CollegeBoard Pre-AP®

Pre-AP® Geometry with Statistics

TEACHER RESOURCES

Units 1 and 2

ABOUT COLLEGE BOARD

College Board is a mission-driven not-for-profit organization that connects students to college success and opportunity. Founded in 1900, College Board was created to expand access to higher education. Today, the membership association is made up of over 6,000 of the world's leading educational institutions and is dedicated to promoting excellence and equity in education. Each year, College Board helps more than seven million students prepare for a successful transition to college through programs and services in college readiness and college success—including the SAT® and the Advanced Placement Program®. The organization also serves the education community through research and advocacy on behalf of students, educators, and schools.

For further information, visit **www.collegeboard.org**.

PRE-AP EQUITY AND ACCESS POLICY

College Board believes that all students deserve engaging, relevant, and challenging grade-level coursework. Access to this type of coursework increases opportunities for all students, including groups that have been traditionally underrepresented in AP and college classrooms. Therefore, the Pre-AP program is dedicated to collaborating with educators across the country to ensure all students have the supports to succeed in appropriately challenging classroom experiences that allow students to learn and grow. It is only through a sustained commitment to equitable preparation, access, and support that true excellence can be achieved for all students, and the Pre-AP course designation requires this commitment.

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The sentence-writing strategies used in Pre-AP lessons are based upon The Writing Revolution, Inc., a national nonprofit organization that trains educators to implement The Hochman Method, an evidence-based approach to teaching writing. The strategies included in Pre-AP materials are meant to support students' writing, critical thinking, and content understanding, but they do not represent The Writing Revolution's full, comprehensive approach to teaching writing. More information can be found at **www.thewritingrevolution.org**.

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UNIT 2

Tools and Techniques of Geometric Measurement

Acknowledgments

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Introduction to Pre-AP Geometry with Statistics

About Pre-AP

Introduction to Pre-AP

Every student deserves classroom opportunities to learn, grow, and succeed. College Board developed Pre-AP® to deliver on this simple premise. Pre-AP courses are designed to support all students across varying levels of readiness. They are not honors or advanced courses.

Participation in Pre-AP courses allows students to slow down and focus on the most essential and relevant concepts and skills. Students have frequent opportunities to engage deeply with texts, sources, and data as well as compelling higher-order questions and problems. Across Pre-AP courses, students experience shared instructional practices and routines that help them develop and strengthen the important critical thinking skills they will need to employ in high school, college, and life. Students and teachers can see progress and opportunities for growth through varied classroom assessments that provide clear and meaningful feedback at key checkpoints throughout each course.

DEVELOPING THE PRE-AP COURSES

Pre-AP courses are carefully developed in partnership with experienced educators, including middle school, high school, and college faculty. Pre-AP educator committees work closely with College Board to ensure that the course resources define, illustrate, and measure grade-level-appropriate learning in a clear, accessible, and engaging way. College Board also gathers feedback from a variety of stakeholders, including Pre-AP partner schools from across the nation who have participated in multiyear pilots of select courses. Data and feedback from partner schools, educator committees, and advisory panels are carefully considered to ensure that Pre-AP courses provide all students with grade-level-appropriate learning experiences that place them on a path to college and career readiness.

PRE-AP EDUCATOR NETWORK

Similar to the way in which teachers of Advanced Placement® (AP®) courses can become more deeply involved in the program by becoming AP Readers or workshop consultants, Pre-AP teachers also have opportunities to become active in their educator network. Each year, College Board expands and strengthens the Pre-AP National Faculty—the team of educators who facilitate Pre-AP Readiness Workshops and Pre-AP Summer Institutes. Pre-AP teachers can also become curriculum and assessment contributors by working with College Board to design, review, or pilot the course resources.

