first quadrant is to the right of the y-axis and the hypotenuse of the reference triangle in the third quadrant is to the left of the y-axis.

- This is a good time for students to practice what they have learned about using the lengths of the sides of reference triangles to determine the coordinates of a point on a circle centered at the origin. There are several practice problems included on **Handout 4T.5.B: Exploring Reference Triangles**.

UNIT 4T

Assess and Reflect on the Lesson

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

Suppose a circle centered at the origin has a radius of 5 units. Determine the coordinates of the point on the circle where the terminal ray of each of the following angle measures intersects the circle.

(a) $\frac{\pi}{2}$ radians

$$\left(5\cos\left(\frac{\pi}{2}\right),\ 5\sin\left(\frac{\pi}{2}\right)\right) = (0,\ 5)$$

(b) 4 radians

$$(5\cos(4),\ 5\sin(4)) \approx (-3.268,\ -3.784)$$

(c) $-\frac{11\pi}{6}$ radians

$$\left(5\cos\left(-\frac{11\pi}{6}\right), 5\sin\left(-\frac{11\pi}{6}\right)\right) \approx (4.33, 2.5)$$

HANDOUTS

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 4T.5.A: Determining the Coordinates of Points on the Unit Circle

PART 1

1. (a) The coordinates of the starting position are $(1, 0)$.

   (b) The coordinates of the point on the ride that is farthest north are $(0, 1)$. The coordinates of the point on the ride that is farthest west are $(-1, 0)$. The coordinates of the point on the ride that is farthest south are $(0, -1)$.

   (c) If the x-coordinate is negative, the swing is between one-quarter and three-quarters of a revolution of the ride. If the y-coordinate is negative, the swing is between one-half and one revolution of the ride.

   (d) The exact coordinates of the location of the water jet are $(\cos(0.5),\ \sin(0.5))$. The approximate coordinates of the location of the camera are $(0.878,\ 0.479)$.

PART 2

2. If the ride has a radius of 30 feet, then the camera should be located a distance of $30\cos(0.5) \approx 26.327$ feet east of the center of the ride and $30\sin(0.5) \approx 14.383$ feet north of the center of the ride.

3. The coordinates of the second location at which the rider is above the Lazy River are $(30\cos(\pi - 0.5),\ 30\sin(\pi - 0.5))$, which is approximately $(-26.327, 14.383)$.

Handout 4T.5.B: Exploring Reference Triangles

1. (a) If point A had coordinates $\left(\frac{12}{13}, \frac{5}{13}\right)$ then $\sin(\beta) = \frac{5}{13}$ and $\cos(\beta) = \frac{12}{13}$.

   (b) If point A had coordinates $\left(-\frac{4}{5}, -\frac{3}{5}\right)$ then $\sin(\beta) = -\frac{3}{5}$ and $\cos(\beta) = -\frac{4}{5}$.

   (c) If point A had coordinates $\left(-\frac{8}{17}, \frac{15}{17}\right)$ then $\sin(\beta) = \frac{15}{17}$ and $\cos(\beta) = -\frac{8}{17}$.

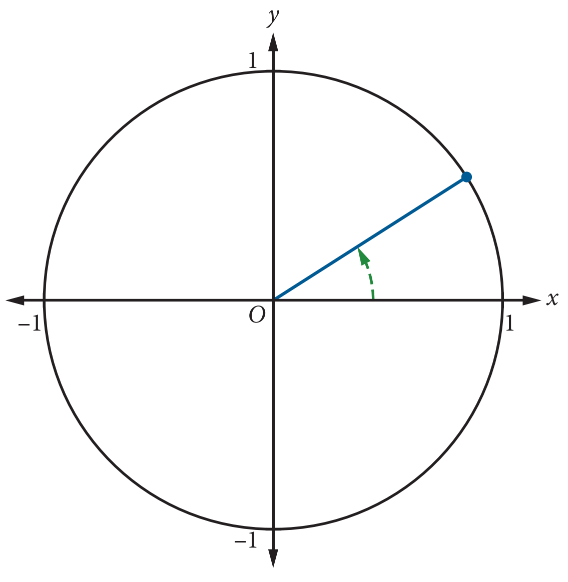
2. (a) If a circle has a radius of 2 and the measure of α is $\frac{\pi}{4}$, then $(x, y) = \left(2\cos\left(\frac{\pi}{4}\right),\ 2\sin\left(\frac{\pi}{4}\right)\right)$.

   (b) If a circle has a radius of 3 and the measure of α is $-\frac{7\pi}{10}$, then $(x, y) = \left(3\cos\left(-\frac{7\pi}{10}\right),\ 3\sin\left(-\frac{7\pi}{10}\right)\right)$.

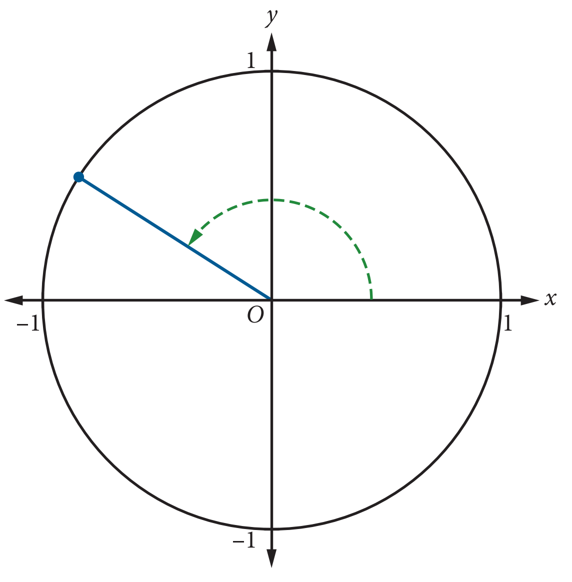
   (c) If a circle has a radius of 1 and the measure of α is 15 radians, then $(x, y) = (\cos(15),\ \sin(15))$.

3. Four different angle measures that have a reference angle of $\frac{\pi}{9}$ radians are $\frac{8\pi}{9}$, $\frac{10\pi}{9}$, $\frac{17\pi}{9}$, and $\frac{19\pi}{9}$. The angle that the terminal ray of each of these angles forms with the positive or negative x-axis measures $\frac{\pi}{9}$ radians.

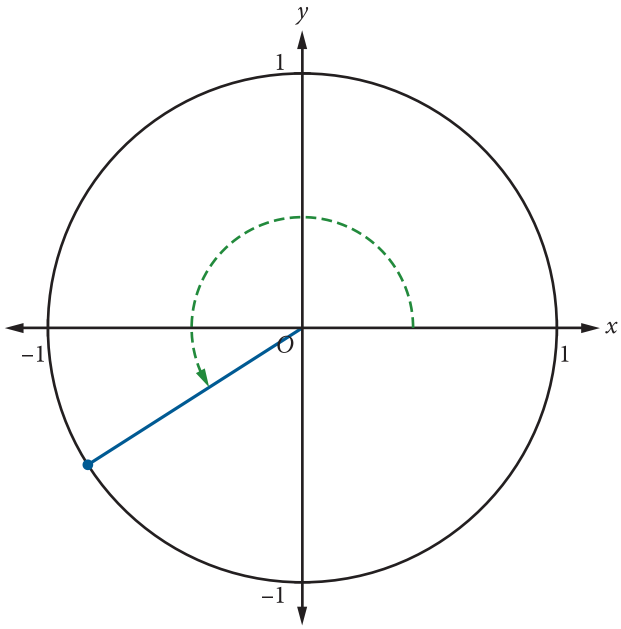
$\theta = \frac{\pi}{9}$:



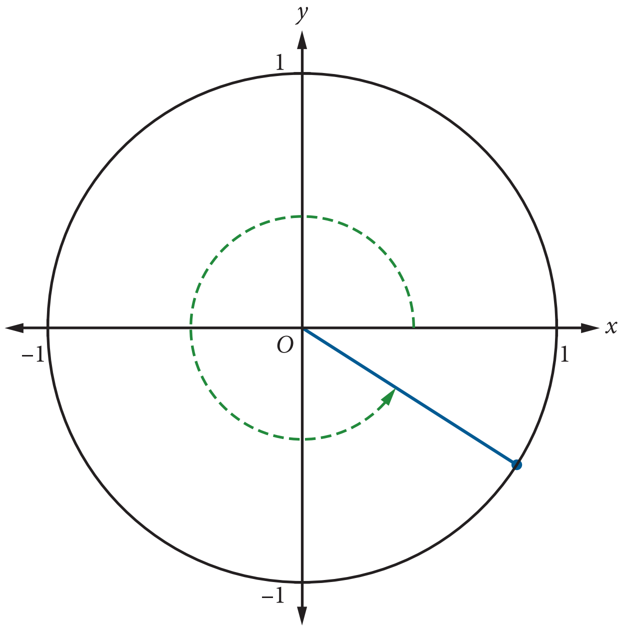
$\theta = \frac{8\pi}{9}$:



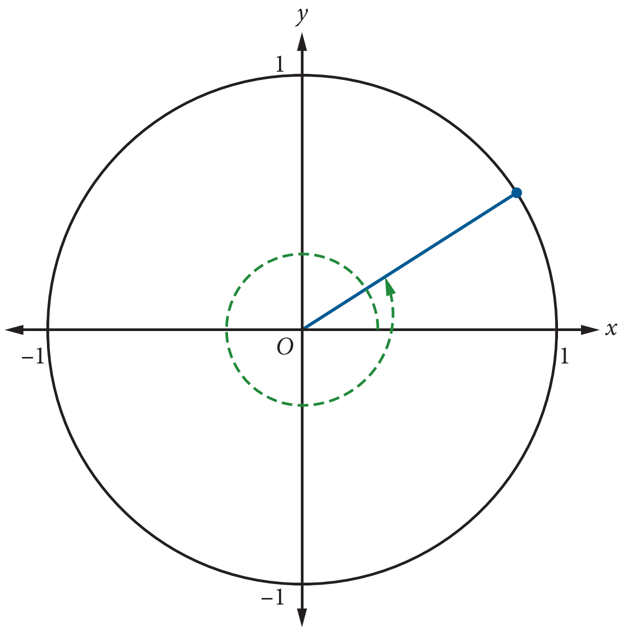
$\theta = \frac{10\pi}{9}$:



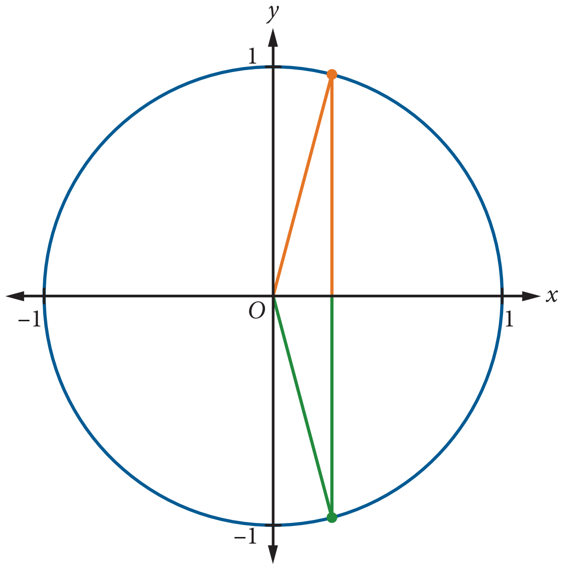
$\theta = \frac{17\pi}{9}$:



$\theta = \frac{19\pi}{9}$:

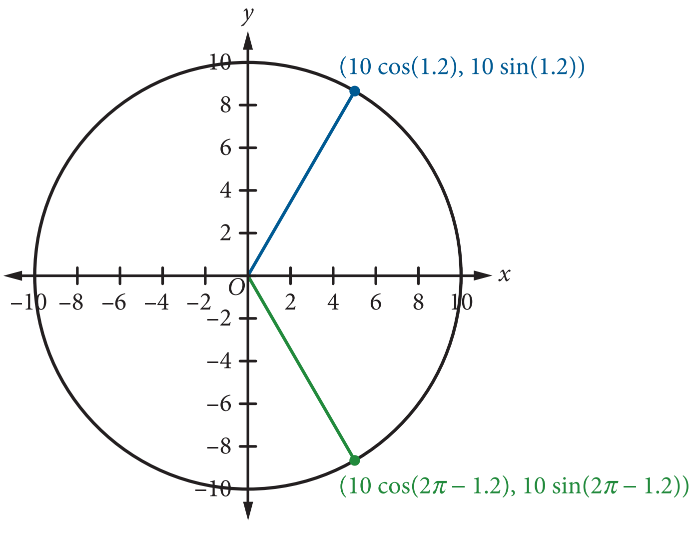


4. The terminal ray of an angle in standard position that measures $\frac{8\pi}{5}$ would lie in quadrant IV. Its reference angle would be $\frac{2\pi}{5}$. Because the angle is in the fourth quadrant, its reference angle is determined by the angle measure that the terminal ray makes with the positive x-axis. This angle can be reflected across the x-axis to create a congruent reference triangle as in the following figure:



5. (a) The coordinates of the point on a circle with a radius of 10 units that corresponds to an angle of 1.2 radians are $(x, y) = (10\cos(1.2),\ 10\sin(1.2))$.

   (b) The other point on the circle that has the same x-coordinate as the point in problem 5(a) is located in quadrant IV. That means that the angle measure of this point is $(2\pi - 1.2)$ radians. The y-coordinate of that point is $y = 10\sin(2\pi - 1.2)$. A diagram of these points on the circle follows:



LESSON 4T.6

Common Reference Triangles in a Unit Circle

LEARNING OBJECTIVES

4T.1.4 Determine the exact coordinates of any point on a circle centered at the origin.

4T.1.7 Solve problems involving trigonometric identities.

LESSON OVERVIEW

CONTENT FOCUS

This lesson continues the exploration of unit circles that students began in Lesson 4T.5. Students use their understanding of the relationship among the lengths of the sides of right triangles to develop the Pythagorean identity for trigonometry. Then they determine the exact values of the sine and cosine of commonly used angles: 30°, 45°, 60°, and 90°, or equivalently $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$. These values are used to establish the exact coordinates of points on a unit circle whose associated reference angles are congruent to one of the commonly used angles.

LESSON DESCRIPTION

Part 1: Exploring the Pythagorean Theorem
In this part of the lesson, students use their prior knowledge of the relationship among the lengths of the sides of a right triangle to derive the Pythagorean identity, $\sin^2(\theta)+\cos^2(\theta)=1$.

AREAS OF FOCUS

- Connections Among Multiple Representations
- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~75 minutes

LESSON SEQUENCE

- This lesson is part of a lesson sequence (~600 minutes total) that includes Lessons 4T.1 through 4T.7.

MATERIALS

- calculator
- straightedge (optional)
- compass (optional)

HANDOUTS

Lesson

- 4T.6.A: Investigating Right Triangles in a Unit Circle – Group 1
- 4T.6.B: Investigating Right Triangles in a Unit Circle – Group 2
- 4T.6.C: Investigating Right Triangles in a Unit Circle – Group 3
- 4T.6.D: Common Angles in a Unit Circle

Part 2: Investigating Right Triangles in a Unit Circle

In this part of the lesson, students use their knowledge of equilateral, isosceles, and right triangles to determine the exact values of the sine and cosine of angles that commonly appear in trigonometry: 30°, 45°, 60°, and 90°, or equivalently $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$. This culminates in students determining the exact coordinates of the points on a unit circle whose reference angles include one of these commonly used angles.

FORMATIVE ASSESSMENT GOAL

This lesson prepares students to complete the following formative assessment activity.

A circle centered at the origin has a radius of 1.

(a) At what points on the circle does the y-coordinate have an absolute value that is exactly half the length of the radius? For each point, list both of its coordinates.

(b) At what points on the circle does the x-coordinate have an absolute value that is exactly half the length of the radius? For each point, list both of its coordinates.

(c) At what points on the circle is the x-coordinate equal to the y-coordinate? For each point, list both of its coordinates.

Part 1: Exploring the Pythagorean Theorem

~30 MIN.

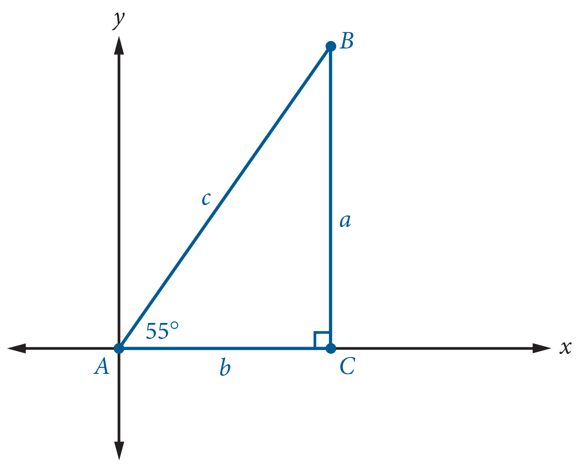
In this part of the lesson, students use their prior knowledge of the relationship among the lengths of the sides of a right triangle to derive the Pythagorean identity, $\sin^2(\theta) + \cos^2(\theta) = 1$.

Instructional Rationale

It is not necessary for students to rationalize the denominators of the exact numerical values they determine in this part of the lesson. Because many textbooks and assessments use the rationalized expression, it may be helpful to show students how to rationalize denominators of fractions that include square roots. It is essential for students to understand that rationalizing the denominator is the same process as writing an equivalent fraction, so both forms of the number express the same value. That is, there is no reason why $\frac{\sqrt{2}}{2}$ is necessarily a better answer than $\frac{1}{\sqrt{2}}$, because they are equivalent.

Student Task

Triangle ABC is a right triangle with legs of length a and b, and hypotenuse of length c.



1. What is the relationship between side lengths a, b, and c?
2. Suppose point A is at the origin and is the center of a circle whose radius is 1. What is the exact value of a? What is the exact value of b? What is the exact value of c?
3. Rewrite the relationship from problem 1 in terms of your answers from problem 2. What do you notice?

Facilitating the Task

- First, display the image in a central location where all students can access it. Then have students work in pairs to answer the three questions.
- As you circulate around the room, be sure that students have correctly identified that the Pythagorean theorem expresses the relationship among the lengths of the sides of the right triangle. That is, $a^2 + b^2 = c^2$.
- If students seem to struggle answering problem 2, this is an excellent opportunity to provide some just-in-time review of the relationship between the sine of an angle and the signed vertical distance of the associated point on a unit circle from the horizontal axis and the relationship between the cosine of an angle and the signed horizontal distance of the associated point on a unit circle from the vertical axis. Some questions you can use follow:
  - How can we determine the height of point B, which is equal to a? Which trigonometric function gives the height of a point on the circle?

    The sine function gives the signed vertical distance of the point from the origin, so $a = \sin(55°)$.
  - How can we determine the horizontal distance that point C is from the y-axis, which is equal to b? Which trigonometric function gives the horizontal distance from the origin of a point on the circle?

    The cosine function gives the signed horizontal distance of the point from the origin, so $b = \cos(55°)$.
- Once students get to problem 3, you may notice that they are unsure of how to square a trigonometric expression. At this point, you can encourage students to write the equation with grouping symbols, $[\sin(55°)]^2 + [\cos(55°)]^2 = 1$, rather than introduce the notation $\sin^2(55°) + \cos^2(55°) = 1$.
- Allow pairs of students some time to compare their work with that of another pair of students. Encourage them to verify with a calculator that the equation they wrote in problem 3 is a true statement. You may have to remind students to make sure their calculator is in degree mode.

Summarizing the Task

- To ensure that students are prepared for the next part of the lesson, they should recognize that the equation they wrote is true for any angle, not just for 55°. The activity that follows helps students develop this understanding.
- Have each pair of students pick three more angle measures other than 55°. Encourage them to pick measures in degrees and in radians and then use a

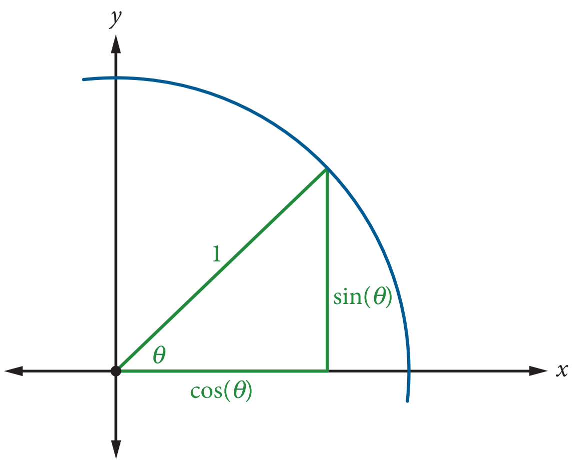
calculator to verify that the equation $\sin^2(\theta)+\cos^2(\theta)=1$ is true for their angle measures. Be sure students have the appropriate mode selected in their calculator for each angle.

- Have students report out their angle measures and whether the equation was a true statement. You can record students' findings in a central location accessible by all students in the class. Ask students to make a conjecture about the relationships they observe.
  - What do you notice about relationships we found?

    It appears that the sum of the squares of sine and cosine of an angle is the same regardless of the angle measure.
  - Make a conjecture about the relationship you observe.

    The sum of the squares of the sine and cosine of any angle is equal to 1.
  - How could we prove that this relationship is true for all angles?

    We could write the equation using a general angle, like θ, and determine if the Pythagorean relationship is true.
- If no students suggest drawing a triangle with an arbitrary angle and showing that the Pythagorean relationship is true, you may need to introduce this approach yourself.
- You can display a reference triangle in a unit circle, as shown in the figure, and ask students some questions like the ones that follow:



Meeting Learners' Needs

Some students might have a hard time conceptualizing what it means to square the output of a sinusoidal function. You can connect this way of thinking to their work in Unit 2, where students composed two functions. In this lesson, they determine the output of the sine function and the cosine function and then square each of these values before adding them. The common notation for this relationship, $\sin^2(\theta)+\cos^2(\theta)=1$, may cause confusion initially because the notation for squaring occurs between the abbreviation of the sinusoidal function and the argument of that function. It may be easier for students to recognize the composition of these functions if the identity is expressed as $[\sin(\theta)]^2+[\cos(\theta)]^2=1$.

- What is the length of the hypotenuse of the reference triangle?

  This is a unit circle, and the hypotenuse of the reference triangle is a radius of the circle, which has a length of 1 unit.

- Why is it correct to describe the length of the vertical leg of the reference triangle as $\sin(\theta)$?

  The signed vertical distance of a point on a unit circle from the horizontal axis is given by the sine of the associated angle.

- Why is it correct to describe the length of the horizontal leg of the reference triangle as $\cos(\theta)$?

  The signed horizontal distance of a point on a unit circle from the vertical axis is given by the cosine of the associated angle.

- How can we use the Pythagorean theorem to express the relationship between the legs of the reference triangle, in terms of the sine and the cosine of the associated angle?

  We can express this relationship as $\sin^2(\theta) + \cos^2(\theta) = 1$.

Guiding Student Thinking

Some students may be unsure whether the Pythagorean relationship is true for angles in the second, third, or fourth quadrants. Encourage them to draw an angle in standard position whose terminal ray does not lie in the first quadrant and use a reference triangle to observe that the values of the sine and cosine of their angle can be determined by the sine and cosine of the reference angle and the appropriate sign(s), and that squaring these values will yield a sum of 1.

- Let students know that the relationship $\sin^2(\theta) + \cos^2(\theta) = 1$ is true for the measure of any angle, whether expressed in degrees or radians, and is called the *Pythagorean identity*.

Part 2: Investigating Right Triangles in a Unit Circle

~45 MIN.

In this part of the lesson, students use their knowledge of equilateral, isosceles, and right triangles to determine the exact values of the sine and cosine of angles that commonly appear in trigonometry: 30°, 45°, 60°, and 90°, or equivalently $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$, and $\frac{\pi}{2}$. This culminates in students determining the exact coordinates of the points on a unit circle whose reference angles include one of these commonly used angles.

Student Task

The task for this section is a jigsaw activity, whose components are provided on **Handout 4T.6.A: Investigating Right Triangles in a Unit Circle – Group 1**, **Handout 4T.6.B: Investigating Right Triangles in a Unit Circle – Group 2**, and **Handout 4T.6.C: Investigating Right Triangles in a Unit Circle – Group 3**. The final part of the task has students collect the discoveries they made about the sine and cosine of commonly used angles in a summary chart, which they then use to label a unit circle with angle measures and their corresponding coordinates. The summary table and unit circle diagram are provided on **Handout 4T.6.D: Common Angles in a Unit Circle**. Possible solutions to the handouts are provided in the Assess and Reflect section of the lesson.

Facilitating the Task

- Let students know that in this part of the lesson they will use their knowledge of triangles from geometry to investigate the connection between reference triangles and the coordinates of points on the unit circle.
- Have students count off in threes. Then regroup the students so that all students whose number is 1 are together, etc. Explain that each group is going to become an expert on the sine and cosine of a specific angle measure, and they need to be prepared to share their expertise with a small group of other experts.
- Distribute Handout 4T.6.A $\left(60°; \frac{\pi}{3}\right)$ to Group 1, Handout 4T.6.B $\left(30°; \frac{\pi}{6}\right)$ to Group 2, and Handout 4T.6.C $\left(45°; \frac{\pi}{4}\right)$ to Group 3.
- Allow students time to work collaboratively in their groups to answer the problems on the handouts. Encourage students to record everything they know about the given triangle including the relationships between angle measures and side lengths.

Classroom Ideas
To better manage the size of the groups and give all students an opportunity to meaningfully contribute to their shared understanding, you might need to create multiple groups of 30-degree experts, multiple groups of 45-degree experts, and multiple groups of 60-degree experts.

- The reference triangles for the 60° group (Group 1) and the 30° group (Group 2) are not shown on the handout. Encourage students to draw the reference triangles to help them answer the questions.
- On each handout, problem 3 asks students to determine the other points on the circle that have reference angles congruent to their specified angle. Students many need some support to determine these other points. You can use this as an opportunity to reinforce some of the key characteristics of reference triangles.
  - How is a reference triangle drawn with respect to the x-axis?

    A reference triangle is drawn so that the horizontal leg lies on the positive or negative x-axis and the vertical leg is perpendicular to the x-axis.
  - How is a reference triangle in quadrant II, III, or IV related to a reference triangle in quadrant I?

    A reference triangle in quadrant II, III, or IV is congruent to a reference triangle in quadrant I. They have the same reference angle. The triangle in quadrant II, III, or IV is a reflection of the quadrant I triangle over one or both axes.
  - What is a reference angle?

    A reference angle is the acute angle that the terminal ray of the angle in standard position makes with the negative x-axis in quadrants II and III and with the positive x-axis in quadrants I and IV.

Instructional Rationale

Encourage students to employ their knowledge of right triangles and reference triangles to determine the exact values of sine and cosine for commonly used angles instead of trying to memorize a long list of values. Repeatedly using reference triangles to determine the values of the sine and cosine of commonly used angles helps students develop a durable understanding of trigonometry that will benefit them in future courses. It also helps students internalize that trigonometry is a mathematical tool that relates triangles and circles in the coordinate plane. The summary table and diagram on Handout 4T.6.D should be used to help students capture their thinking rather than to encourage memorization.

Summarizing the Task

- To complete the jigsaw activity, regroup the students into discussion groups so that at least one student from each of the expert groups is represented in the discussion group. If possible, you could have two students from each expert group represented in each discussion group. This would reduce the responsibility that a single student might feel to convey angle information to their classmates since the responsibility would be shared.
- As students start to share their observations with each other, distribute Handout 4T.6.D and have students complete the table with the values of the sine and cosine of the angles.
- Students may notice that there are no experts for the 90° set of angles. Each discussion group should work collaboratively to determine the values of the sine and cosine of these angles. Encourage students to use the unit circle diagram on Handout 4T.6.D to help them determine the values.
- Before concluding the lesson, have a few groups of students share their summary tables and unit circle diagrams with the class. You could add the summary table and unit circle diagram to your classroom's word wall if you have one.

UNIT 4T

Assess and Reflect on the Lesson

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

A circle centered at the origin has a radius of 1.

(a) At what points on the circle does the y-coordinate have an absolute value that is exactly half the length of the radius? For each point, list both of its coordinates.

The y-coordinate has an absolute value that is half the length of the radius when the terminal ray of an angle in standard position measures $\frac{\pi}{6}$ radians, $\frac{5\pi}{6}$ radians, $\frac{7\pi}{6}$ radians, or $\frac{11\pi}{6}$ radians. The coordinates of the points that correspond to these angles are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$, and $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ respectively.

(b) At what points on the circle does the x-coordinate have an absolute value that is exactly half the length of the radius? For each point, list both of its coordinates.

The x-coordinate is half the length of the radius when the terminal ray of an angle in standard position measures $\frac{\pi}{3}$ radians, $\frac{2\pi}{3}$ radians, $\frac{4\pi}{3}$ radians, or $\frac{5\pi}{3}$ radians. The coordinates of the points that correspond to these angles are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ respectively.

(c) At what points on the circle is the x-coordinate equal to the y-coordinate? For each point, list both of its coordinates.

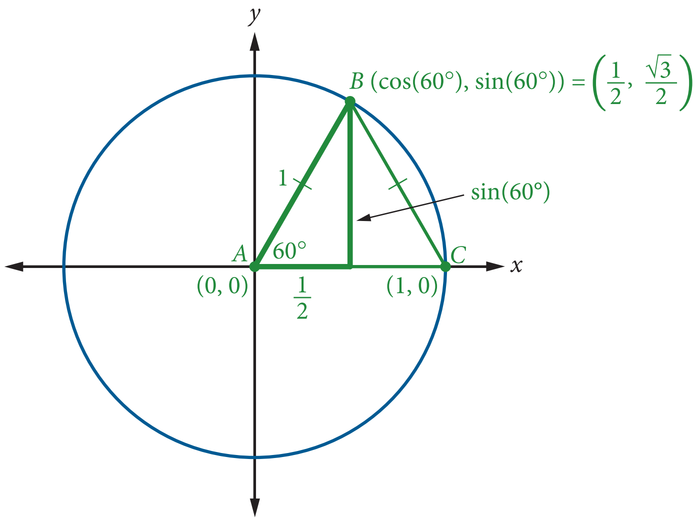
The x-coordinate is equal to the y-coordinate when the terminal ray of an angle in standard position measures $\frac{\pi}{4}$ radians or $\frac{5\pi}{4}$ radians. The coordinates of the points that correspond to these angles are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ respectively.

HANDOUTS

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 4T.6.A: Investigating Right Triangles in a Unit Circle – Group 1

The following figure shows a possible reference triangle with angle measurements and side lengths included.



1. The measure of $\angle BAC$ is $60°$, which is equivalent to $\frac{\pi}{3}$ radians. This is because the measures of the angles in a triangle must sum to $180°$ and the angles in an equilateral triangle are congruent.
2. The coordinates of point B are $(\cos(60°),\ \sin(60°))$. These coordinates can also be expressed as $\left(\cos\left(\frac{\pi}{3}\right),\ \sin\left(\frac{\pi}{3}\right)\right)$ or $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. These coordinates are approximately equal to $(0.5,\ 0.866)$. The exact value for $\sin\left(\frac{\pi}{3}\right)$ can be determined using the Pythagorean theorem:

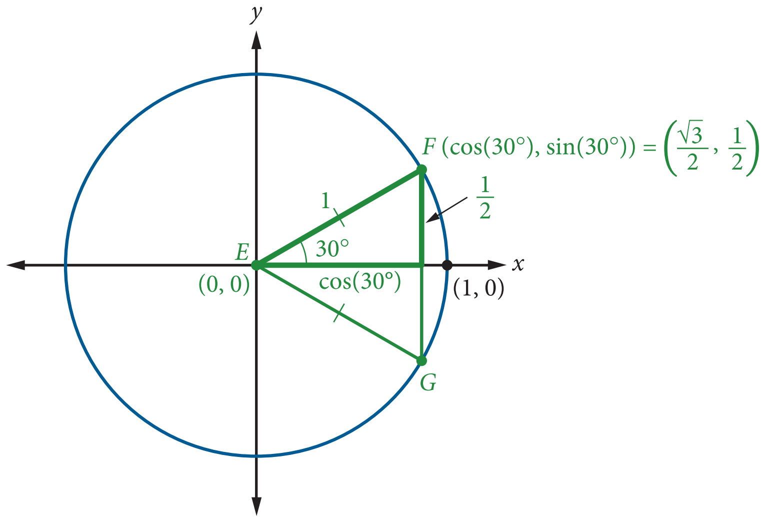
$$\left(\frac{1}{2}\right)^2 + [\sin(60°)]^2 = 1$$
$$\frac{1}{4} + [\sin(60°)]^2 = 1$$
$$[\sin(60°)]^2 = \frac{3}{4}$$
$$\sin(60°) = \frac{\sqrt{3}}{2}$$

UNIT 4T

3. The other angles that have reference triangles that are congruent to the one for point B are $120°$ or $\frac{2\pi}{3}$ radians, $240°$ or $\frac{4\pi}{3}$ radians, and $300°$ or $\frac{5\pi}{3}$ radians. The coordinates of the points on a unit circle that correspond to those angles are
$(\cos(120°),\ \sin(120°)) = \left(\cos\left(\frac{2\pi}{3}\right),\ \sin\left(\frac{2\pi}{3}\right)\right) = \left(-\frac{1}{2},\ \frac{\sqrt{3}}{2}\right)$,
$(\cos(240°),\ \sin(240°)) = \left(\cos\left(\frac{4\pi}{3}\right),\ \sin\left(\frac{4\pi}{3}\right)\right) = \left(-\frac{1}{2},\ -\frac{\sqrt{3}}{2}\right)$, and
$(\cos(300°),\ \sin(300°)) = \left(\cos\left(\frac{5\pi}{3}\right),\ \sin\left(\frac{5\pi}{3}\right)\right) = \left(\frac{1}{2},\ -\frac{\sqrt{3}}{2}\right)$.

Handout 4T.6.B: Investigating Right Triangles in a Unit Circle – Group 2

The following figure shows a possible reference triangle with angle measurements and side lengths included.



1. The measure of θ is 30 degrees, which is equivalent to $\frac{\pi}{6}$ radians. This is because an angle of an equilateral triangle was bisected, so its measure is one half of 60 degrees.
2. The coordinates of point F are $(\cos(30°),\ \sin(30°))$. These coordinates can also be expressed as $\left(\cos\left(\frac{\pi}{6}\right),\ \sin\left(\frac{\pi}{6}\right)\right)$ or $\left(\frac{\sqrt{3}}{2},\ \frac{1}{2}\right)$. These coordinates are approximately equal to $(0.866,\ 0.5)$. The exact value for $\cos\left(\frac{\pi}{6}\right)$ can be determined using the Pythagorean theorem:

$$\left(\frac{1}{2}\right)^2+\left[\cos(30°)\right]^2=1$$
$$\frac{1}{4}+\left[\cos(30°)\right]^2=1$$
$$\left[\cos(30°)\right]^2=\frac{3}{4}$$
$$\cos(30°)=\frac{\sqrt{3}}{2}$$

3. The other angles that have reference triangles that are congruent to the one for point F are $150°$ or $\frac{5\pi}{6}$ radians, $210°$ or $\frac{7\pi}{6}$ radians, and $330°$ or $\frac{11\pi}{6}$ radians. The coordinates of the points on a unit circle that correspond to those angles are

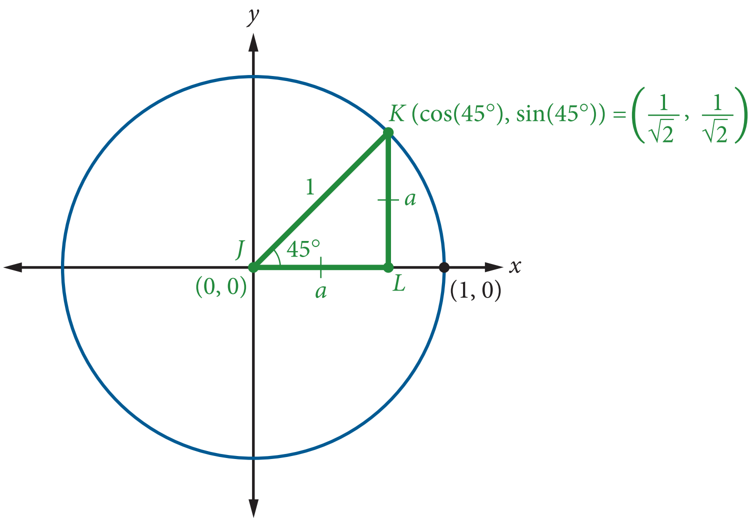
$\left(\cos(150°),\ \sin(150°)\right)=\left(\cos\left(\frac{5\pi}{6}\right),\ \sin\left(\frac{5\pi}{6}\right)\right)=\left(-\frac{\sqrt{3}}{2},\ \frac{1}{2}\right)$,

$\left(\cos(210°),\ \sin(210°)\right)=\left(\cos\left(\frac{7\pi}{6}\right),\ \sin\left(\frac{7\pi}{6}\right)\right)=\left(-\frac{\sqrt{3}}{2},\ -\frac{1}{2}\right)$, and

$\left(\cos(330°),\ \sin(330°)\right)=\left(\cos\left(\frac{11\pi}{6}\right),\ \sin\left(\frac{11\pi}{6}\right)\right)=\left(\frac{\sqrt{3}}{2},\ -\frac{1}{2}\right)$.

Handout 4T.6.C: Investigating Right Triangles in a Unit Circle – Group 3

The following figure shows a possible reference triangle with angle measurements and side lengths included.