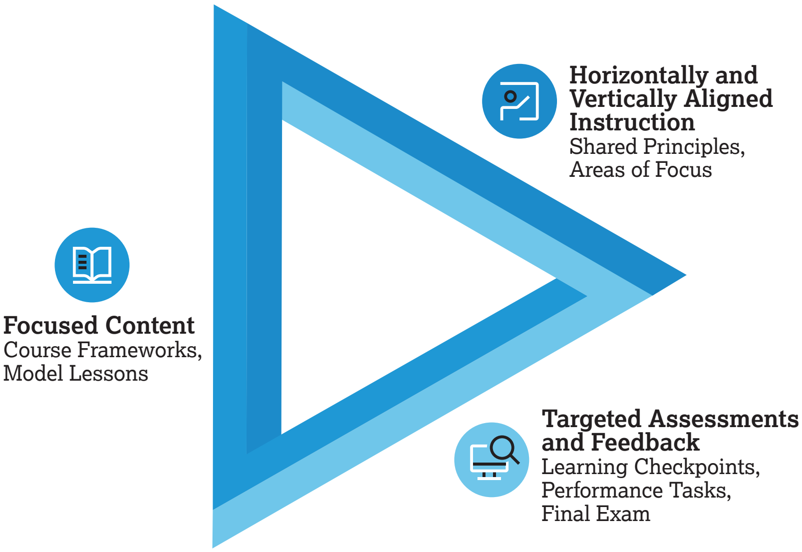
HOW TO GET INVOLVED

Schools and districts interested in learning more about participating in Pre-AP should visit **preap.org/join** or contact us at **preap@collegeboard.org**.

Teachers interested in becoming members of Pre-AP National Faculty or participating in content development should visit **preap.org/national-faculty** or contact us at **preap@collegeboard.org**.

Pre-AP Approach to Teaching and Learning

Pre-AP courses invite all students to learn, grow, and succeed through focused content, horizontally and vertically aligned instruction, and targeted assessments for learning. The Pre-AP approach to teaching and learning, as described below, is not overly complex, yet the combined strength results in powerful and lasting benefits for both teachers and students. This is our theory of action.



FOCUSED CONTENT

Pre-AP courses focus deeply on a limited number of concepts and skills with the broadest relevance for high school coursework and college and career success. The course framework serves as the foundation of the course and defines these prioritized concepts and skills. Pre-AP model lessons and assessments are based directly on this focused framework. The course design provides students and teachers with intentional permission to slow down and focus.

HORIZONTALLY AND VERTICALLY ALIGNED INSTRUCTION

Shared principles cut across all Pre-AP courses and disciplines. Each course is also aligned to discipline-specific areas of focus that prioritize the critical reasoning skills and practices central to that discipline.

SHARED PRINCIPLES

All Pre-AP courses share the following set of research-supported instructional principles. Classrooms that regularly focus on these cross-disciplinary principles allow students to effectively extend their content knowledge while strengthening their critical thinking skills. When students are enrolled in multiple Pre-AP courses, the horizontal alignment of the shared principles provides students and teachers across disciplines with a shared language for their learning and investigation, and multiple opportunities to practice and grow. The critical reasoning and problem-solving tools students develop through these shared principles are highly valued in college coursework and in the workplace.

Close Observation and Analysis

Students are provided time to carefully observe one data set, text image, performance piece, or problem before being asked to explain, analyze, or evaluate. This creates a safe entry point to simply express what they notice and what they wonder. It also encourages students to slow down and capture relevant details with intentionality to support more meaningful analysis, rather than rush to completion at the expense of understanding.

Higher-Order Questioning

Students engage with questions designed to encourage thinking that is elevated beyond simple memorization and recall. Higher-order questions require students to make predictions, synthesize, evaluate, and compare. As students grapple with these questions, they learn that being inquisitive promotes extended thinking and leads to deeper understanding.

Evidence-Based Writing

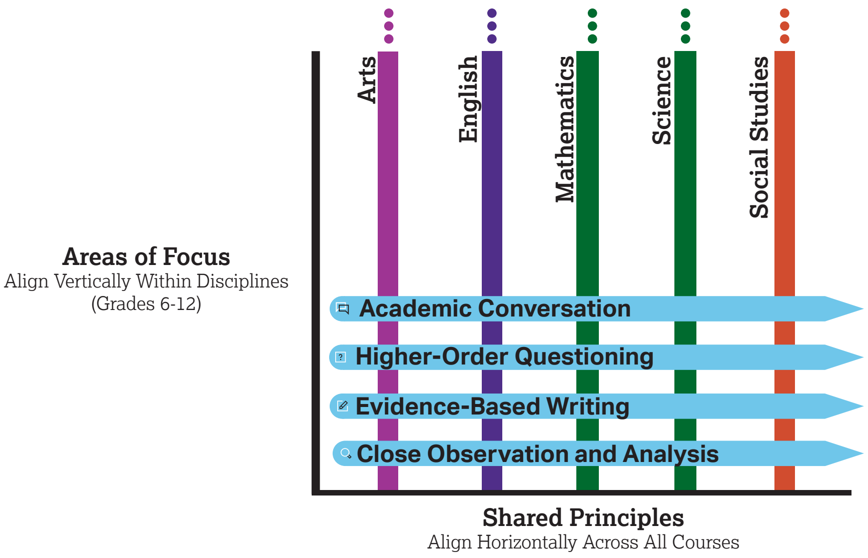
With strategic support, students frequently engage in writing coherent arguments from relevant and valid sources of evidence. Pre-AP courses embrace a purposeful and scaffolded approach to writing that begins with a focus on precise and effective sentences before progressing to longer forms of writing.