1. The measure of $\angle KJL$ is 45°, which is equivalent to $\frac{\pi}{4}$ radians. This is because the base angles of the isosceles right triangle must be congruent, so their measures are equal.
2. The coordinates of point K are $(\cos(45°), \sin(45°))$. These coordinates can also be expressed as $\left(\cos\left(\frac{\pi}{4}\right), \sin\left(\frac{\pi}{4}\right)\right)$ or $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. These coordinates are approximately equal to $(0.707, 0.707)$. The exact values can be determined using the Pythagorean theorem:

$$a^2 + a^2 = 1$$
$$2a^2 = 1$$
$$a^2 = \frac{1}{2}$$
$$a = \frac{1}{\sqrt{2}}$$

3. The other angles that have reference triangles that are congruent to the one for point K are 135° or $\frac{3\pi}{4}$ radians, 225° or $\frac{5\pi}{4}$ radians, and 315° or $\frac{7\pi}{4}$ radians. The coordinates of the points on a unit circle that correspond to these angles are
$(\cos(135°), \sin(135°)) = \left(\cos\left(\frac{3\pi}{4}\right), \sin\left(\frac{3\pi}{4}\right)\right) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$,
$(\cos(225°), \sin(225°)) = \left(\cos\left(\frac{5\pi}{4}\right), \sin\left(\frac{5\pi}{4}\right)\right) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, and
$(\cos(315°), \sin(315°)) = \left(\cos\left(\frac{7\pi}{4}\right), \sin\left(\frac{7\pi}{4}\right)\right) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

UNIT 4T

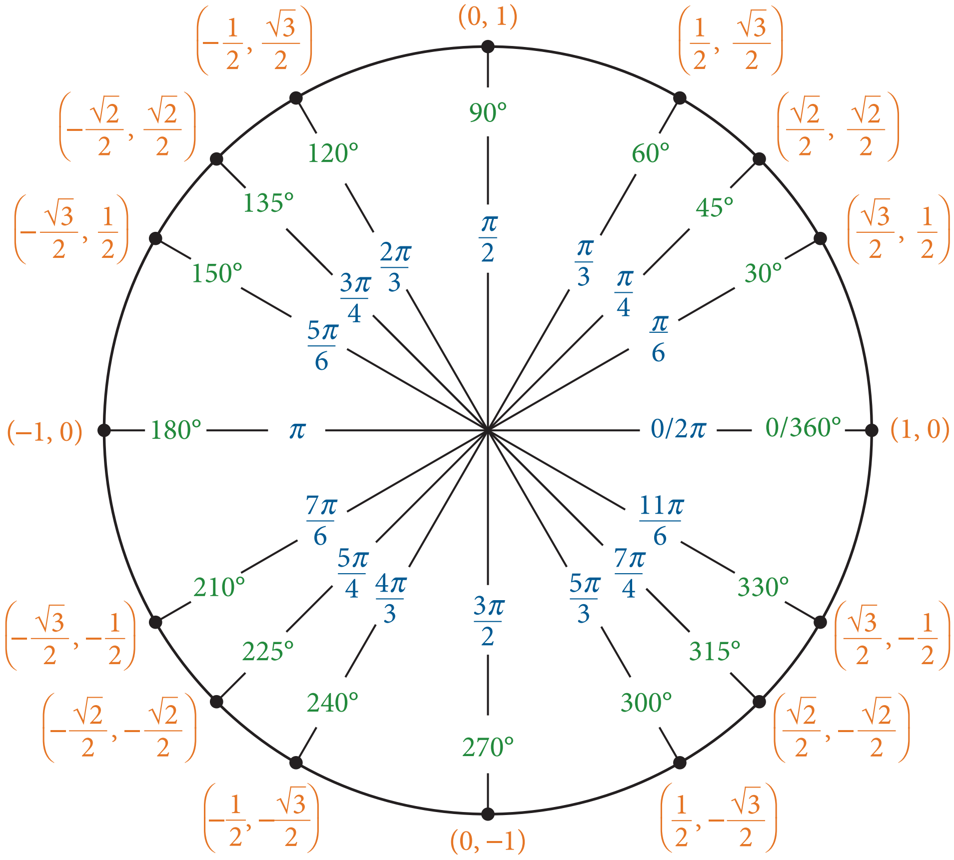
Handout 4T.6.D: Common Angles in a Unit Circle

θ	$\sin(\theta)$	$\cos(\theta)$
0 radians 0 degrees	0	1
$\frac{\pi}{2}$ radians 90 degrees	1	0
π radians 180 degrees	0	-1
$\frac{3\pi}{2}$ radians 270 degrees	-1	0

θ	$\sin(\theta)$	$\cos(\theta)$
$\frac{\pi}{4}$ radians 45 degrees	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{3\pi}{4}$ radians 135 degrees	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
$\frac{5\pi}{4}$ radians 225 degrees	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
$\frac{7\pi}{4}$ radians 315 degrees	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$

θ	$\sin(\theta)$	$\cos(\theta)$
$\frac{\pi}{6}$ radians 30 degrees	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{5\pi}{6}$ radians 150 degrees	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$\frac{7\pi}{6}$ radians 210 degrees	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$\frac{11\pi}{6}$ radians 330 degrees	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

θ	$\sin(\theta)$	$\cos(\theta)$
$\frac{\pi}{3}$ radians 60 degrees	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{2\pi}{3}$ radians 120 degrees	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\frac{4\pi}{3}$ radians 240 degrees	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\frac{5\pi}{3}$ radians 300 degrees	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$


(0, 1)
90°
$\frac{\pi}{2}$
$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
60°
$\frac{\pi}{3}$
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
45°
$\frac{\pi}{4}$
$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
30°
$\frac{\pi}{6}$
(1, 0)
0/360°
$0/2\pi$
$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
330°
$\frac{11\pi}{6}$
$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
315°
$\frac{7\pi}{4}$
$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
300°
$\frac{5\pi}{3}$
(0, −1)
270°
$\frac{3\pi}{2}$
$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
240°
$\frac{4\pi}{3}$
$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
225°
$\frac{5\pi}{4}$
$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
210°
$\frac{7\pi}{6}$
(−1, 0)
180°
π
$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
150°
$\frac{5\pi}{6}$
$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
135°
$\frac{3\pi}{4}$
$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
120°
$\frac{2\pi}{3}$

LESSON 4T.7

A Model for Periodic Phenomena

LEARNING OBJECTIVES

4T.1.5 Identify key characteristics of a sinusoidal function.

4T.1.6 Construct a sinusoidal function to model a periodic phenomenon that has a specified frequency, period, amplitude, and phase shift.

LESSON OVERVIEW

CONTENT FOCUS

Sinusoidal functions are excellent models for scenarios that involve circular motion because the sine and cosine functions relate angle measures to vertical or horizontal distances in a circle. These functions can also be used to model other periodic phenomena in which data values repeat in predictable ways. In this lesson, students explore a real-world model of circular motion to make sense of transformations of a sinusoidal function. Then they apply their understanding of the transformations of a sinusoidal function to make sense of a periodic phenomenon that does not involve circular motion.

LESSON DESCRIPTION

Part 1: Analyzing a Circular Motion Context

In this part of the lesson, students analyze a real-world context that involves circular motion. The Ferris wheel context is similar to scenarios that students explored in previous lessons, but the quantities involved are slightly different. In Lesson 4T.4, students used the sine function to relate the height of a point on a circle above the horizontal diameter with the measure of an angle

AREAS OF FOCUS

- Connections Among Multiple Representations
- Greater Authenticity of Applications and Modeling

SUGGESTED TIMING

~120 minutes

LESSON SEQUENCE

- This lesson is part of a lesson sequence (~600 minutes total) that includes Lessons 4T.1 through 4T.7.

MATERIALS

- individual whiteboards
- graph paper (optional)
- graphing utility, such as **Desmos.com**

HANDOUTS

Lesson

- 4T.7.A: Translations of a Circular Motion Model

Practice

- 4T.7.B: Modeling Periodic Phenomena

in standard position. Students now relate the height of a Ferris wheel rider above the ground with the length of time they spend on the ride. This part of the lesson culminates in students constructing a graphical representation of the context and arguing that a sinusoidal function would be an appropriate model.

Part 2: Transforming a Sinusoidal Function
In this part of the lesson, students develop a conceptual understanding of the transformation of the sine function using the form $f(x) = a\sin\big(b(x+c)\big) + d$ in the context of the Ferris wheel problem from Part 1. Through their work in this part of the lesson, students define the terms amplitude, phase shift, and period as they relate to transformations of sinusoidal functions.

Part 3: Interpreting a Model for Periodic Phenomena
In this part of the lesson, students use sinusoidal functions to model real-world periodic phenomena that do not involve circular motion. Students interpret the parameters of the algebraic representation of a sinusoidal function to make sense of the scenario.

FORMATIVE ASSESSMENT GOAL

This lesson prepares students to complete the following formative assessment activity.

The depth of the ocean water at the end of an industrial pier near Pensacola, Florida, changes with the tides. Suppose that on a particular day, high tide occurs at 4:15 a.m. with a water depth of 5.2 meters. Low tide occurs at 10:30 a.m. with a water depth of 2 meters.

(a) Write an algebraic representation of a sinusoidal function that models the depth of the water as a function of the number of hours after midnight.

(b) For your function, explain how you determined the values of the parameters.

(c) Use your function to determine the depth of the water at noon.

Part 1: Analyzing a Circular Motion Context

~20 MIN.

In this part of the lesson, students analyze a real-world context that involves circular motion. The Ferris wheel context is similar to scenarios that students explored in previous lessons, but the quantities involved are slightly different. In Lesson 4T.4, students used the sine function to relate the height of a point on a circle above the horizontal diameter with the measure of an angle in standard position. Students now relate the height of a Ferris wheel rider above the ground with the length of time they spend on the ride. This part of the lesson culminates in students constructing a graphical representation of the context and arguing that a sinusoidal function would be an appropriate model.

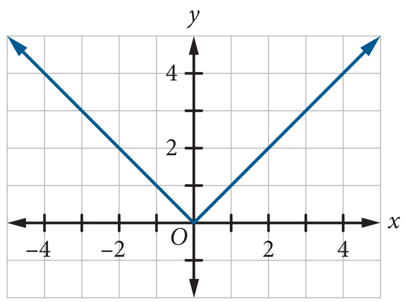
Warm-Up

- As a just-in-time review of additive and multiplicative transformations, have students engage in a brief analysis of the function $f(x) = |x|$.
- If individual whiteboards are available, have students graph each of the following functions and describe how it is a transformation of the parent function, $f(x) = |x|$.
  - Let $f(x) = |x|$. What kind of function is this? What special characteristics does it have?

    This is the absolute value function. The graph has a V shape. The parent function lies on or above the x-axis, with the vertex of the V at the origin.

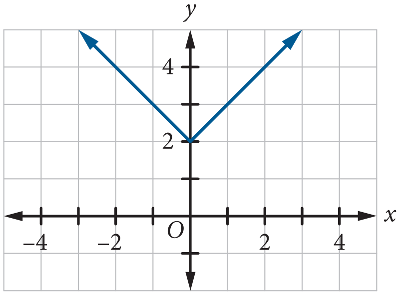
  - Graph $y = f(x)$. What does the graph look like?

    The graph of $y = f(x)$ follows:



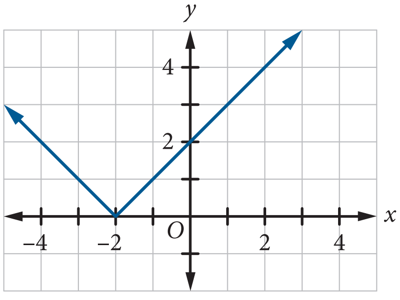
  - Graph $y = f(x) + 2$. What kind of transformation is this? How does the graph compare to the graph of the parent function?

    This is an additive transformation that results in a vertical translation of the graph of $f(x)$ by 2 units up.



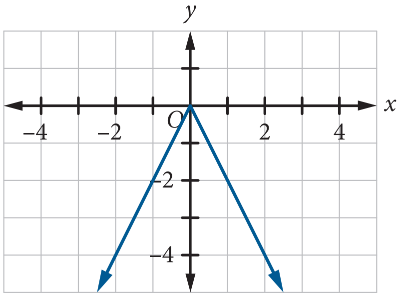
- Graph $y = f(x+2)$. What kind of transformation is this? How does the graph compare to the graph of the parent function?

  This is an additive transformation that results in a horizontal translation of the graph of $f(x)$ by 2 units to the left.



- Graph $y = -2f(x)$. What kind of transformation is this? How does the graph compare to the graph of the parent function?

  This is a multiplicative transformation that results in a vertical scaling of the graph of $f(x)$ by a factor of 2. Because the scaling factor is negative, it also results in a reflection across the x-axis.



Student Task

The student task is provided on **Handout 4T.7.A: Translations of a Circular Motion Model**. In this task, students develop both graphical and algebraic representations of the function that models a person's height above the ground in terms of the number of minutes after the person boarded the Ferris wheel. Solutions for the handout are provided in the Assess and Reflect section of the lesson.

Facilitating the Task

- Begin by providing a copy of Handout 4T.7.A to each student. Make sure students have access to whiteboards and a graphing utility such as Desmos. Allow them time to closely read the handout and ask clarifying questions about the context.
- Have students work with a partner or in a small group of three to complete problems 1–4 on Handout 4T.7.A.
- As you circulate around the room while students work, you can ask questions to make sure they understand how to extract information from the context and incorporate it into the function model. Some questions that you could use follow:
  - How are the maximum and minimum heights Mira reaches related to the function?

    The maximum and minimum heights represent the range of the function.
  - Suppose that Mira completes two revolutions on the Ferris wheel. What would be the contextual domain of the function?

    The contextual domain is $0 \leq t \leq 60$, where t is the number of minutes since she boarded the Ferris wheel.

Meeting Learners' Needs

As students answer problem 1 on the handout, you may find that some of them need to mark a diagram of the Ferris wheel to keep track of the distances involved. You can encourage them to make their own drawings if the provided figure is not sufficient for their needs.

Classroom Ideas

Have students construct their graphs for problem 4 on personal whiteboards so that they can easily share their graphs with the class.

- Based on the paragraph, what are some points you can be certain are on the graph of the function f?

  The point $(0,\ 30)$ represents the time in minutes after boarding and height in feet when Mira gets on the Ferris wheel, and the point $(30,\ 30)$ represents the time in minutes after boarding and height in feet when she is back at the bottom of the Ferris wheel after completing one revolution. The points $(7.5,\ 290)$, $(22.5,\ 290)$, $(37.5,\ 290)$, and $(52.5,\ 290)$ represent the time in minutes after boarding and height in feet when she is at the same height as the center of the Ferris wheel. The points $(15,\ 550)$ and $(45,\ 290)$ represent the time in minutes after boarding and height in feet when she is at the maximum height of the Ferris wheel.

Meeting Learners' Needs

For students who are unsure of how to construct a graph using the information they determine in problems 1 through 3, you could suggest that they construct a table of values of the time in minutes after boarding and the height in feet that Mira is above the ground. A numerical approach can help focus students on the relevant information and reinforce the connections between graphical and numerical representations of a function.

Summarizing the Task

- As students complete their graphs for problem 4, have each pair of students or small group compare their graphs with those of another pair of students or small group. Encourage students to focus on similarities and differences between the groups' graphs and note any changes they want to make to their own function model.
- After student pairs or small groups compare their graphs, have several groups share their graphs with the class.
- Be sure to bring attention to the specific characteristics of this function, such as the oscillating behavior of the graph, the ordered pairs that repeat on the graph, and the relationship between the diameter of the Ferris wheel and the distance between the maximum and minimum output values.
- Have students share their answers to problem 6, in which they describe the type of function they think would best model the scenario. You can anticipate that students might reason that a sine function is appropriate because the context being modeled is the height that Mira is above the ground, while other students might reason that a cosine function is appropriate because this graph more closely resembles the graph of a cosine function. Be sure to validate both ways of thinking but explain to students that it is more appropriate to choose a sine function as the base

function of our model. This is because the phenomenon being modeled is a vertical distance, and vertical distances are traditionally expressed using the sine function for angles in standard position.

- At this point in the lesson, students are ready to build an algebraic model of this scenario, which is the focus of Part 2 of the lesson.

Part 2: Transforming a Sinusoidal Function

~40 MIN.

In this part of the lesson, students develop a conceptual understanding of the transformation of the sine function using the form $f(x) = a\sin(b(x+c)) + d$ in the context of the Ferris wheel problem from Part 1. Through their work in this part of the lesson, students define the terms amplitude, phase shift, and period as they relate to transformations of sinusoidal functions.

Instructional Rationale

It is intentional that this lesson uses the form $f(x) = a\sin(b(x+c)) + d$ in which the b parameter is factored out of the argument of the function. Some texts use a form of the function that looks like $f(x) = a\sin(bx+c) + d$, where the c parameter is the product of the horizontal dilation parameter and the horizontal translation parameter. To help students develop a better understanding of the meaning of each parameter, and to connect this lesson with the understanding of additive and multiplicative transformations that students developed in Unit 2, this lesson separates the horizontal scaling parameter and the horizontal translation parameter in the form $f(x) = a\sin(b(x+c)) + d$.

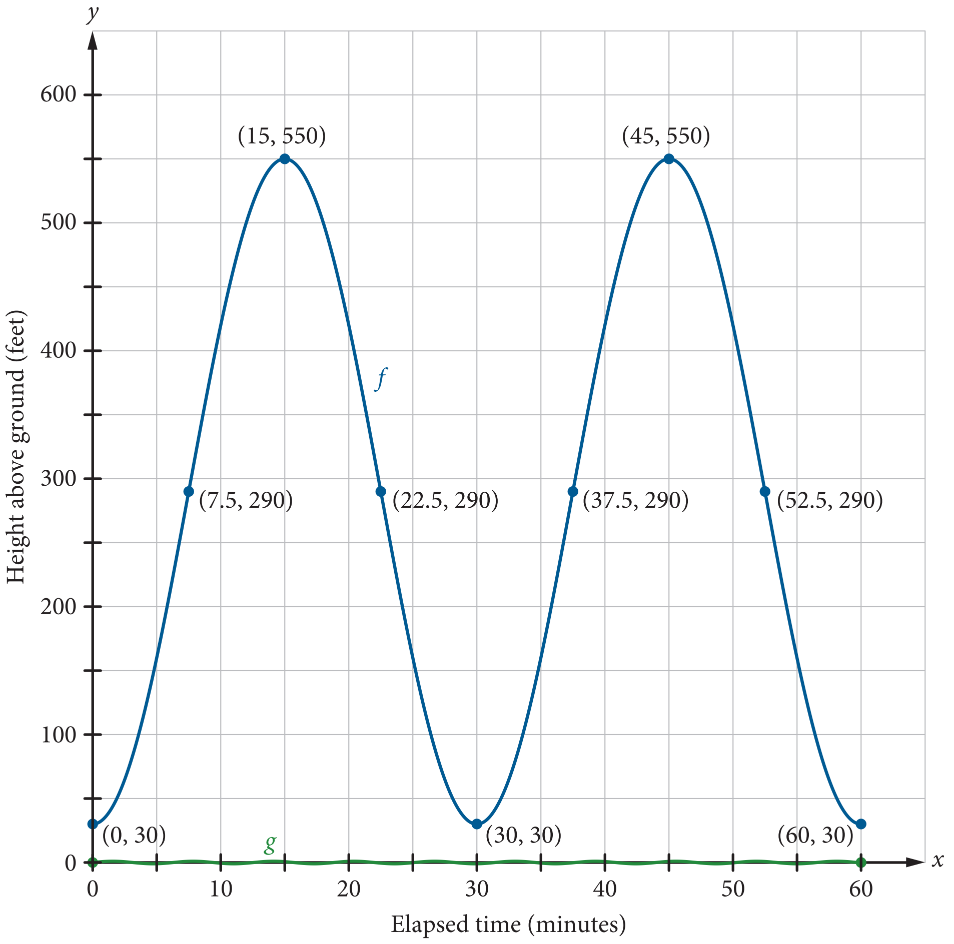
Student Task

We know that a sine function would be an appropriate model for this scenario. How could we make modifications to the sine parent function so that it matches the Ferris wheel context? What transformations should we use to construct an algebraic representation of the function model?

Facilitating the Task

- Begin by having students experiment with additive and multiplicative transformations using a graphing utility. Encourage them to think about how the transformations affect the graph of $g(x) = \sin(x)$. Students do not need to create a complete list of the transformations they need or identify the exact parameters of the function. They should simply brainstorm about the types of transformations they might need and think about why those transformations might be appropriate.

- As students determine what kinds of transformations they might use to modify $g(x) = \sin(x)$ so that it adequately models the Ferris wheel context, have them share their observations with the class. They could also share things they wonder about transformations of sinusoidal functions. This is an opportunity to gather their questions and understandings, so you know what they are before continuing to build the algebraic model.
- Display the graph of the context that students constructed in Part 1 of the lesson. It may also be helpful to display the graph of $g(x) = \sin(x)$ on the graph of the function, although it may be barely visible as it oscillates above and below the x-axis with maximum and minimum values of 1 and -1, respectively.



- Engage students with guided questions to help them determine the values of the parameters for a transformation of the sine function. As they answer these questions, have them modify the algebraic representation of $g(x) = \sin(x)$ so that they can compare it to their graph of f. You can use questions like the ones that follow to structure the discussion.

UNIT 4T

Instructional Rationale

The discussion to determine the values of the parameters of the sine function transformation is structured so that students first consider the vertical translation and then the vertical scaling (amplitude). It is usually a little easier for students to consider the vertically oriented parameters from the graph. Next, students consider the horizontal scaling (period) and then the horizontal translation (phase shift). The horizontally oriented parameters are usually a little more difficult to determine. Be sure to have students use a graphing utility and refer to the graph they constructed in Part 1 as they engage in the discussion. Connecting the graphical representation with the algebraic representation helps students make sense of the values of the parameters. It is not necessary to introduce the terminology to students before they have a chance to explore the scenario and make sense of the parameters. It is better to allow them to develop their understanding, and then provide them with the appropriate mathematical vocabulary toward the end of the lesson.

- Examine how the graphs of functions f and g oscillate. What is the horizontal line over which each graph oscillates?

  The horizontal line over which the graph of g oscillates is $y = 0$. The horizontal line over which the graph of f oscillates is $y = 290$.

- What kind of transformation should we use so that g oscillates over the same horizontal line as f? How can we transform function g?

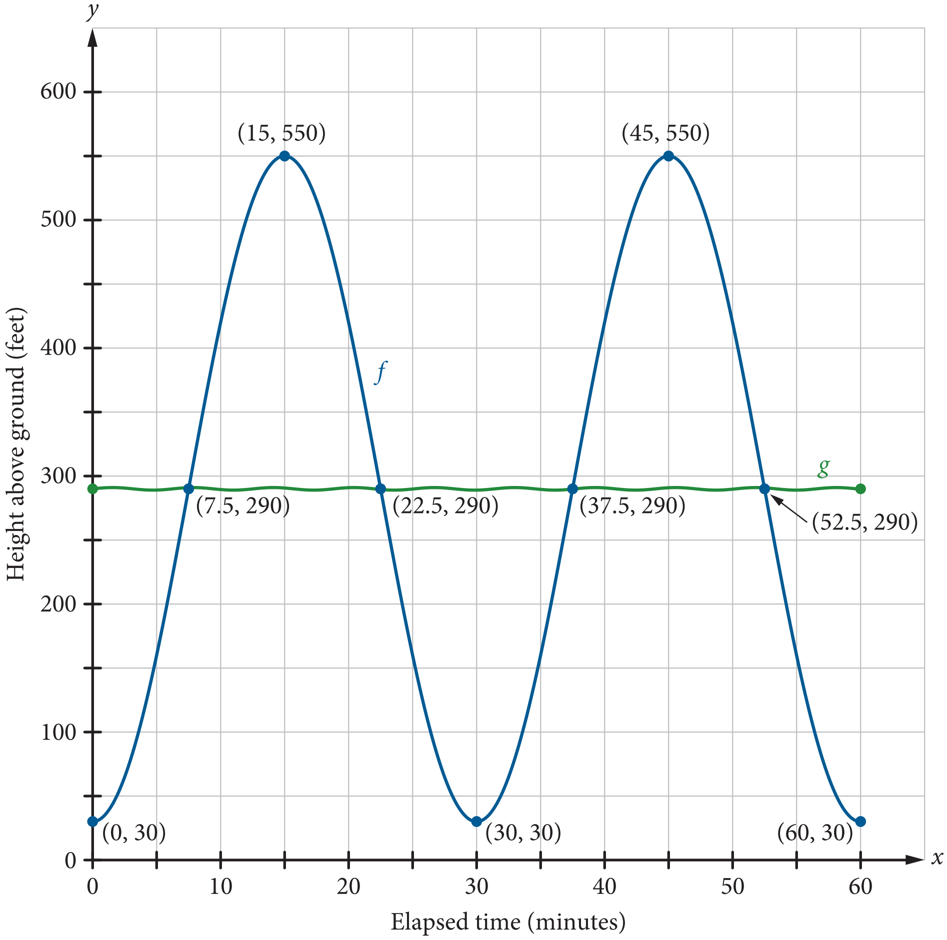
  We should use an additive transformation that affects the output of the function. A vertical translation by 290 transforms the graph of g so that it oscillates over the same horizontal line as the graph of f. We can accomplish this transformation by adding 290 to $\sin(x)$.

Guiding Student Thinking

You may need to help students recognize that a vertical translation is the correct transformation. The magnitude of the vertical translation can be determined by comparing the location of a common feature, like a peak or trough, of the graphs.

- Make the adjustment to your sine function to create a new function g that is $g(x) = \sin(x) + 290$ and check the graph to make sure that f and g oscillate over the same horizonal line $y = 290$.

  The graphs of the new g and f follow.



- Let's look at the maximum and minimum values of f and g. What kind of transformation should we use to make the minimum and maximum values of g match the minimum and maximum values for f?

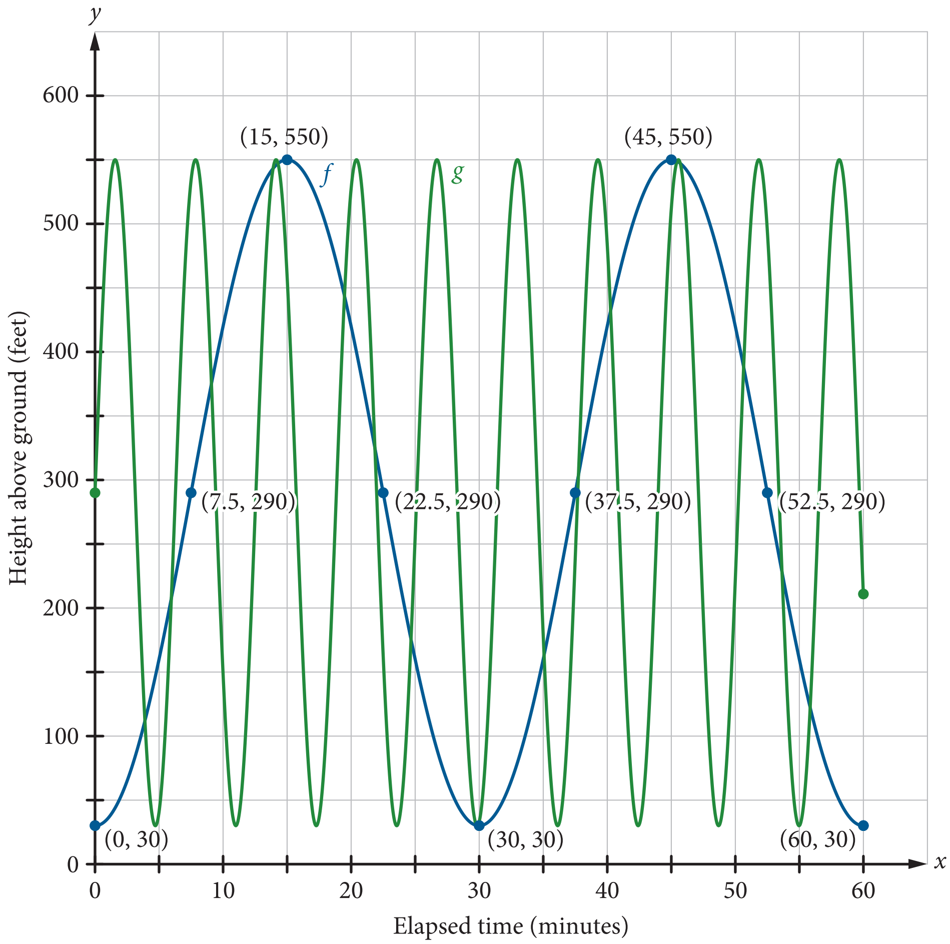
  We should use a multiplicative transformation that affects the output of the function. A vertical dilation by a factor of 260 transforms the graph of g so that it has the same minimum and maximum values as the graph of f. We can accomplish this transformation by multiplying $\sin(x)$ by 260.

Guiding Student Thinking

Some students may suggest that the vertical scaling factor should be 550 because that is the maximum value. It is important for these students to attempt to use a scale factor of 550 to observe that it does not result in the correct scaling of the graph. You may need to help students reason that, because the vertical scale factor is multiplied by the output of sine function before the vertical translation is added, the magnitude of the scale factor only has to be half the difference of the maximum and minimum values. That is, the vertical scale factor has the effect of vertically stretching the graph from its center line.

- Make the adjustment to your sine function to create a new function g that is $g(x) = 260\sin(x) + 290$ and check the graph to make sure that f and g have the same minimum and maximum values.

  The graphs of the new g and f follow.



- What other transformations should we consider to make the graph of g match the graph of f?

  We should consider a horizontal dilation and/or horizontal translation.

- Let's compare how often f and g complete one full cycle. This will help us determine the horizontal scale factor we should use. Over what interval of the domain does g complete one full cycle? Over what interval of the domain does f complete one full cycle?

  Function f completes one full cycle every 30 minutes. Function g completes one full cycle every 2π radians.

Guiding Student Thinking

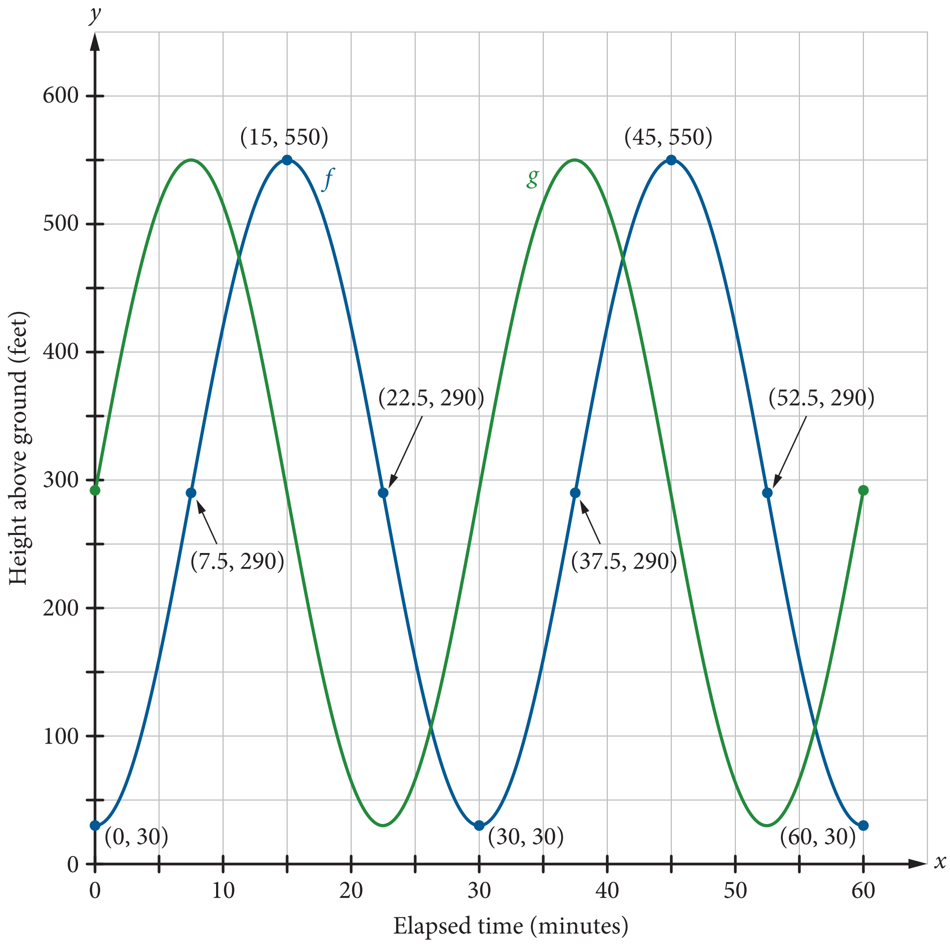
You will have to help guide students through some reasoning to help them make sense of how to determine the horizontal scale factor:

- The Ferris wheel context, modeled by function f, has a cycle of 30 minutes because it takes 30 minutes for Mira to complete one rotation of the Ferris wheel. That is, for any starting location it takes 30 minutes on the Ferris wheel for Mira to return to the same location.
- Function g has a cycle of 2π because it takes 2π radians to complete one full rotation of a circle. That is, an angle in standard position must sweep out 2π radians before its terminal ray returns to the same location at which it started.
- To get the correct scale factor we can reason that the endpoints of one cycle for f are 0 minutes and 30 minutes while the endpoints of one cycle for g are 0 radians and 2π radians.

The reasoning in the bullets above suggests that there is a proportional relationship between the cycle of f and the cycle of g. Since we are transforming g so that it matches f, we should use the scale factor $\frac{2\pi}{30}$ for the input of g. This scale factor has the effect of transforming the units of the input from minutes to radians. The scale factor could be expressed as $\frac{\pi}{15}$.

- Make the adjustment to your sine function to create a new function g that is $g(x) = 260\sin\left(\frac{\pi}{15}x\right) + 290$ and check the graph to make sure that f and g have the same cycle.

  The graphs of the new g and f follow.



- What is the final transformation that we need to use so that the graph of our algebraic function g matches the graph of f?

  The last transformation we should use is a horizontal translation.

- How could we determine the amount of the horizontal translation that we need to use to make the graph of g coincide with f?

  We could compare the x-values of the minimum or maximum values of g and f to determine the difference between them. The x-values of f are 7.5 units to the right of the x-values of g, so we should subtract 7.5 from x in the input of g.

Guiding Student Thinking

You may have to guide students through their reasoning as they determine the magnitude of the horizontal translation:

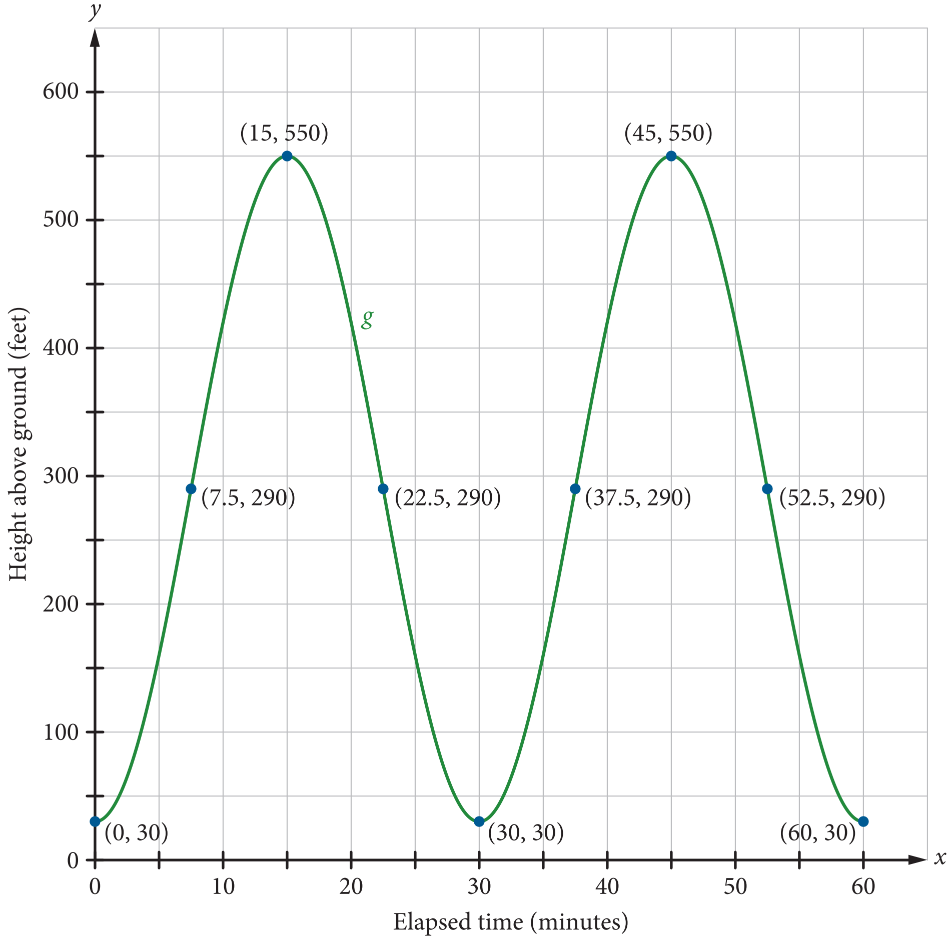
- A graphical analysis of the graph of *f* and the graph of $g(x) = 260\sin\left(\frac{\pi}{15}x\right) + 290$ reveals that if the graph of *g* were translated to the right by 7.5 units, then the two graphs coincide.
- It makes sense that the horizontal translation is 7.5 units because it takes 7.5 minutes for Mira to reach the height of the center of the Ferris wheel for the first time.
- The output of the sine function is the height above the horizontal diameter of the circle. When Mira's car has a height of 0 feet above the center of the Ferris wheel, it has been in motion for 7.5 minutes.
- The input value for *f*, 7.5 minutes, should correspond to the input value of *g*, 0 radians. This suggests that we should subtract 7.5 from the input of *g*.

The best way for students to make sense of this transformation is to apply it to *x* before the horizontal scaling. Therefore, the most appropriate way to express the transformation algebraically is $g(x) = 260\sin\left(\frac{\pi}{15}(x - 7.5)\right) + 290$.

- Make the adjustment to your sine function to create a new function *g* that is $g(x) = 260\sin\left(\frac{\pi}{15}(x - 7.5)\right) + 290$ and check the graph to make sure that *f* and *g* have the same cycle.

  The graphs of the new *g* and *f* follow.

UNIT 4T



- What should we conclude about our completed function g, defined by $g(x) = 260\sin\left(\frac{\pi}{15}(x - 7.5)\right) + 290$? What is its relationship to function f?

  Function g, defined as $g(x) = 260\sin\left(\frac{\pi}{15}(x - 7.5)\right) + 290$, is the algebraic representation of f.

Summarizing the Task

- Once students have determined an algebraic representation of function f, it is appropriate to introduce some terminology related to trigonometric functions. The new vocabulary is important to discuss with students, but it is critical that they understand how to determine the values of the parameters of the function and what they mean for both the real-world context and the graphical representation of the function.
  - Transformations of sinusoidal functions have the algebraic representation $f(x) = a\sin(b(x + c)) + d$ or $f(x) = a\cos(b(x + c)) + d$.