Academic Conversation

Through peer-to-peer dialogue, students' ideas are explored, challenged, and refined. As students engage in academic conversation, they come to see the value in being open to new ideas and modifying their own ideas based on new information. Students grow as they frequently practice this type of respectful dialogue and critique and learn to recognize that all voices, including their own, deserve to be heard.

AREAS OF FOCUS

The areas of focus are discipline-specific reasoning skills that students develop and leverage as they engage with content. Whereas the shared principles promote horizontal alignment across disciplines, the areas of focus provide vertical alignment within a discipline, giving students the opportunity to strengthen and deepen their work with these skills in subsequent courses in the same discipline.



For information about the Pre-AP mathematics areas of focus, see page 15.

TARGETED ASSESSMENTS FOR LEARNING

Pre-AP courses include strategically designed classroom assessments that serve as tools for understanding progress and identifying areas that need more support. The assessments provide frequent and meaningful feedback for both teachers and students across each unit of the course and for the course as a whole. For more information about assessments in Pre-AP Geometry with Statistics, see page 60.

Pre-AP Professional Learning

Pre-AP teachers are required to engage in two professional learning opportunities. The first requirement is designed to help prepare them to teach their specific course. There are two options to meet the first requirement: the Pre-AP Summer Institute (Pre-APSI) and the Online Foundational Module Series. Both options provide continuing education units to educators who complete them.

- The Pre-AP Summer Institute is a four-day collaborative experience that empowers participants to prepare and plan for their Pre-AP course. While attending, teachers engage with Pre-AP course frameworks, shared principles, areas of focus, and sample model lessons. Participants are given supportive planning time where they work with peers to begin to build their Pre-AP course plan.
- The Online Foundational Module Series is available to all teachers of Pre-AP courses. This 12- to 20-hour course supports teachers in preparing for their Pre-AP course. Teachers explore course materials and experience model lessons from the student's point of view. They also begin to plan and build their own course so they are ready on day one of instruction.

The second professional learning requirement is to complete at least one of the Online Performance Task Scoring Modules, which offer guidance and practice applying Pre-AP scoring guidelines to student work.

About the Course

Introduction to Pre-AP Geometry with Statistics

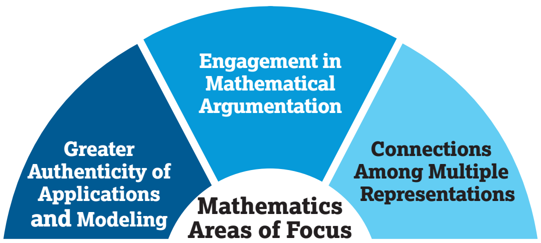
Pre-AP Geometry with Statistics is designed to provide students with a meaningful conceptual bridge between algebra and geometry to deepen their understanding of mathematics. Students often struggle to see the connections among their mathematics courses. In this course, students are expected to use the mathematical knowledge and skills they have developed previously to problem solve across the domains of algebra, geometry, and statistics.

Rather than seeking to cover all topics traditionally included in a standard geometry or introductory statistics textbook, this course focuses on the foundational geometric and statistical knowledge and skills that matter most for college and career readiness. The Pre-AP Geometry with Statistics Course Framework highlights how to guide students to connect core ideas within and across the units of the course, promoting a coherent understanding of measurement.

The components of this course have been crafted to prepare not only the next generation of mathematicians, scientists, programmers, statisticians, and engineers, but also a broader base of mathematically informed citizens who are well equipped to respond to the array of mathematics-related issues that impact our lives at the personal, local, and global levels.