- The *midline* of the function is the horizontal line over which the graph of the sinusoidal function oscillates. In the algebraic representation of the function, the parameter d indicates the location of the midline, $y = d$, and the vertical translation of the graph. For the Ferris wheel scenario, the parameter has the value 290, which is the difference between the center of the Ferris wheel and the ground.
- The *amplitude* of a sinusoidal function describes the vertical scaling of the graph. In the algebraic representation of the function, the absolute value of parameter a indicates the amplitude. For the Ferris wheel scenario, this parameter has the value 260, which is also the length of the radius of the Ferris wheel.
- The *period* of a sinusoidal function is the length of the interval of the input values over which the function completes one full cycle. In the algebraic representation of the function, the absolute value of the b parameter can be used to determine the period. The proportional relationship between the period of the sine or cosine function and the period of a periodic phenomenon can be expressed as $b = \frac{2\pi}{\text{period}}$. For the Ferris wheel scenario, the parameter b has the value $\frac{2\pi}{30} = \frac{\pi}{15}$ because the Ferris wheel completes one full cycle in 30 minutes.
- The *phase shift* of a sinusoidal function describes the horizontal translation of the graph. In the algebraic representation of the function, the parameter c indicates the horizontal translation. The horizontal translation can often be determined by comparing the graphs of two functions—the model of the contextual scenario and the graph of the function with all appropriate transformations except for the horizontal translation—to determine the distance between the x-values of their minimum or maximum values. For the Ferris wheel scenario, this parameter has the value 7.5, which indicates how long a rider is on the Ferris wheel before they reach the height of the center of the Ferris wheel.

- At this point, you may want to have students work through another example of constructing a sinusoidal function model of circular motion. You could use problem 1 on **Handout 4T.7.B: Modeling Periodic Phenomena** for this purpose.
- You could provide students with time to summarize their new understandings about sinusoidal functions by creating a graphic vocabulary organizer (Handout 1.1) for Transformations of Sinusoidal Functions.
- In Part 3 of the lesson, students are given an algebraic representation of a function that models a real-world context and asked to interpret the meaning of the parameters.

Part 3: Interpreting a Model for Periodic Phenomena

~60 MIN.

In this part of the lesson, students use sinusoidal functions to model real-world periodic phenomena that do not involve circular motion. Students interpret the parameters of the algebraic representation of a sinusoidal function to make sense of the scenario.

Instructional Rationale

Because either the sine or the cosine function can be used to model periodic phenomena that do not involve circular motion, it is important for students to compare the graphs of cosine and sine functions using a graphing utility and adjust the parameters appropriately. Experimenting with the parameters helps them observe that the graphs of the sine and cosine functions are phase shifts of each other.

Student Task

An amateur oceanographer records the height of the tide in Ocean City, New Jersey, every hour and models her data with the sinusoidal function g, defined as $g(x) = 1.4\cos\big(0.524(x-0.725)\big) + 2.1$, where x is the number of hours since midnight and $g(x)$ is the height of the tide in feet.

(a) Construct a graph of g over an appropriate domain. What domain did you use and why?

(b) What is the period of g? What does this tell us about the phenomenon?

(c) What are the minimum and maximum values of g? Explain what they mean in the context of the problem.

(d) What is the amplitude of g? What does this tell us about the context?

(e) It is possible to express g as a sine function instead of a cosine function. What is an algebraic representation of g expressed with a sine function?

Facilitating the Task

- Begin by displaying the prompt and questions of the Student Task. Students should have access to a graphing utility, such as Desmos, to successfully engage with this problem. Allow them time to closely read the prompt and ask clarifying questions about the context.
- It is important to be sensitive to the fact that some students may not have much familiarity with oceans. To ensure that all students can develop appropriate mental models of this phenomenon, you could consider showing a short video about ocean tides or explaining to students that throughout the course of a day the tide—the

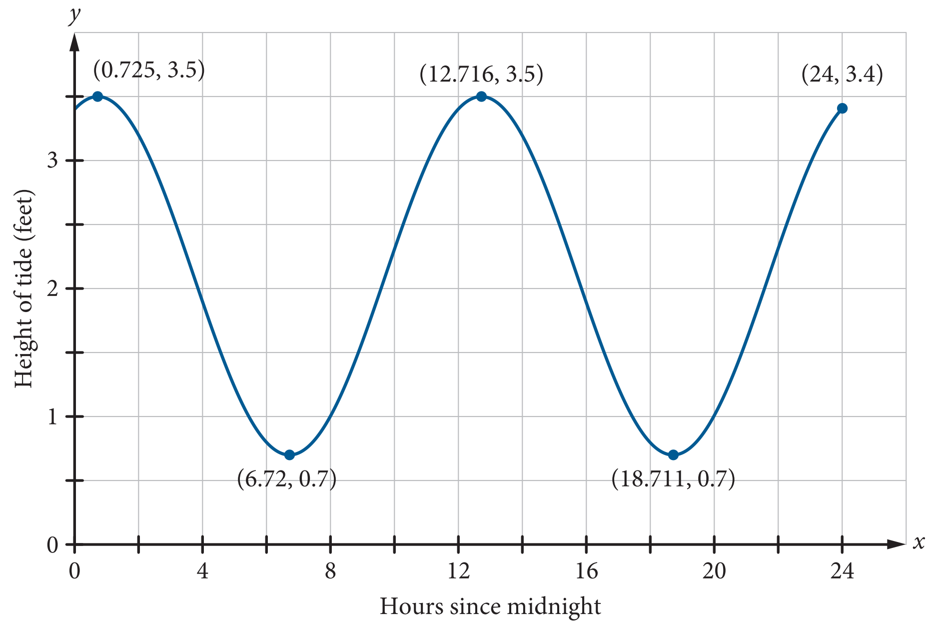
level of the water in the ocean—increases and decreases. There are two high tides and two low tides each day. People visiting the beach for a long period of time often keep track of the tides to make sure they don't set up their belongings too close to the water's edge, so their belongings do not get wet during high tide.

- Have students work with a partner or in small groups to answer the questions about the high-tide scenario. The problem is designed to help students formalize their understanding of how the parameters of the algebraic representation of a function can be used to make sense of the context.

Summarizing the Task

- As they complete the questions, have each pair of students or small group compare their solutions with those of another pair of students or small group. Allow time for them to discuss and refine their solutions, if necessary.
- Be sure to debrief solutions with students to address any concerns they have about interpreting the parameters of a cosine function.
  - Construct a graph of g over an appropriate domain. What domain did you use and why?

    A graph of g over a domain of one day is shown in the figure. It makes sense to graph the function over one day because there are two high tides and two low tides in one day.



- What is the period of function g? What does this tell us about the situation?

  The b parameter of the function is 0.524, which means that the relationship $0.524 = \dfrac{2\pi}{\text{period}}$ can be used to determine the period. The period is $\dfrac{2\pi}{0.524} \approx 12$. This makes sense in the context of the problem because pairs of high tides (and pairs of low tides) are about 12 hours apart.

- What are the minimum and maximum values of g? Explain what they mean in the context of the problem.

  The graph indicates that the maximum value of the function is 3.5, which means that the maximum height of the tide is 3.5 feet. The minimum value of the function is 0.7, which means that the minimum height of the tide is 0.7 feet.

- What is the amplitude of g? What does this tell us about the context?

  The a parameter of the function is 1.4. This indicates that the amplitude is 1.4 feet. It also tells us that twice the amplitude, 2.8 feet, is the difference between the heights of the high and low tide.

- It is possible to express g as a sine function instead of a cosine function. What is an algebraic representation of g expressed with a sine function?

  There are many sine functions that produce the same graph as $g(x) = 1.4\cos(0.524(x - 0.725)) + 2.1$. One such function is $f(x) = 1.4\sin(0.524(x + 2.273)) + 2.1$.

- Have students share the algebraic representations of their sine functions with the class. It is valuable to note that graphs of the sine and cosine functions are phase shifts of each other.
- If students have not yet summarized their understanding of transformations of sinusoidal functions through a graphic vocabulary organizer, this would be an appropriate time to have them complete one.
- At this point, you may want to have students construct a sinusoidal function model of another periodic phenomenon that does not involve circular motion. You could use problem 2 on Handout 4T.7.B for this purpose.

Meeting Learners' Needs

If time allows, you could challenge students to determine the magnitude of the phase shift between the graphs of the parent sine and cosine functions. That is, have students investigate what horizontal translation needs to be applied to the graph of $y = \sin(x)$ so that it coincides with the graph of $y = \cos(x)$, and vice versa. After some experimentation, they should determine that a horizontal translation by about 1.57 units is correct. This magnitude corresponds to $\dfrac{\pi}{2}$.

UNIT 4T

Assess and Reflect on the Lesson

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

The depth of the ocean water at the end of an industrial pier near Pensacola, Florida, changes with the tides. Suppose that on a particular day, high tide occurs at 4:15 a.m. with a water depth of 5.2 meters. Low tide occurs at 10:30 a.m. with a water depth of 2 meters.

(a) Write an algebraic representation of a sinusoidal function that models the depth of the water as a function of the number of hours after midnight.

One possible function is $f(x) = 1.6\cos(0.503(x - 4.25)) + 3.6$, where x is the number of hours since midnight and $f(x)$ is the depth of the water in meters.

(b) For your function, explain how you determined the values of the parameters.

To determine the midline (vertical translation) we can find the value halfway between the maximum and minimum values, which is 3.6. To determine the amplitude, we can determine half the difference of the maximum and minimum values, which is 1.6. To determine the period, we can determine that if it is 6 hours and 15 minutes between high and low tides, then it is 12.5 hours for one full cycle. This means the value of the b parameter is determined by $\frac{2\pi}{12.5} \approx 0.503$. To determine the phase shift, we can compare the graphs of the function $g(x) = 1.6\cos(0.503x) + 3.6$ with a sketch of the graph of f to observe that the x-values differ by 4.25 units.

(c) Use your function to determine the depth of the water at noon.

The graph of the function shows that at noon, 12 hours after midnight, the height of the tide is 2.437 meters.

HANDOUTS

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 4T.7.A: Translations of a Circular Motion Model

1. (a) The maximum height that a car on the Ferris wheel reaches is 550 feet.

   (b) The diameter of the Ferris wheel is 520 feet. The radius of the Ferris wheel is 260 feet.

(c) The ground is 30 feet from the bottom of the Ferris wheel.

(d) The center of the Ferris wheel is 290 feet from the ground.

2. (a) Mira will reach a maximum height of 550 feet above the ground.

(b) Mira will reach a minimum height of 30 feet above the ground.

(c) Mira's maximum and minimum heights are the maximum and minimum values of the function f, which models her height above the ground.

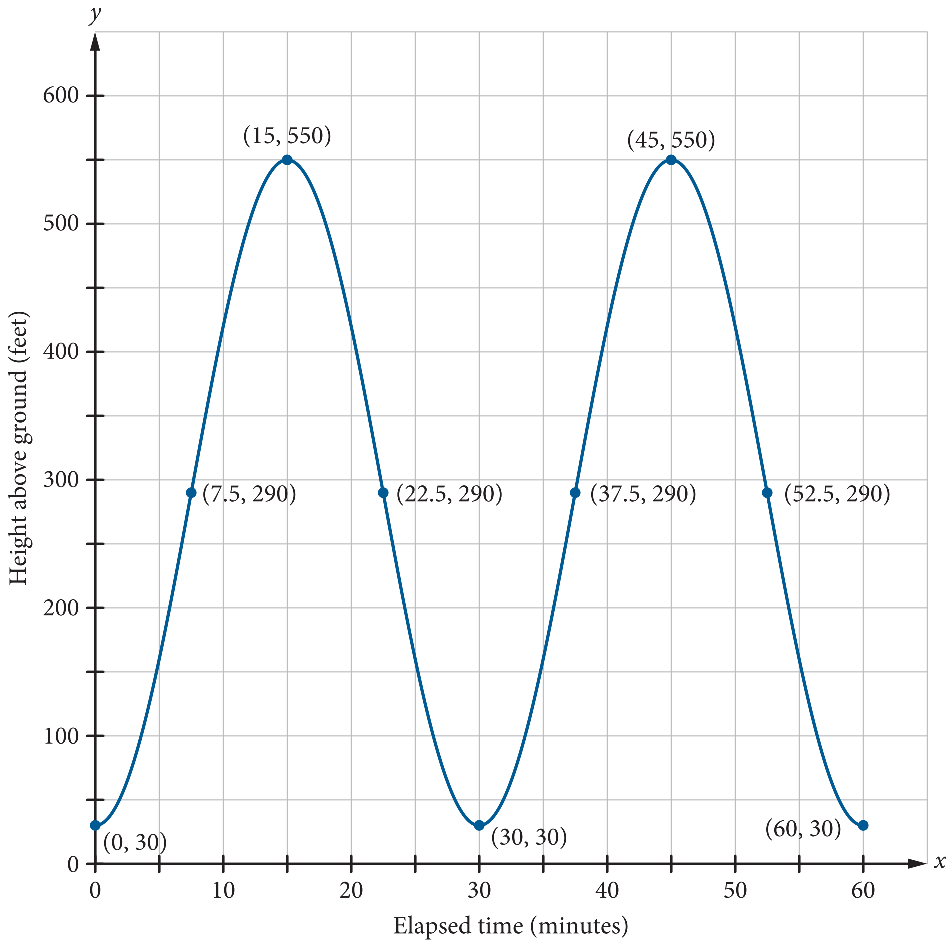
3. (a) Mira would be at the same height as the center of the Ferris wheel 7.5 minutes after she first boards the ride.

(b) Mira would reach the maximum height above the ground after 15 minutes.

(c) Mira would reach the minimum height above the ground after 30 minutes.

(d) If one ride consists of two revolutions that last a total of one hour, then a reasonable interval of input values for the function is from 0 to 60 minutes.

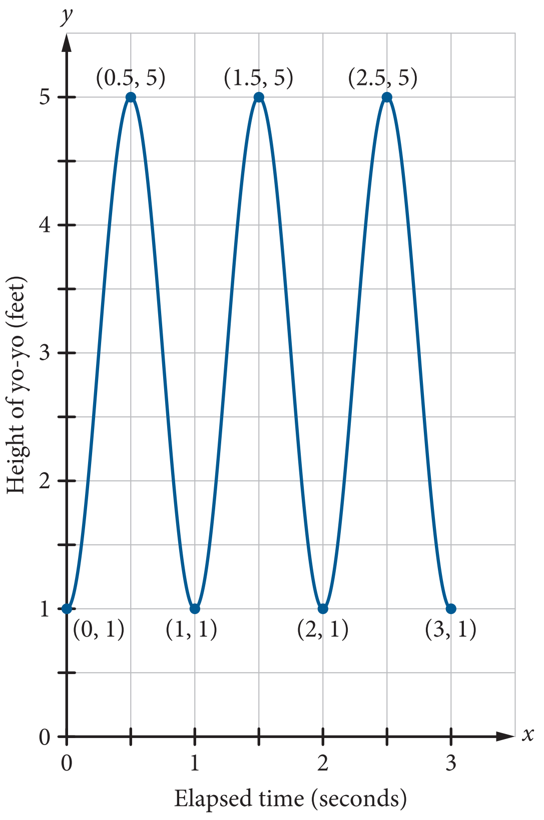
4. A graph that models Mira's height above the ground in terms of the number of minutes elapsed is shown in the figure.



5. If Mira completes a second revolution on the Ferris wheel, then the minimum and maximum values will be the same. Also, the time it takes the Ferris wheel to complete a second revolution would be the same.
6. It is reasonable to conclude that a sinusoidal function would produce a graph like the one constructed in problem 4 because sinusoidal functions have graphs that are wave-like and repeat in predictable patterns.

Handout 4T.7.B: Modeling Periodic Phenomena

1. (a) A graph of the function follows.



(b) The vertical translation of the function is 3 units up because 3 is halfway between the minimum and maximum values of the function. The amplitude of the function is 2 feet because that is half the difference between the minimum and maximum values. The period of the function is 1 second because the trick is performed three times in 3 seconds, so one cycle occurs over 1 second. For a transformation of a sine function, the phase shift is 0.25 units to the right. For a transformation of a cosine function, the phase shift is 0.5 units to the right.

(c) An algebraic representation of the function could be $y = 2\sin(2\pi(x - 0.25)) + 3$ or $y = 2\cos(2\pi(x - 0.5)) + 3$.

Guiding Student Thinking

You can support students in determining the solutions for question 1(d) by encouraging them to graph their function model without horizontal translation and then use the graph to determine a reasonable value for the phase shift.

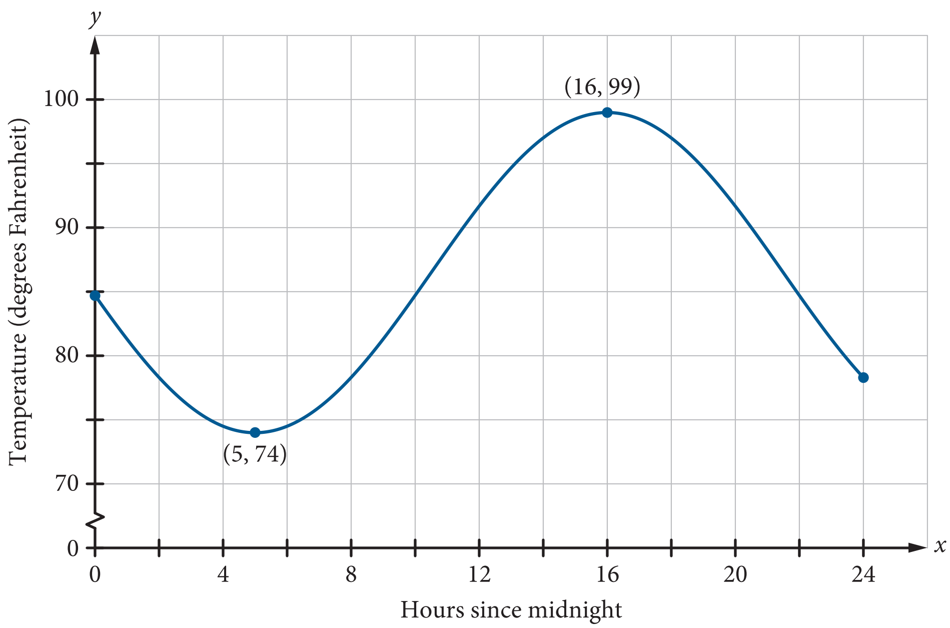
(d) The height of the yo-yo at 0.75 seconds is 3 feet.

(e) All the times between 0 seconds and 3 seconds that the yo-yo is exactly 4 feet above the ground are $\frac{1}{3}, \frac{2}{3}, 1\frac{1}{3}, 1\frac{2}{3}, 2\frac{1}{3},$ and $2\frac{2}{3}$ seconds.

Guiding Student Thinking

You can support students in determining the solutions for question 1(e) by encouraging them to use the graph of their scenario to reason when the yo-yo is exactly 4 feet above the ground. It is not expected that students would explicitly solve either of the equations $2\sin(2\pi(x - 0.25)) + 3 = 4$ or $2\cos(2\pi(x - 0.5)) + 3 = 4$ to determine the answer to the problem.

2. (a) A graph of the function follows.



(b) This phenomenon is well modeled by a sinusoidal function because the output of the function oscillates between 74 and 99.

(c) The vertical translation of the function is 86.5 because that is halfway between the minimum and maximum values. The amplitude is 12.5 because that is half the difference between the maximum and minimum values. The period is 22 because the length of half of a cycle is 11 hours. A possible phase shift for a sine function is 10.5 units to the right and a possible phase shift for a cosine function is 16 units to the right. Therefore, an algebraic representation of the function could be $y = 12.5\sin\left(\frac{2\pi}{22}(x - 10.5)\right) + 86.5$ or $y = 12.5\cos\left(\frac{2\pi}{22}(x - 16)\right) + 86.5$.