PRE-AP MATHEMATICS AREAS OF FOCUS

The Pre-AP mathematics areas of focus, shown below, are mathematical practices that students develop and leverage as they engage with content. They were identified through educator feedback and research about where students and teachers need the most curriculum support. These areas of focus are vertically aligned to the mathematical practices embedded in other mathematics courses in high school, including AP, and in college, giving students multiple opportunities to strengthen and deepen their work with these skills throughout their educational career. They also support and align to the AP Calculus Mathematical Practices, the AP Statistics Course Skills, and the mathematical practices listed in various state standards.



Greater Authenticity of Applications and Modeling

Students create and use mathematical models to understand and explain authentic scenarios.

Mathematical modeling is a process that helps people analyze and explain the world. In Pre-AP Geometry with Statistics, students explore real-world contexts where mathematics can be used to make sense of a situation. They engage in the modeling process by making choices about what aspects of the situation to model, assessing how well the model represents the available data, drawing conclusions from their model, justifying decisions they make through the process, and identifying what the model helps clarify and what it does not.

In addition to mathematical modeling, Pre-AP Geometry with Statistics students engage in mathematics through authentic applications. Applications are similar to modeling problems in that they are drawn from real-world phenomena, but they differ because the applications dictate the appropriate mathematics to use to solve the problem. Pre-AP Geometry with Statistics balances these two types of real-world tasks.

Engagement in Mathematical Argumentation

Students use evidence to craft mathematical conjectures and prove or disprove them.

Reasoning and proof lie at the heart of the discipline of mathematics. Mathematics is both a way of thinking and a set of tools for solving problems. Pre-AP Geometry with Statistics students gain proficiency in deductively reasoning with axioms and theorems to reach logical conclusions. Students also develop skills in using statistical and probabilistic reasoning to make sense of data and craft assertions using data as evidence and support. Students learn how to quantify chance and make inferences about populations. Through these two different types of mathematical argumentation, students learn how to be critical of their own reasoning and the reasoning of others.

Connections Among Multiple Representations

Students represent mathematical concepts in a variety of forms and move fluently among the forms.

Pre-AP Geometry with Statistics students explore how to weave together multiple representations of geometric and statistics concepts. Every mathematical representation illuminates certain characteristics of a concept while also obscuring other aspects. Often, geometric reasoning is used to make sense of algebraic calculations. Likewise, algebraic techniques can be used to solve problems involving geometry. Patterns in data can emerge by depicting the data visually. Statistical calculations are important and valuable, but they make more sense to students when they are conceptually grounded in and related to graphical representations of data. With experience that continues to develop in Pre-AP Geometry with Statistics, students become equipped with a nuanced understanding of which representations best serve a particular purpose.

PRE-AP GEOMETRY WITH STATISTICS AND CAREER READINESS

The Pre-AP Geometry with Statistics course resources are designed to expose students to a wide range of career opportunities that depend on geometry and statistics knowledge and skills. Examples include not only field-specific specialty careers such as mathematicians or statisticians, but also other endeavors where geometry and statistics knowledge is relevant, such as architects, carpenters, engineers, mechanics, actuaries, and programmers.

Career clusters that involve geometry and statistics, along with examples of careers in mathematics or related to mathematics, are provided below and on the following page. Teachers should consider discussing these with students throughout the year to promote motivation and engagement.

Career Clusters Involving Mathematics
agriculture, food, and natural resources
architecture and construction
arts, A/V technology, and communications
business management and administration
finance
government and public administration
health science
information technology
manufacturing
marketing
STEM (science, technology, engineering, and math)
transportation, distribution, and logistics

Examples of Geometry Related Careers	Examples of Statistics Related Careers
animator architect cartographer drafter mathematician mathematics teacher professor programmer surveyor	drafter economist financial analyst mathematics teacher meteorologist professor programmer research analyst statistician

Source for Career Clusters: "Advanced Placement and Career and Technical Education: Working Together." Advance CTE and the College Board. October 2018. **https://careertech.org/resource/ap-cte-working-together**.

For more information about careers that involve mathematics, teachers and students can visit and explore the College Board's Big Future resources: **https://bigfuture.collegeboard.org/majors/math-statistics-mathematics**.

SUMMARY OF RESOURCES AND SUPPORTS

Teachers are strongly encouraged to take advantage of the full set of resources and supports for Pre-AP Geometry with Statistics, which is summarized below. Some of these resources must be used for a course to receive the Pre-AP Course Designation. To learn more about the requirements for course designation, see details below and on page 68.

COURSE FRAMEWORK

The framework defines what students should know and be able to do by the end of the course. It serves as an anchor for model lessons and assessments, and it is the primary document teachers can use to align instruction to course content. **Use of the course framework is required**. *For more details see page 24.*

MODEL LESSONS

Teacher resources, available in print and online, include a robust set of model lessons that demonstrate how to translate the course framework, shared principles, and areas of focus into daily instruction. **Use of the model lessons is encouraged but not required**. *For more details see page 58.*

LEARNING CHECKPOINTS

Accessed through Pre-AP Classroom (the Pre-AP digital platform), these short formative assessments provide insight into student progress. They are automatically scored and include multiple-choice and technology-enhanced items with rationales that explain correct and incorrect answers. **Use of one learning checkpoint per unit is required**. *For more details see page 60.*

PERFORMANCE TASKS

Available in the printed teacher resources as well as on Pre-AP Classroom, performance tasks allow students to demonstrate their learning through extended problem-solving, writing, analysis, and/or reasoning tasks. Scoring guidelines are provided to inform teacher scoring, with additional practice and feedback suggestions available in online modules on Pre-AP Classroom. **Use of each unit's performance task is required**. *For more details see page 61.*

PRACTICE PERFORMANCE TASKS

Available in the student resources, with supporting materials in the teacher resources, these tasks provide an opportunity for students to practice applying skills and knowledge as they would in a performance task, but in a more scaffolded environment. **Use of the practice performance tasks is encouraged but not required**. *For more details see page 62.*

FINAL EXAM

Accessed through Pre-AP Classroom, the final exam serves as a classroom-based, summative assessment designed to measure students' success in learning and applying the knowledge and skills articulated in the course framework. **Administration of the final exam is encouraged but not required**. *For more details see page 63.*

PROFESSIONAL LEARNING

Both the four-day Pre-AP Summer Institute (Pre-APSI) and the Online Foundational Module Series support teachers in preparing and planning to teach their Pre-AP course. **All Pre-AP teachers are required to either attend the Pre-AP Summer Institute or complete the module series. In addition, teachers are required to complete at least one Online Performance Task Scoring module**. *For more details see page 11.*

Course Map

PLAN

The course map shows how components are positioned throughout the course. As the map indicates, the course is designed to be taught over 140 class periods (based on 45-minute class periods), for a total of 28 weeks.

Model lessons are included for approximately 50% of the total instructional time, with the percentage varying by unit. Each unit is divided into key concepts.

TEACH

The model lessons demonstrate how the Pre-AP shared principles and mathematics areas of focus come to life in the classroom.

Shared Principles
Close observation and analysis
Higher-order questioning
Evidence-based writing
Academic conversation

Areas of Focus
Greater authenticity of applications and modeling
Engagement in mathematical argumentation
Connections among multiple representations

ASSESS AND REFLECT

Each unit includes two learning checkpoints and a performance task. These formative assessments are designed to provide meaningful feedback for both teachers and students.

Note: The final exam, offered during a six-week window in the spring, is not represented on the map.

UNIT 1 Measurement in Data

~35 Class Periods

Pre-AP model lessons provided for 100% of instructional time in this unit

KEY CONCEPT 1.1
The Shape of Data

Learning Checkpoint 1

KEY CONCEPT 1.2
Chance Events

KEY CONCEPT 1.3
Inferences from Data

Learning Checkpoint 2

Performance Task for Unit 1

UNIT 2 Tools and Techniques of Geometric Measurement

~35 Class Periods

Pre-AP model lessons provided for approximately 50% of instructional time in this unit

KEY CONCEPT 2.1

Measurement in Geometry

Learning Checkpoint 1

KEY CONCEPT 2.2

Parallel and Perpendicular Lines

KEY CONCEPT 2.3

Measurement in Right Triangles

Learning Checkpoint 2

Performance Task for Unit 2

UNIT 3 Measurement in Congruent and Similar Figures

~35 Class Periods

Pre-AP model lessons provided for approximately 30% of instructional time in this unit