Guiding Student Thinking

You can support students in determining the solutions for question 2(c) by encouraging them to graph their function model without horizontal translation and then use the graph to determine a reasonable value for the phase shift.

Unit 4T

Performance Task

PERFORMANCE TASK

Modeling Hours of Sunlight with a Trigonometric Function

LEARNING OBJECTIVES

4T.1.3 Construct a representation of a sinusoidal function.

4T.1.5 Identify key characteristics of a sinusoidal function.

PERFORMANCE TASK DESCRIPTION

In this performance task, students model the number of hours of sunlight at different times of year in Anchorage, Alaska. Students construct a sinusoidal function model and then use their model to answer questions about the scenario. Sinusoidal functions are appropriate models for periodic phenomena because the parameters of these functions can be used to make sense of such scenarios. This task requires students to form conjectures about the parameters of a sinusoidal function model, and then compare their conjectures to the regression equation they develop. The task also introduces students to techniques for finding solutions of trigonometric equations. Students are not expected to solve trigonometric equations by hand. Instead, they estimate solutions using a graph of the function model and the periodicity of sinusoidal functions.

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Connections Among Multiple Representations

SUGGESTED TIMING

~45 minutes

MATERIALS

- access to a graphing utility, like **Desmos.com**

HANDOUT

- Unit 4T Performance Task: Modeling Hours of Sunlight with a Trigonometric Function

AP Connections

This performance task supports AP preparation through alignment to the following AP Calculus Mathematical Practices:

- **2.B** Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.

This performance task supports AP preparation through alignment to the following AP Statistics Course Skills:

- **4.B** Interpret statistical calculations and findings to assign meaning or assess a claim.

ELICITING PRIOR KNOWLEDGE

The goal of this task is for students to demonstrate their understanding of sinusoidal functions as models for periodic phenomena. The task requires students to use their graphing utility to determine a regression equation for a data set that exhibits periodic behavior. The task also requires students to interpret the parameters of their sinusoidal function models.

- To begin, introduce students to the performance task.
- To prepare students to engage in the task, you could ask the following questions:
  - The rise and fall of water levels in coastal regions is due primarily to two forces: the force of gravitation exerted by the moon (and sun) upon Earth, and centrifugal forces produced by the rotation of Earth and the moon. The time at which the water level is at its greatest height is called high tide, while the time at which the water level is at its lowest height is called low tide. On January 1, 2020, high tide for Miami Beach, Florida occurred at 1:00 a.m. when the water level was 2.0 feet. Exactly 6 hours later, low tide occurred when the water level was 0.4 feet. Assume that the next high tide occurred at exactly 1:00 p.m. and that this tidal cycle repeats. If we want to model this scenario with a function, what would be the independent variable and what would be the dependent variable? Explain your choices.

    The independent variable would be the time in hours since midnight, and the dependent variable would be the water level in feet.

  - Why would a sinusoidal function be an appropriate model for the relationship between these variables?

    A sinusoidal function would be an appropriate model for this relationship because the water level rises and falls in a predictable way and repeats periodically.

  - Write a sinusoidal function model for the data and explain the meaning of the parameters in the context of this scenario.

    A general sinusoidal function has the form $y = a\sin(b(x-c)) + d$ or $y = a\cos(b(x-c)) + d$. The amplitude is half the difference between the minimum and maximum value, so $a = 0.8$. The vertical shift, or midline, is halfway between the minimum and maximum values, so $d = 1.2$. For this scenario, the period is the length of the interval of the time (independent variable) over which the height (dependent variable) completes one cycle of repeating values. So, the period is 12 hours because the cycle of heights repeats every 12 hours. The parameter b is the ratio of the period of an unscaled sinusoidal function, 2π, and the period of the scenario being modeled, 12. Therefore, $b = \frac{2\pi}{12} \approx 0.524$. A graphical analysis of a sine function with this period, amplitude, and vertical translation shows the phase shift to be a horizontal translation of -1.998 hours, so $c = -1.998$. A graphical analysis of a cosine function with this period, amplitude, and vertical translation shows that the phase shift is a horizontal translation of 1 hour, so $c = 1$. Therefore, a sinusoidal function model for this scenario could be $y = 0.8\sin(0.524(x + 1.998)) + 1.2$ or $y = 0.8\cos(0.524(x - 1)) + 1.2$.

- If students struggle with the warm-up questions, it could indicate that they are not yet fully prepared to engage in the performance task. You may find it beneficial to provide a just-in-time review of modeling contextual scenarios with sinusoidal functions. Lessons 4T.4 and 4T.6 have sample questions you can use to help students get ready for the performance task.

SUPPORTING STUDENTS

Here are a few possible implementation strategies you can use to help students engage with the task. You should rely on knowledge of your students and your professional expertise to determine how to provide appropriate scaffolds while maintaining the cognitive demand of the task.

- **Previewing the Task:** To support students in identifying key features of the problem, you could display the introductory text and data and have students read them before they start working. Allow some time for students to ask clarifying questions about anything they read or observe, which may prevent students from working on the task with incorrect interpretations of the given information.
- **Collaboration:** This performance task could be particularly challenging for students if they have not engaged in data analysis problems at regular intervals throughout the course. Many students would benefit from working with a partner so that they can discuss their approach to the task and facilitate the data entry and other technology considerations.
- **Chunking the Task:** To support students who struggle with time management and may be overwhelmed by large tasks, you could chunk the task into several parts. For this task, parts (a) and (b) should be completed together. Next, parts (c) and (d) should be completed together. Finally, part (e) could be completed separately from the other parts or completed together with parts (c) and (d).

SCORING STUDENT WORK

Because this is a performance task and not a practice performance task, it is recommended that teachers score their students' work, rather than have each student score a peer's work. Be sure to use the results of the performance task to identify patterns and trends that can inform further instruction about modeling real-world scenarios with trigonometric functions.

Modeling Hours of Sunlight with a Trigonometric Function

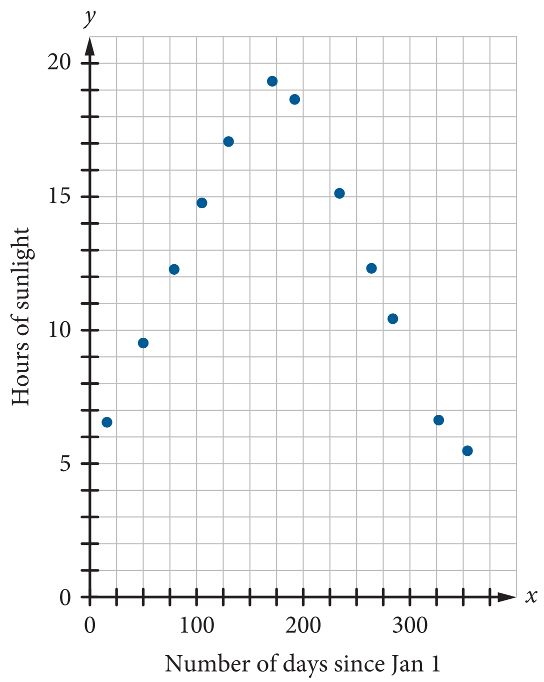
The following table shows the number of hours of sunlight in Anchorage, Alaska, for selected days of each year. The day number indicates the number of days since the start of the year, so day number 1 corresponds to January 1, day number 32 corresponds to February 1, and so on.

Day Number	Hours of Sunlight
16	6.55
50	9.52
79	12.28
105	14.77
130	17.07
171	19.33
192	18.65
234	15.13
264	12.32
284	10.43
327	6.63
354	5.48

There are 15 possible points for this performance task.

Student Stimulus and Part (a)

(a) Construct a scatterplot of the number of hours of sunlight versus the number of days since the start of the year. Make sure to identify which variable is displayed on which axis. Explain why a sinusoidal function would be an appropriate choice to model the data.

Sample Solution	Points Possible
A scatterplot of the data follows.  The variables in this problem are the day number, or the number of days since the start of the year, and the number of hours of sunlight in a day. The day number should be on the horizontal axis and the number of hours of sunlight should be on the vertical axis. The number of hours of sunlight increases from January to June and then decreases from June to December. A sinusoidal function would be an appropriate model because the number of hours of sunlight repeats each year in a predictable pattern.	**3 points maximum** 1 point for constructing a scatterplot (with or without a graphing utility or other technology) 1 point for including axes labels or descriptions of axes assignments 1 point for explaining why a sinusoidal function would be an appropriate model for the data

Student Stimulus and Part (b)

(b) Use the scatterplot to make conjectures about the amplitude, period, vertical translation, and phase shift of a sinusoidal function model of the data. Explain your reasoning in terms of the number of days since the start of the year and the number of hours of sunlight for that day in Anchorage, Alaska. Write a function model using parameters you determine from your conjectured amplitude, period, vertical translation, and phase shift.

Sample Solution	Points Possible
A general sinusoidal function model has the form $y = a\sin(b(x-c)) + d$ where a is the vertical dilation or amplitude, b is the horizontal dilation, which determines the period, c is the horizontal translation or phase shift, and d is the vertical translation. Based on the data table, the maximum number of hours of sunlight is about 19.33 and the minimum number of hours of sunlight is about 5.48. Therefore, a reasonable value for the amplitude is 6.925. Because the cycle of hours of sunlight repeats predictably each year, the period of the function is 365 days. Therefore, parameter b is the ratio of 2π and 365, or about 0.017. The vertical shift of the function is about 12.405, because this is halfway between the maximum and minimum hours of sunlight. Without the horizontal dilation, the sinusoidal function model has the form $y = 6.925\sin\left(\frac{2\pi}{365}(x-c)\right) + 12.405$. The graph of the model has its maximum at $x = 91.25$ rather than at $x = 171$. Therefore, a reasonable value for the phase shift is 79.75. *Scoring note*: Students can use a general sinusoidal model of the form $y = a\cos(b(x-c)) + d$ instead to earn all possible points for this part.	**4 points maximum** 1 point for determining a reasonable value for the amplitude and providing an explanation 1 point for determining a reasonable value for the period and providing an explanation 1 point for determining a reasonable value for the vertical shift and providing an explanation 1 point for determining a reasonable value for the phase shift and providing an explanation

Student Stimulus and Part (c)

(c) Use a graphing utility to calculate a regression equation for your model. Compare the graphs of the regression equation and your function model from part (b). Explain which graph models the data better.

Sample Solution	Points Possible
The sinusoidal regression equation is: $$y \approx 6.614\sin(0.017(x-78.166))+12.229$$ The regression equation better models the data because it more closely estimates the values of the data points than my function model. *Scoring note*: Students can state that either the regression equation or their initial model is better, but they must provide a reasonable explanation to support their response to receive the second point.	**2 points maximum** 1 point for determining a sinusoidal regression equation 1 point for comparing the regression equation and the initial model and providing a reasonable explanation

Student Stimulus and Part (d)

(d) The summer solstice occurs on June 20, or the 171st day of the year. Use your scatterplot and your function model to describe the significance of the summer solstice. Then compare the average rates of change for the time interval from the beginning of the year to the solstice and the time interval from the solstice to the end of the year. How do these average rates of change support your description of the significance of the summer solstice?

Sample Solution	Points Possible
Based on the graph and the data, the summer solstice is the day of the year with the greatest number of hours of sunlight. The time interval from day 1 to day 171 has a positive average rate of change and the time interval from day 171 to day 365 has a negative average rate of change. In context, this means that from January 1 through June 20, the number of hours of sunlight in a day is increasing, while from June 20 through December 31, the number of hours of sunlight is decreasing. Because the average rate of change switches from positive to negative at day 171, the summer solstice must be a maximum value.	**3 points maximum** 1 point for identifying that the number of hours of sunlight for the summer solstice is a maximum 1 point for describing the two average rates of changes 1 point for connecting the sign change in the average rates of change with the maximum value

UNIT 4T

Student Stimulus and Part (e)

(e) There are two days of the year that have 11 hours of sunlight in Anchorage, Alaska. Set up an equation that could be used to determine the first day of the year that has 11 hours of sunlight. Instead of solving this equation, use the graph of the function model to estimate the first day of the year that has 11 hours of sunlight. Then describe the key features of the function that you can use to determine the second day of the year that has 11 hours of sunlight.

Sample Solution	Points Possible
The equation $6.614\sin(0.017(x-78.166))+12.229=11$ can be solved to find the number of days since the start of the year with 11 hours of sunlight. Using the graph, we can estimate that $y=11$ when $x\approx 67$. Therefore, the 67th day of the year, or March 8, will have 11 hours of sunlight. The symmetry of the graph around the summer solstice can be used to estimate the other day with 11 hours of sunlight. Since there are 104 days from day 67 to day 171, we can determine that the $171 + 104 = 275$th day of the year, or October 2, will have 11 hours of sunlight as well.	**3 points maximum** 1 point for setting the y-value of the function model equal to 11 1 point for estimating the value of x for which $y=11$ from the graph 1 point for describing the symmetry of the sinusoidal function and for determining the second day of the year with 11 hours of sunlight using the distance from the first day of the year with 11 hours of sunlight to the summer solstice

PROVIDING FEEDBACK ON STUDENT WORK

After scoring your students' work, it is important to identify trends in their responses to inform further instruction. These trends should include topics that students consistently displayed mastery of, as well as conceptual errors that students commonly made. Possible trends and suggested guidance for each part of the task follow, although the patterns you observe in your classroom may differ.

(a) Students may find it challenging to explain why a sinusoidal function is an appropriate model for this data. The key observations for students to make are that the number of hours of sunlight oscillates predictably over the course of a year, and that this pattern repeats every year. This means that a sinusoidal function is an appropriate choice.

Teacher Notes and Reflections

(b) This part of the task is critical for students to develop an understanding of the meaning of the parameters of a sinusoidal function. Support hesitant students by having them spend additional time making sense of the problem and reasoning through the values of the parameters and what each represents graphically. As long as the values of the parameters that students determine are reasonable and students support their values with explanations, they should receive all possible points. They will have an opportunity in part (c) to revisit and revise their model.

Teacher Notes and Reflections

UNIT 4T

(c) Some students may have difficulty evaluating which model is better, or they may believe that a regression equation will always be better, but not be able to provide a reason why. This is a good opportunity for students to examine pairs of function models graphed with scatterplots of data sets. Students can craft arguments about which functions are appropriate or inappropriate models of the data.

Teacher Notes and Reflections

(d) This part of the task requires students to make connections between the average rate of change and the maximum and minimum values of the function. If students struggle to make these connections, they may benefit from revisiting the average rate of change problems presented in Key Concept 1.2, and analyzing the graphs of polynomial functions, such as those presented in Lesson 3.4.

Teacher Notes and Reflections

(e) If students are using a graphing utility, they may want to find the intersection points of the line $y = 11$ and the sinusoidal function model. Encourage them to reason through the symmetry of the graph to first estimate the x-coordinate of the second intersection point and then use the graph to confirm their conjecture.

Teacher Notes and Reflections

Try to assure students that converting their score into a percentage does not provide an accurate measure of how they performed on the task. You can use the following suggested score interpretations with students to discuss their performance.

Points Received	How Students Should Interpret Their Score
13 to 15 points	"I know all of these algebraic concepts really well. This is top-level work. (A)"
10 to 12 points	"I know all of these algebraic concepts well, but I made a few mistakes. This is above-average work. (B)"
7 to 9 points	"I know some of these algebraic concepts well, but not all of them. This is average-level work. (C)"
4 to 6 points	"I know only a little bit about these algebraic concepts. This is below-average work. (D)"
0 to 3 points	"I don't know much about these algebraic concepts at all. This is not passing work. (F)"

PERFORMANCE TASK

Modeling Hours of Sunlight with a Trigonometric Function

The following table shows the number of hours of sunlight in Anchorage, Alaska, for selected days of each year. The day number indicates the number of days since the start of the year, so day number 1 corresponds to January 1, day number 32 corresponds to February 1, and so on.

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192	18.65
234	15.13
264	12.32
284	10.43
327	6.63
354	5.48

(a) Construct a scatterplot of the number of hours of sunlight versus the number of days since the start of the year. Make sure to identify which variable is displayed on which axis. Explain why a sinusoidal function would be an appropriate choice to model the data.

(b) Use the scatterplot to make conjectures about the amplitude, period, vertical translation, and phase shift of a sinusoidal function model of the data. Explain your reasoning in terms of the number of days since the start of the year and the number of hours of sunlight for that day in Anchorage, Alaska. Write a function model using parameters you determine from your conjectured amplitude, period, vertical translation, and phase shift.

(c) Use a graphing utility to calculate a regression equation for your model. Compare the graphs of the regression equation and your function model from part (b). Explain which graph models the data better.

(d) The summer solstice occurs on June 20, or the 171st day of the year. Use your scatterplot and your function model to describe the significance of the summer solstice. Then compare the average rates of change for the time interval from the beginning of the year to the solstice and the time interval from the solstice to the end of the year. How do these average rates of change support your description of the significance of the summer solstice?

(e) There are two days of the year that have 11 hours of sunlight in Anchorage, Alaska. Set up an equation that could be used to determine the first day of the year that has 11 hours of sunlight. Instead of solving this equation, use the graph of the function model to estimate the first day of the year that has 11 hours of sunlight. Then describe the key features of the function that you can use to determine the second day of the year that has 11 hours of sunlight.

UNIT 4T

LESSON 4T.8

The Tangent Function

LEARNING OBJECTIVES

4T.2.1 Construct a representation of a tangent function.

4T.2.2 Identify key characteristics and values of functions that are defined by quotients of sinusoidal functions.

LESSON OVERVIEW

CONTENT FOCUS

In this lesson, students define the tangent function as the relationship between the measure of the angle that a line forms with the positive x-axis and the slope of that line. Students explore the relationship between the tangent function and the sine and cosine functions, which leads to an alternate, equivalent definition of the tangent function. They also investigate the characteristics of the tangent function, including the asymptotic behavior of its graph and the periodic nature of its output values.

AREAS OF FOCUS

- Connections Among Multiple Representation
- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~120 minutes

MATERIALS

- graphing utility, like **Desmos.com**

HANDOUT

Lesson

- 4T.8: Modeling the Slope of a Ray of Light

LESSON DESCRIPTION

Part 1: Defining the Tangent Function

In this part of the lesson, students explore the relationship between the slope of a ray of light emitted from a lighthouse and the angle it makes with a nearby bridge. This context is designed to help students observe that there is a function relationship between the slope of a line and the angle that the line makes with the positive x-axis. This part of the lesson culminates in students defining the tangent function.