KEY CONCEPT 3.1

Transformations of Points in a Plane

KEY CONCEPT 3.2

Congruent and Similar Polygons

Learning Checkpoint 1

KEY CONCEPT 3.3

Measurement of Lengths and Angles in Circles

Learning Checkpoint 2

Performance Task for Unit 3

UNIT 4 Measurement in Two and Three Dimensions

~35 Class Periods

Pre-AP model lessons provided for approximately 10% of instructional time in this unit

KEY CONCEPT 4.1

Area as a Two-Dimensional Measurement

KEY CONCEPT 4.2

Learning Objectives 4.2.1–4.2.4

Volume as a Three-Dimensional Measurement

Learning Checkpoint 1

KEY CONCEPT 4.2 (*continued*)

Learning Objectives 4.2.5–4.2.7

Volume as a Three-Dimensional Measurement

KEY CONCEPT 4.3

Measurements of Spheres

Learning Checkpoint 2

Performance Task for Unit 4

Pre-AP Geometry with Statistics Course Framework

INTRODUCTION

Based on the Understanding by Design® (Wiggins and McTighe) model, the Pre-AP Geometry with Statistics Course Framework is back mapped from AP expectations and aligned to essential grade-level expectations. The course framework serves as a teacher's blueprint for the Pre-AP Geometry with Statistics instructional resources and assessments.

The course framework was designed to meet the following criteria:

- **Focused:** The framework provides a deep focus on a limited number of concepts and skills that have the broadest relevance for later high school, college, and career success.
- **Measurable:** The framework's learning objectives are observable and measurable statements about the knowledge and skills students should develop in the course.
- **Manageable:** The framework is manageable for a full year of instruction, fosters the ability to explore concepts in depth, and enables room for additional local or state standards to be addressed where appropriate.
- **Accessible:** The framework's learning objectives are designed to provide all students, across varying levels of readiness, with opportunities to learn, grow, and succeed.

COURSE FRAMEWORK COMPONENTS

The Pre-AP Geometry with Statistics Course Framework includes the following components:

Big Ideas

The big ideas are recurring themes that allow students to create meaningful connections between course concepts. Revisiting the big ideas throughout the course and applying them in a variety of contexts allows students to develop deeper conceptual understandings.

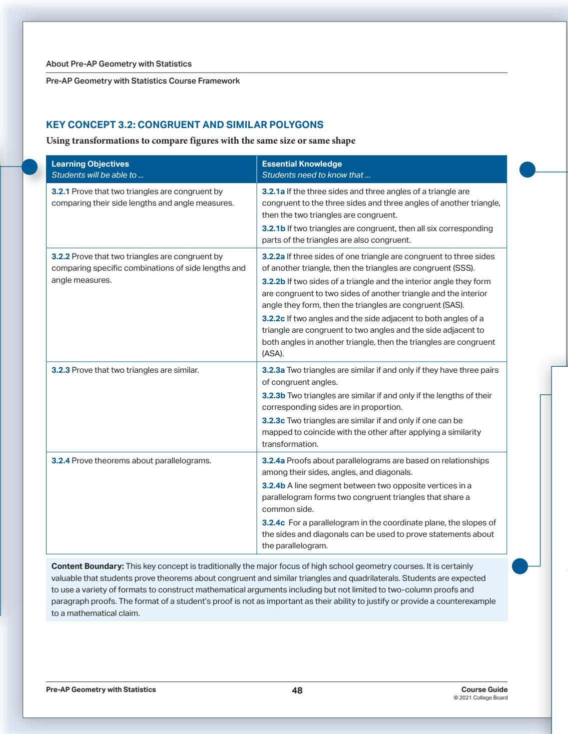
Enduring Understandings

Each unit focuses on a small set of enduring understandings. These are the long-term takeaways related to the big ideas that leave a lasting impression on students. Students build and earn these understandings over time by exploring and applying course content throughout the year.

Key Concepts

To support teacher planning and instruction, each unit is organized by key concepts. Each key concept includes relevant **learning objectives** and **essential knowledge statements** and may also include **content boundary and cross connection statements**. These are illustrated and defined below.

Learning Objectives:
These objectives define what a student needs to be able to do with essential knowledge to progress toward the enduring understandings. The learning objectives serve as actionable targets for instruction and assessment.



About Pre-AP Geometry with Statistics

Pre-AP Geometry with Statistics Course Framework

KEY CONCEPT 3.2: CONGRUENT AND SIMILAR POLYGONS

Using transformations to compare figures with the same size or same shape

Learning Objectives *Students will be able to …*	Essential Knowledge *Students need to know that …*
3.2.1 Prove that two triangles are congruent by comparing their side lengths and angle measures.	**3.2.1a** If the three sides and three angles of a triangle are congruent to the three sides and three angles of another triangle, then the two triangles are congruent. **3.2.1b** If two triangles are congruent, then all six corresponding parts of the triangles are also congruent.
3.2.2 Prove that two triangles are congruent by comparing specific combinations of side lengths and angle measures.	**3.2.2a** If three sides of one triangle are congruent to three sides of another triangle, then the triangles are congruent (SSS). **3.2.2b** If two sides of a triangle and the interior angle they form are congruent to two sides of another triangle and the interior angle they form, then the triangles are congruent (SAS). **3.2.2c** If two angles and the side adjacent to both angles of a triangle are congruent to two angles and the side adjacent to both angles in another triangle, then the triangles are congruent (ASA).
3.2.3 Prove that two triangles are similar.	**3.2.3a** Two triangles are similar if and only if they have three pairs of congruent angles. **3.2.3b** Two triangles are similar if and only if the lengths of their corresponding sides are in proportion. **3.2.3c** Two triangles are similar if and only if one can be mapped to coincide with the other after applying a similarity transformation.
3.2.4 Prove theorems about parallelograms.	**3.2.4a** Proofs about parallelograms are based on relationships among their sides, angles, and diagonals. **3.2.4b** A line segment between two opposite vertices in a parallelogram forms two congruent triangles that share a common side. **3.2.4c** For a parallelogram in the coordinate plane, the slopes of the sides and diagonals can be used to prove statements about the parallelogram.

Content Boundary: This key concept is traditionally the major focus of high school geometry courses. It is certainly valuable that students prove theorems about congruent and similar triangles and quadrilaterals. Students are expected to use a variety of formats to construct mathematical arguments including but not limited to two-column proofs and paragraph proofs. The format of a student's proof is not as important as their ability to justify or provide a counterexample to a mathematical claim.

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Essential Knowledge Statements:
Each essential knowledge statement is linked to a learning objective. One or more essential knowledge statements describe the knowledge required to perform each learning objective.

Content Boundary and Cross Connection Statements:
When needed, content boundary statements provide additional clarity about the content and skills that lie within versus outside of the scope of this course.

Cross connection statements highlight important connections that should be made between key concepts within and across the units.

BIG IDEAS IN PRE-AP GEOMETRY WITH STATISTICS

While the Pre-AP Geometry with Statistics framework is organized into four core units of study, the content is grounded in three big ideas, which are cross-cutting concepts that build conceptual understanding and spiral throughout the course. Since these ideas cut across units, they serve as the underlying foundation for the enduring understandings, key concepts, and learning objectives that make up the focus of each unit. A deep and productive understanding in Pre-AP Geometry with Statistics relies on these three big ideas:

- **Measurement:** Measurement is the quantification of features of an object or a phenomenon. In geometry, measuring objects allows us to draw meaningful conclusions about those objects. Measurement provides relatable real-world applications in one, two, and three dimensions.
- **Transformation:** A transformation is a function, which means that it associates one set of objects with another. When a mathematical object is transformed, some of its measurements change while other measurements do not change. Congruence and similarity are defined through transformations, which puts the focus on measurements that are affected by transformations and those that are not. An understanding how data distributions are affected by transformations enhances the connections between probability and statistics.
- **Comparison and Composition:** Throughout mathematics, new and more complex concepts are understood in terms of simpler, previously explored concepts. In geometry, this mode of thinking allows for the deconstruction of two- and three-dimensional shapes for further investigation. This interpretation relies on the recognition that complex objects are composed of, and can be compared to, simpler objects. For statistics, this means using measures of center and spread to characterize complex data distributions.