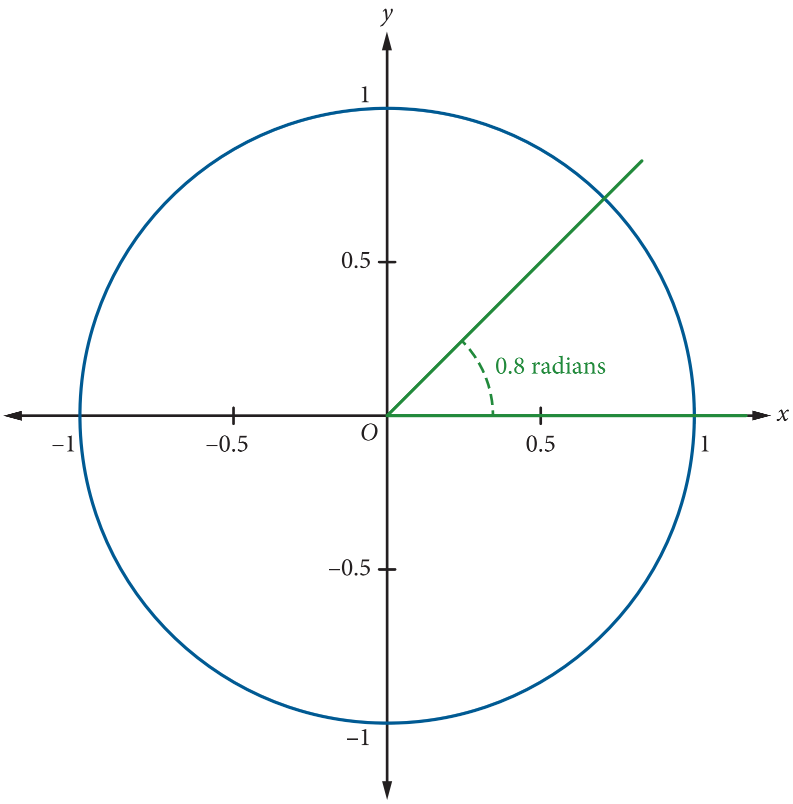
Part 2: Describing Characteristics of the Tangent Function

In this part of the lesson, students explore the characteristics of the tangent function by analyzing its graph. Students investigate the domain and range of the tangent function and identify two of its key features: the asymptotic behavior of the function's graph at odd integer multiples of $\frac{\pi}{2}$ and the periodic nature of its output values.

FORMATIVE ASSESSMENT GOAL

This lesson prepares students to complete the following formative assessment activity.

The central angle in this figure has a measure of 0.8 radians.



(a) What is the slope of the terminal ray shown in the figure?

(b) Sketch an angle whose terminal ray has the same slope as the angle shown but is not coterminal to it.

(c) Determine the measure of the angle you sketched in part (b).

(d) What is the relationship between the measures of the two angles you sketched?

(e) What is the measure of an angle whose terminal ray is the opposite of the angle in the figure?

Part 1: Defining the Tangent Function

~45 MIN.

In this part of the lesson, students explore the relationship between the slope of a ray of light emitted from a lighthouse and the angle it makes with a nearby bridge. This context is designed to help students observe that there is a function relationship between the slope of a line and the angle that the line makes with the positive x-axis. This part of the lesson culminates in students defining the tangent function.

Warm-Up

- This brief warm-up is intended to get students thinking about how the slope of a line through the origin is related to the coordinates of a non-origin point the line passes through.
  - What is the slope of the line that passes through the points $(0, 0)$ and $(2, 1)$?

    The slope of the line is $\frac{1}{2}$.
  - What is the slope of the line that passes through the points $(0, 0)$ and $(-2, 5)$?

    The slope of the line is $\frac{5}{-2}$.
  - What is the slope of the line that passes through the points $(0, 0)$ and $(3, -7)$?

    The slope of the line is $\frac{-7}{3}$.
  - What is the slope of the line that passes through the points $(0, 0)$ and (x, y)?

    The slope of the line is $\frac{y}{x}$. This is because the change in y is the y-value and the change in x is the x-value.
- The takeaway for this warm-up should be that a line that contains the points $(0, 0)$ and (x, y) has slope $\frac{y}{x}$.

Student Task

The student task is provided on **Handout 4T.8: Modeling the Slope of a Ray of Light**. In this task, students explore the context of a ray of light emitted from a lighthouse and use trigonometry to determine the slope of the ray of light. Solutions for the handout are provided in the Assess and Reflect section of the lesson.

Facilitating the Task

- Begin by providing Handout 4T.8 to each student and access to a graphing utility such as Desmos. Arrange students in pairs or in

Classroom Ideas

If students are not familiar with a lighthouse beacon you can consider showing them a video or the Desmos animation **preap.org/Desmos-Lighthouse**.

small groups of three and allow them time to closely read the handout and ask clarifying questions about the context.

- To get students thinking about the context, ask students to write a list of the aspects of the ray of light that do and do not change as the beacon rotates. Students might consider the angle measure the ray makes relative to the bridge, the brightness of the light, or the steepness of the ray of light relative to the ground. If no student suggests it, ask if the slope of the ray of light relative to the bridge changes as the beacon rotates.
- You may want to explain to students that since the task involves the slope of the ray of light, the figure on Handout 4T.8 is situated on a coordinate plane so that it is possible to describe points along the circular path of the ray. Let students know that the ray of light is the terminal ray of an angle in standard position, with the bridge serving as the initial ray.
- Now have students work collaboratively on the problems on Handout 4T.8. If pairs of students finish quickly, have them compare their solutions with those of another pair of students and allow them some time to make revisions, if necessary.
- As students reach problem 5 on the handout, you may want to suggest that they sketch various lines with positive, negative, zero, and undefined slopes to help them visualize the slope in each quadrant.

Meeting Learners' Needs
If students are struggling to recognize that any slope must occur twice in one revolution for each angle measure, you can use the Desmos animation **preap.org/Desmos-Lighthouse** to help illustrate this idea.

Summarizing the Task

- Ask students some debriefing questions like the ones that follow, to spur a conversation about the task:
  - How can we express the slope of a line through the origin using trigonometry?

    Because the slope of a line through the origin can be determined by the ratio of the y-coordinate to the x-coordinate, we can express the slope of the line as the ratio of the sine of the angle to the cosine of the angle.
  - Is it true that the slope of a line through the origin is a function of the angle measure? Why or why not?

    It is true because each slope occurs twice in one revolution, but each angle measure corresponds to only one slope. Therefore, the slope of a line through the origin is a function of the angle measure.

- What did you observe about terminal rays (lines through the origin) that have positive slopes, negative slopes, zero slopes, or undefined slopes?

  Terminal rays that lie in quadrants I and III have positive slopes while rays that lie in quadrants II and IV have negative slopes. Terminal rays that coincide with the x-axis have slopes of zero and terminal rays that coincide with the y-axis have undefined slopes.

- At this point, students are ready to define this function. The *tangent function* relates the measure of an angle in standard position to the slope of the terminal ray of the angle. For an angle in standard position, the tangent of the angle's measure is equal to the slope of the terminal ray. Because the slope of the terminal ray of an angle that measures θ radians can be determined by the ratio $\frac{\sin(\theta)}{\cos(\theta)}$, then the tangent of an angle's measure can also be expressed as $\frac{\sin(\theta)}{\cos(\theta)}$.
- In Part 2 of the lesson, students explore the behavior of the graph of the tangent function.

Part 2: Describing Characteristics of the Tangent Function

~45 MIN.

In this part of the lesson, students explore the characteristics of the tangent function by analyzing its graph. Students investigate the domain and range of the tangent function and identify two of its key features: the asymptotic behavior of the function's graph at odd integer multiples of $\frac{\pi}{2}$ and the periodic nature of its output values.

Student Task

Let's investigate the tangent function.

(a) What is the domain of the function? How do you know? Why does the domain have this property?

(b) What is the range of the function? How do you know? Why does the range have this property?

(c) What does the graph of the tangent function look like? What is the behavior of the graph?

(d) Is the tangent function periodic? If so, what is the period?

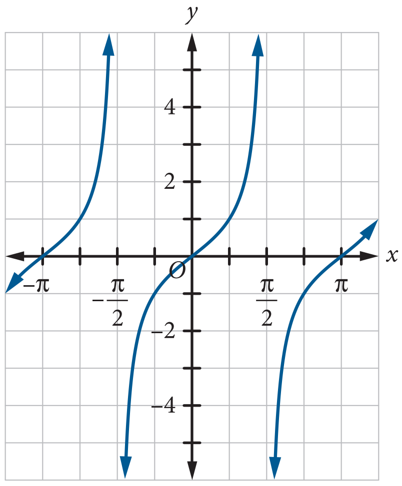
Facilitating the Task

- Be sure to display the task in a central location accessible to all students.
- Allow students some time to work with a partner or a small group to answer the questions.
- Encourage students to try to reason through problems (a) and (b) without looking at the graph of the tangent function and confirm or revise their answers after they graph the function in problem (c).
- Some students may benefit from creating a table of values for the slopes of the terminal rays of various angles. If students have access to a scientific calculator, they can use it to determine the slopes of the terminal rays of various angles. Then they can plot the ordered pairs formed by the angle measure and the tangent of the angle measure to get a sense of the domain, range, and graph of the function. You may need to remind students to investigate angle measures from 0 to 2π radians.
- One way to help students numerically observe that the tangent function has asymptotic behavior when the input is close to $\frac{\pi}{2}$ would be to have them create a table of values with angle measures close to but less than $\frac{\pi}{2}$, and angle measures close to but greater than $\frac{\pi}{2}$. Then they can repeat the process for angle measures close to $\frac{3\pi}{2}$ and $\frac{5\pi}{2}$. It might be easier for students to use decimal values rather than fractions close to $\frac{\pi}{2}$. Some example tables of values are shown.

θ	$\tan(\theta)$
1.55	48.078
1.56	92.62
1.565	172.521
1.569	556.691
1.57	1255.766
$\frac{\pi}{2}$	Undefined
1.571	−4909.826
1.572	−830.790
1.58	−108.649
1.59	−52.067
1.6	−34.233

θ	$\tan(\theta)$
4.70	80.713
4.705	135.334
4.709	295.073
4.71	418.587
4.712	2570.824
$\frac{3\pi}{2}$	Undefined
4.7125	−9007.417
4.715	−382.991
4.72	−131.386
4.73	−56.777
4.8	−11.385

- As students reach problem (c) and graph the tangent function with a graphing utility, encourage them to informally describe the graph and make note of any observations they make or wonderings they have.
- Students may have some sense that the tangent function is periodic because one way to express the tangent of an angle measure is as a ratio of the sine and cosine of that angle measure. Other students may not think that the tangent function is periodic because the graph does not repeat in the same manner as the graphs of the sine and cosine functions. It is important to capture all conjectures that students make about the periodicity of the tangent function so they can investigate the claims.



Summarizing the Task

- To summarize the task, engage students in a conversation about their observations and wonderings about the tangent function and its graph. You can use questions like the following:
  - What did you determine to be the domain of the tangent function? How did you determine this?

    The domain of the tangent function is all real numbers except odd integer multiples of $\frac{\pi}{2}$. A graphical analysis indicates breaks in the graph at odd integer multiples of $\frac{\pi}{2}$. A numerical analysis shows that when the input values are close to an odd integer multiple of $\frac{\pi}{2}$, the tangent takes on very large positive or large negative values.

- What values are excluded from the domain of the tangent function? Does that make sense to you? Why or why not?

  Odd integer multiples of $\frac{\pi}{2}$ are excluded from the domain of the tangent function. This makes sense because the tangent function can be thought of as the slope of a terminal ray, and when the terminal ray is vertical at an angle measure of an odd integer multiple of $\frac{\pi}{2}$, the slope is undefined.

- What did you determine to be the range of the tangent function? How did you determine this?

  The range of the tangent function is all real numbers. A graphical analysis shows that the range of y-values is from positive infinity to negative infinity.

- What values are excluded from the range of the tangent function? Does that make sense to you? Why or why not?

  There are no values excluded from the range of the tangent. This makes sense because there are no real number values that cannot be the slope of a line.

- What does the graph of the tangent function look like?

  The graph of the tangent looks like a series of elongated S shapes, a little like the graph of $y = x^3$, which repeats at regular intervals.

- What behavior does the graph of the tangent function exhibit near odd integer multiples of $\frac{\pi}{2}$? Why is that the case?

  At odd integer multiples of $\frac{\pi}{2}$, the tangent function is asymptotic and approaches positive infinity or negative infinity. This is because as a line becomes more vertical as the angle increases toward an odd integer multiple of $\frac{\pi}{2}$, its slope increases without bound, and as a line becomes more vertical as the slope decreases toward an odd integer multiple of $\frac{\pi}{2}$, its slope decreases without bound.

- Is the tangent function periodic? Why or why not?

  The tangent function is periodic because the graph repeats at regular intervals. Also, the slope of a line through the origin cycles from negative but nearly vertical to zero to positive but nearly vertical, and then repeats.

- What is the period of the tangent function? How do you know?

  The period of the tangent function is π because the function takes on values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. That is a distance of π radians.

- What are similarities in the characteristics of the tangent function and characteristics of the two sinusoidal functions, sine and cosine?

  The tangent function is periodic, just like sine and cosine. Also, the tangent function is defined in terms of aspects of circular motion, just like sine and cosine. The input of the tangent function is an angle measure, just like sine and cosine.

- What are differences in the characteristics of the tangent function and characteristics of the two sinusoidal functions, sine and cosine?

  The domain of the tangent function is different than the domain of sine and cosine functions. The range of tangent function is different than the range of sine and cosine functions. The period of tangent function is different than the period of sine and cosine functions.

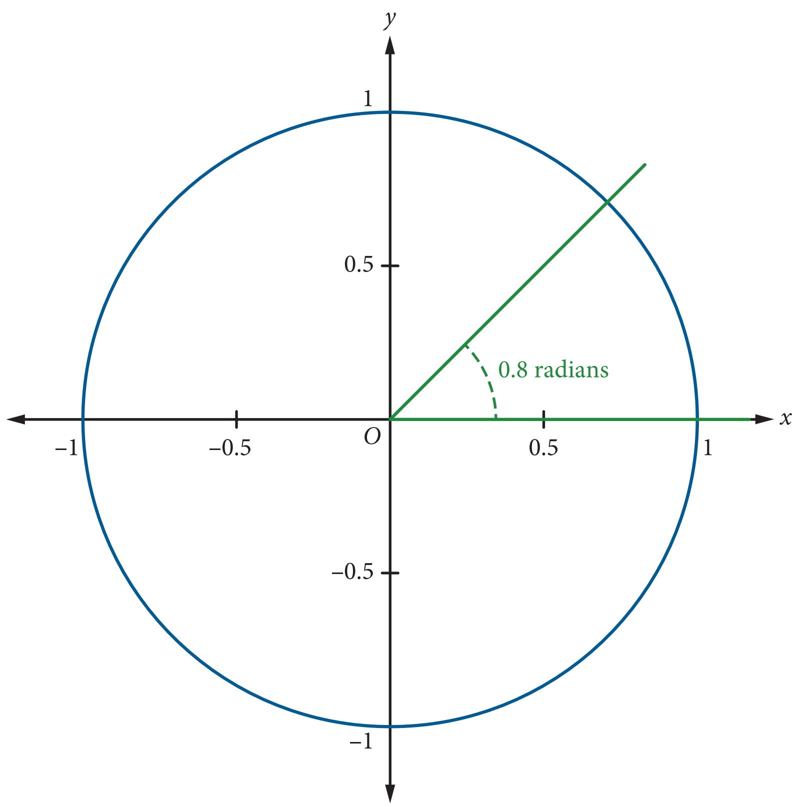
- This is a good opportunity for students to create a vocabulary graphic organizer for the tangent function. It is important that they identify some key features of the tangent function:
  - the period of the tangent function is π, when the input is measured in radians,
  - the domain excludes angle measures where the slope of the terminal ray is undefined, which are odd integer multiples of $\frac{\pi}{2}$, expressed algebraically as $(2k+1)\frac{\pi}{2}$ where k is an integer,
  - the graph has vertical asymptotes at these values excluded from the domain of the function, and
  - the range of the function is all real numbers because the slope of the terminal ray increases without bound as it becomes increasingly more like a vertical line.

Assess and Reflect on the Lesson

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

The central angle in this figure has a measure of 0.8 radians.

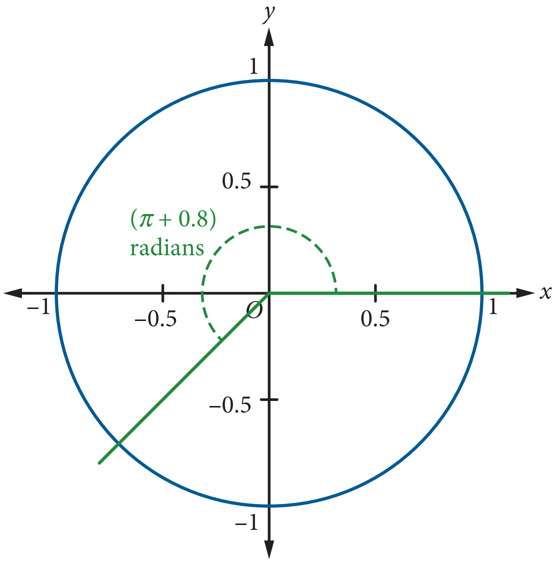


(a) What is the slope of the terminal ray shown in the figure?

The slope of the terminal ray is $\tan(0.8) \approx 1.03$.

(b) Sketch an angle whose terminal ray has the same slope as the angle shown but is not coterminal to it.

An angle whose terminal ray has the same slope as the given angle, but is not coterminal to it, is displayed in the following figure:



(c) Determine the measure of the angle you sketched in part (b).

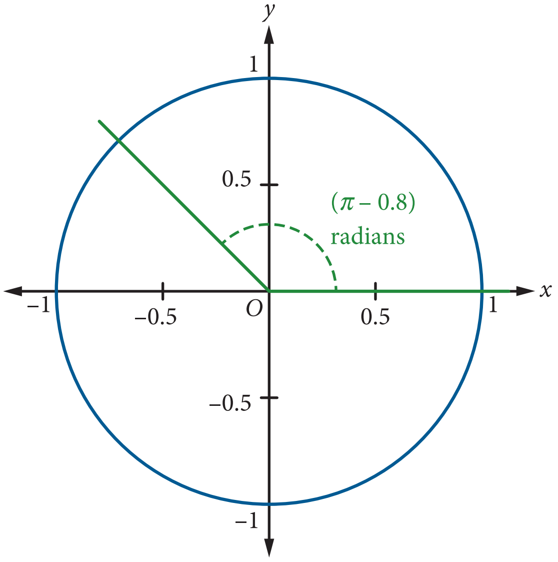
The measure of the angle sketched in part (b) is $(\pi + 0.8)$ radians.

(d) What is the relationship between the measures of the two angles you sketched?

The angle measures differ by π radians.

(e) What is the measure of an angle whose terminal ray is the opposite of the angle in the figure?

An angle whose terminal ray has a slope that is the opposite of the given angle is $(\pi - 0.8)$ radians, as displayed in the following figure.



HANDOUT

To supplement the information within the body of the lesson, additional answers and guidance on the handout are provided below.

Handout 4T.8: Modeling the Slope of a Ray of Light

1. The coordinates of point P are approximately $(0.6216,\ 0.7833)$. We can determine this because the coordinates of the point are given by $(\cos(0.9),\ \sin(0.9))$.
2. The slope of the ray of light is $\frac{0.7833}{0.6216} \approx 1.26$. This can also be expressed as $\frac{\sin(0.9)}{\cos(0.9)}$.
3. The slope of the ray of light after the beacon rotates 2.4 radians can be expressed as $\frac{\sin(2.4)}{\cos(2.4)}$, which is approximately -0.9156.
4. The slope of the ray of light after the beacon rotates 4.7 radians can be expressed as $\frac{\sin(4.7)}{\cos(4.7)}$, which is approximately 80.713.
5. The ray of light forms angles from 0 to 2π radians as it completes a full revolution.
6. The slope of the ray of light is positive when the angle measures are between 0 radians and $\frac{\pi}{2}$ radians, which corresponds to quadrant I, and is positive again when the angle measures are between π radians and $\frac{3\pi}{2}$ radians, which corresponds to quadrant III. The slope of the ray of light is negative when the angle measures are between $\frac{\pi}{2}$ radians and π radians, which corresponds to quadrant II, and is negative again when the angle measures are between $\frac{3\pi}{2}$ radians and 2π radians, which corresponds to quadrant IV. The slope of the ray of light is zero when the angle measures 0 radians or 2π radians. The slope of the ray of light is undefined when the angle measures $\frac{\pi}{2}$ radians or $\frac{3\pi}{2}$ radians.

UNIT 4T

PRACTICE PERFORMANCE TASK

Connecting Circles, Triangles, and Line Segments

LEARNING OBJECTIVES

4T.2.1 Construct a representation of a tangent function.

4T.2.2 Identify key characteristics and values of functions that are defined by quotients of sinusoidal functions.

PRACTICE PERFORMANCE TASK DESCRIPTION

In this practice performance task, students relate the lengths of line segments drawn in a circle on a coordinate plane to the measures of angles in standard position. This practice performance task provides students with an opportunity to connect their understanding of right triangle trigonometry and similarity with the definitions of the sine, cosine, and tangent functions that they developed throughout the unit.

AREAS OF FOCUS

- Engagement in Mathematical Argumentation
- Connections Among Multiple Representations

SUGGESTED TIMING

~45 minutes

HANDOUT

- Unit 4T Practice Performance Task: Connecting Circles, Triangles, and Line Segments

AP Connections

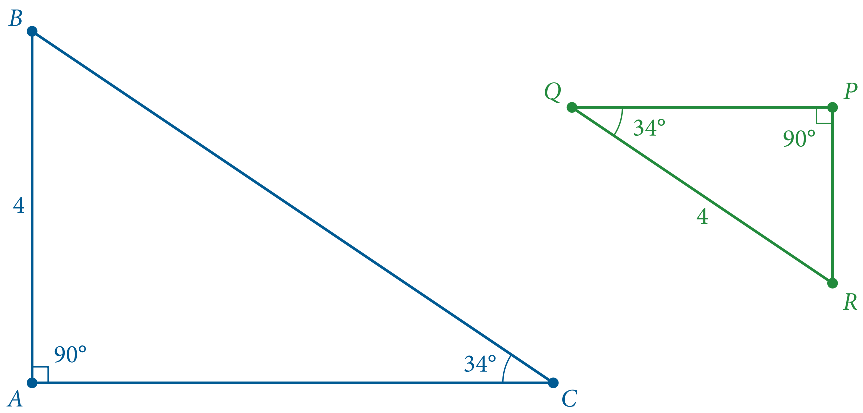
This performance task supports AP preparation through alignment to the following AP Calculus Mathematical Practices:

- **2.B** Identify mathematical information from graphical, numerical, analytical, and/or verbal representations.
- **4.A** Use precise mathematical language.

ELICITING PRIOR KNOWLEDGE

The goal of this task is for students to relate the sine, cosine, and tangent functions to the measures of angles in right triangles on a coordinate plane. This task requires students to engage in mathematical argumentation across algebra and geometry.

- To begin, introduce students to the practice performance task.
- You can prepare students to engage in the task by posing a few questions designed to elicit the knowledge students will need to be successful in this practice performance task, such as:



- Are triangles *ABC* and *PRQ* similar? How do you know?

  Triangles *ABC* and *PRQ* are similar. Both triangles are right and each has a 34° angle, so they must each have a 56° degree angle as well. If the three angles of one triangle are congruent to the three angles of another triangle, then the triangles are congruent.

- How could you determine the length of $\overline{AC}$?

  We could determine the length of $\overline{AC}$ using right triangle trigonometry. That is, $\tan(34°) = \dfrac{4}{\text{length of } \overline{AC}}$. Therefore, length of $\overline{AC} = \dfrac{4}{\tan(34°)} \approx 5.930$.

- Show two different ways to determine the length of $\overline{PR}$.

  One way to determine the length of $\overline{PR}$ is using right triangle trigonometry. That is, $\sin(34°) = \dfrac{\text{length of } \overline{PR}}{4}$. Therefore, length of $\overline{PR} = 4\sin(34°) \approx 2.237$.

  Alternatively, the length of $\overline{PR}$ could be determined with a proportion: $\dfrac{\text{length of } \overline{AB}}{\text{length of } \overline{AC}} = \dfrac{\text{length of } \overline{PR}}{\text{length of } \overline{PQ}}$.

- If students struggle with the warm-up questions, it could indicate that they are not yet fully prepared to engage in the practice performance task. You may find it beneficial to provide a just-in-time review of right triangle trigonometry.

SUPPORTING STUDENTS

Here are a few possible implementation strategies you can use to help students engage with the task. You should rely on your knowledge of your students and your professional expertise to determine how to provide appropriate scaffolds while maintaining the cognitive demand of the task.

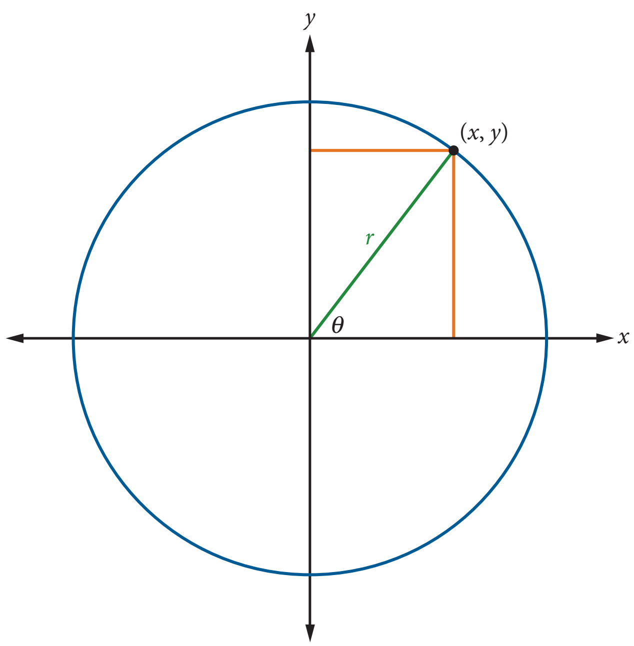
- **Collaboration:** Many students would benefit from discussing this problem with a partner. This is a purely mathematical task that requires substantial reasoning and sense-making as well as some symbolic manipulation to complete. There is ample work and enough potential discussion areas for two students, but in groups of more than two, some students may not have an opportunity to engage meaningfully in the task.
- **Chunking the Task:** You could chunk the task into two parts. Parts (a) and (b) involve line segments contained within a circle and should remind students of their work in Key Concept 4T.1. Parts (c) and (d) involve a line drawn tangent to a circle and rely on right triangle trigonometry and relationships between similar right triangles. This strategy could be particularly helpful for giving students an opportunity to focus on one way of thinking at a time.
- **Notation:** In part (d) of the scoring guidelines of the task, the notation $m\overline{OC}$ is used to indicate the length of $\overline{OC}$. This is consistent with the notation used in Pre-AP Geometry with Statistics. Some curriculum materials use the notation "OC" to mean the length of $\overline{OC}$. You should use whatever notation is most familiar to your students and not burden them with new notation at this time.

SCORING STUDENT WORK

Whether you decide to have students score their own solutions, have students score their classmates' solutions, or score the solutions yourself, you should use the results of the practice performance task to inform further instruction.

Connecting Circles, Triangles, and Line Segments

An acute angle, θ, in standard position is drawn inside a circle with radius r. Horizontal and vertical segments from the x- and y-axes extend to the point (x, y), at which the terminal ray of the angle intersects the unit circle. These two segments, along with angle θ and a circle with radius r, are shown in the following figure:



There are 11 possible points for this practice performance task.

Student Stimulus and Part (a)

(a) In terms of θ, identify each of the following quantities:

- the vertical distance from the point (x, y) to the x-axis
- the horizontal distance from the point (x, y) to the y-axis
- the slope of the terminal ray of the angle in standard position

Sample Solution	Points Possible
The vertical distance from the point (x, y) to the x-axis can be determined by $r\sin(\theta)$ because the circle has radius r and $\sin(\theta)$ is the vertical distance of a point on a unit circle from the x-axis. The horizontal distance from the point (x, y) to the x-axis can be determined by $r\cos(\theta)$ because the circle has radius r and $\cos(\theta)$ is the horizontal distance of a point on a unit circle from the y-axis. The slope of the terminal ray of the angle in standard position can be determined by the ratio $\frac{r\sin(\theta)}{r\cos(\theta)}$, or equivalently, $\tan(\theta)$.	**3 points maximum** 1 point for identifying the vertical distance from (x, y) to the x-axis. 1 point for identifying the horizontal distance from (x, y) to the y-axis 1 point for identifying the slope of the terminal ray

Student Stimulus and Part (b)

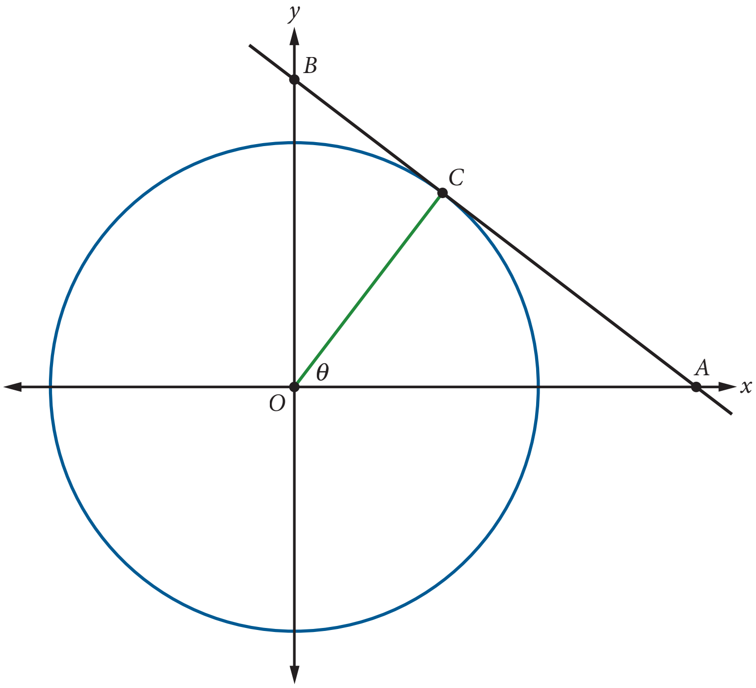
(b) Describe how the figure can be used to explain why the following relationships are true:

- the quotient identity $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$
- the Pythagorean identity $\cos^2(\theta) + \sin^2(\theta) = 1$

Sample Solution	Points Possible
By right triangle trigonometry, the tangent of an angle measure is the ratio of the length of the side opposite the angle to the length of the side adjacent to the angle. This means that $\tan(\theta) = \dfrac{r\sin(\theta)}{r\cos(\theta)} = \dfrac{\sin(\theta)}{\cos(\theta)}$. Equivalently, the tangent of an angle measure in standard position is the slope of the terminal ray. From part (a), the slope of the terminal ray is $\dfrac{r\sin(\theta)}{r\cos(\theta)}$, which implies that $\tan(\theta) = \dfrac{r\sin(\theta)}{r\cos(\theta)} = \dfrac{\sin(\theta)}{\cos(\theta)}$. The Pythagorean identity can be explained by relating the lengths of the sides of the right triangle formed by the terminal ray, the vertical line segment from the intersection point of the terminal ray and the circle to the x-axis, and the horizontal line segment from the origin to the vertical line segment: $$\left[r\cos(\theta)\right]^2 + \left[r\sin(\theta)\right]^2 = r^2$$ This can be simplified as follows: $$\begin{aligned} \left[r\cos(\theta)\right]^2 + \left[r\sin(\theta)\right]^2 &= r^2 \\ r^2\cos^2(\theta) + r^2\sin^2(\theta) &= r^2 \\ r^2\left[\cos^2(\theta) + \sin^2(\theta)\right] &= r^2 \\ \cos^2(\theta) + \sin^2(\theta) &= 1 \end{aligned}$$	**2 points maximum** 1 point for explaining why the quotient identity is true 1 point for explaining why the Pythagorean identity is true

Students use the following information and graph to answer parts (c) and (d).

Consider a circle on a coordinate plane centered at the origin, O. A line is tangent to the circle at point C, which has coordinates (x, y). This tangent line also intersects the x-axis at point A and the y-axis at point B.



Student Stimulus and Part (c)

(c) Explain how the following angles are related to angle θ:

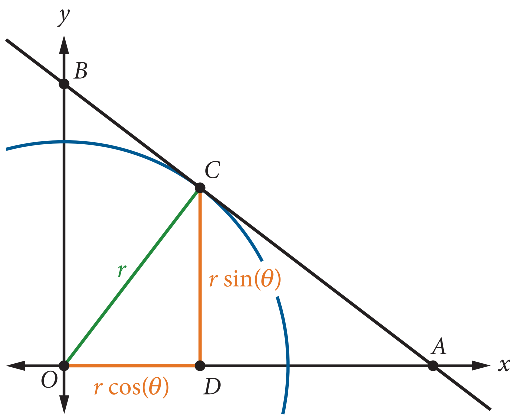
- the acute angle made by tangent line AB and the x-axis, $\angle OAB$
- the acute angle made by tangent line AB and the y-axis, $\angle OBA$

Sample Solution	Points Possible
The acute angle made by the line tangent to the circle and the x-axis, $\angle OAB$, is complementary to angle θ. This is because a line tangent to a circle is perpendicular to the radius at the point of tangency, so $\angle OCA$ is a right angle and triangle COA is a right triangle. Therefore, angle θ and $\angle OAB$ must be complementary. The acute angle made by the line tangent to the circle and the y-axis, $\angle OBA$, is congruent to angle θ. $\angle OCB$ is a right angle because the line tangent to the circle is perpendicular to the radius at the point of tangency. This means that triangle COB is a right triangle. $\angle COB$ must be complementary to θ. Furthermore, $\angle COB$ and $\angle OBA$ must be complementary, which means that $\angle OBA$ must be congruent to θ. Alternative method: Because its legs lie along the coordinate axes, triangle OAB must be a right triangle with a right angle at the origin. Therefore, $\angle OAB$ and $\angle OBA$ must be complementary. We know that $\angle OAB$ is complementary to θ, so $\angle OBA$ must be congruent to θ.	**2 points maximum** 1 point for explaining the relationship of $\angle OAB$ to angle θ 1 point for explaining the relationship of $\angle OBA$ to angle θ

UNIT 4T

Student Stimulus and Part (d)

(d) In terms of θ, determine the lengths of $\overline{AC}$, $\overline{OA}$, $\overline{BC}$, and $\overline{OB}$.

Sample Solution	Points Possible
We can label the intersection point of the vertical line segment through point C and the x-axis as D. We know from part (a) that the length of $\overline{OD} = r\cos(\theta)$ and the length of $\overline{CD} = r\sin(\theta)$.  From the angle relationships we determined in part (c) we can conclude that ΔOCD is similar to ΔOAC. This means that $\frac{m\overline{AO}}{m\overline{OC}} = \frac{m\overline{CO}}{m\overline{OD}}$ and $\frac{m\overline{AC}}{m\overline{CO}} = \frac{m\overline{CD}}{m\overline{DO}}$, which can be used to determine the lengths of $\overline{AO}$ and $\overline{AC}$: $$\frac{m\overline{AO}}{r} = \frac{r}{r\cos(\theta)} \qquad \frac{m\overline{AC}}{r} = \frac{r\sin(\theta)}{r\cos(\theta)}$$ $$m\overline{AO} = \frac{r}{\cos(\theta)} \qquad m\overline{AC} = r\frac{\sin(\theta)}{\cos(\theta)}$$ $$m\overline{AO} = r\sec(\theta) \qquad m\overline{AC} = r\tan(\theta)$$	**4 points maximum** 1 point for determining the length of $\overline{AC}$ 1 point for determining the length of $\overline{OA}$ 1 point for determining the length of $\overline{BC}$ 1 point for determining the length of $\overline{OB}$

Sample Solution	Points Possible
From the angle relationships we determined in part (c) we can conclude that ΔOCD is similar to ΔBOC. This means that $\frac{m\overline{OB}}{m\overline{OC}} = \frac{m\overline{CO}}{m\overline{CD}}$ and $\frac{m\overline{BC}}{m\overline{CO}} = \frac{m\overline{OD}}{m\overline{DC}}$, which can be used to determine the lengths of $\overline{OB}$ and $\overline{BC}$.	

$$\frac{m\overline{OB}}{r} = \frac{r}{r\sin(\theta)} \qquad \frac{m\overline{BC}}{r} = \frac{r\cos(\theta)}{r\sin(\theta)}$$

$$m\overline{OB} = \frac{r}{\sin(\theta)} \qquad m\overline{BC} = r\frac{\cos(\theta)}{\sin(\theta)}$$

$$m\overline{OB} = r\csc(\theta) \qquad m\overline{BC} = r\cot(\theta)$$

Scoring note: It is not necessary for students to use a reciprocal identity in their solutions to receive a point, as long as their work is correct.

PROVIDING FEEDBACK ON STUDENT WORK

Because this is a practice performance task, you could choose to share the scoring guidelines with students before you score their work. This would give students an opportunity to learn what a complete response looks like and allow them to self-assess the completeness and correctness of their answers. Be sure to identify trends in students' responses to inform further instruction. These trends should include topics that students consistently displayed mastery of, as well as conceptual errors that students commonly made. Possible trends and suggested guidance for each part of the task follow, although the patterns you observe in your classroom may differ.

(a) If students struggle with this part of the task, it could mean that they are unfamiliar with the definitions of the sine, cosine, and tangent functions or the sine, cosine, and tangent ratios. Lessons 4T.4, 4T.5, and 4T.8 have several practice problems you could use to review these concepts with students.

Teacher Notes and Reflections

(b) If students are unsure of how to craft their argument to explain the two identities, encourage them to write out their work in symbols along with a brief explanation of their mathematical justifications for their work at each step. This can help them organize their thoughts and help them explain their reasoning in words.

Teacher Notes and Reflections

(c) Some of the right triangles in the figure may be challenging for students to identify. Encourage students to make sketches of the triangles and to keep track of where the angles congruent to θ are located. If this part of the task is too abstract for some students, suggest that they choose a specific numerical value for the measure of angle θ and use that in their diagrams. They can replace the number they chose with θ after they have made sense of the relationship between sides and angles in the triangles.

Teacher Notes and Reflections

(d) If students struggle to identify the similar triangles in this part of the task, you could suggest that they draw the triangles separately and with the same orientation to help them identify the corresponding sides. This arrangement of the figures may help students in correctly constructing the necessary ratios.

Teacher Notes and Reflections

Try to assure students that converting their score into a percentage does not provide an accurate measure of how they performed on the task. You can use the following suggested score interpretations with students to discuss their performance.

Points Received	How Students Should Interpret Their Score
10 or 11 points	"I know all of these algebraic concepts really well. This is top-level work. (A)"
7 to 9 points	"I know all of these algebraic concepts well, but I made a few mistakes. This is above-average work. (B)"
4 to 6 points	"I know some of these algebraic concepts well, but not all of them. This is average-level work. (C)"
2 or 3 points	"I know only a little bit about these algebraic concepts. This is below-average work. (D)"
0 or 1 point	"I don't know much about these algebraic concepts at all. This is not passing work. (F)"