OVERVIEW OF PRE-AP GEOMETRY WITH STATISTICS UNITS AND ENDURING UNDERSTANDINGS

<table>
<tr><th>Unit 1: Measurement in Data</th><th>Unit 2: Tools and Techniques of Geometric Measurement</th></tr>
<tr><td>Statistics are numbers that summarize large data sets by reducing their complexity to a few key values that model their center and spread.
Distributions are functions whose displays are used to analyze data sets.
Probabilistic reasoning allows us to anticipate patterns in data.
The method by which data are collected influences what can be said about the population from which the data were drawn, and how certain those statements are.</td><td>A formal mathematical argument establishes new truths by logically combining previously known facts.
Measuring features of geometric figures is the process of assigning numeric values to attributes of the figures, which allows the attributes to be compared.
Pairs of lines in a plane that never intersect or that intersect at right angles have special geometric and algebraic properties.
Right triangles are simple geometric shapes in which we can relate the measures of acute angles to ratios of their side lengths.</td></tr>
<tr><th>Unit 3: Measurement in Congruent and Similar Figures</th><th>Unit 4: Measurement in Two and Three Dimensions</th></tr>
<tr><td>Transformations are functions that can affect the measurements of a geometric figure.
Congruent figures have equal corresponding angle measures and equal distances between corresponding pairs of points.
Similar figures have equal corresponding angle measurements, and the distances between corresponding pairs of points are proportional.
The geometry of a circle is completely determined by its radius.</td><td>The area of a figure depends on its height and its cross-sectional widths.
The volume of a solid depends on its height and its cross-sectional areas.
The geometry of a sphere is completely determined by its radius.</td></tr>
</table>

Unit 1: Measurement in Data

Suggested Timing: Approximately 7 weeks

This unit offers a sustained and focused examination of statistics and probability to support the development of students' quantitative literacy. Statistics and probability help us perform essential real-world tasks such as making informed choices, deciding between different policies, and weighing competing knowledge claims. While topics of statistics and probability are commonplace in high school geometry courses, students often have limited opportunities to engage in statistical and probabilistic reasoning and sense-making. To move students toward a sophisticated understanding of data, students are expected to think about data sets as distributions which are functions that associate data values with their frequency or their probability. This encourages students to connect their knowledge of functions to concepts of statistics and probability, creating a more complete understanding of mathematics. Throughout the unit, students generate their own data through surveys, experiments, and simulations that investigate some aspect of the real world. They engage in statistical calculations and probabilistic reasoning as methods of analysis to make sense of data and draw inferences about populations. Incorporating statistics and probability in the same course as geometry allows students to experience two distinct forms of argumentation: geometrical reasoning as drawing conclusions with certainty about an ideal mathematical world, and probabilistic reasoning as drawing less-than-certain conclusions about the real world. The conclusions of a probability argument are presented as ranges that have varying degrees of certainty.

ENDURING UNDERSTANDINGS

Students will understand that ...

- Statistics are numbers that summarize large data sets by reducing their complexity to a few key values that model their center and spread.
- Distributions are functions whose displays are used to analyze data sets.
- Probabilistic reasoning allows us to anticipate patterns in data.
- The method by which data are collected influences what can be said about the population from which the data were drawn, and how certain those statements are.

KEY CONCEPTS

- **1.1: The shape of data** – Identifying measures of center and spread to summarize and characterize a data distribution
- **1.2: Chance events** – Exploring patterns in random events to anticipate the likelihood of outcomes
- **1.3: Inferences from data** – Using probability and statistics to make claims about a population

KEY CONCEPT 1.1: THE SHAPE OF DATA

Identifying measures of center and spread to summarize and characterize a data distribution

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
1.1.1 Determine appropriate summary statistics for a quantitative data distribution.	**1.1.1a** A data distribution is a function whose input is each value in a data set and whose output is the corresponding frequency of that value. **1.1.1b** Summary statistics describe the important features of data distributions including identifying a typical value, also called the center of the data, and describing the clustering of the data around the typical value, also called the spread of the data. **1.1.1c** The mean and the median summarize a data distribution by identifying a typical value, or center, of the distribution. The mean and median have the same units as the values in the data distribution. **1.1.1d** The standard deviation, interquartile range, and range summarize a data distribution by quantifying the variability, or spread, of the data set. The standard deviation, interquartile range, and range have the same units as the values in the data distribution.
1.1.2 Create a graphical representation of a quantitative data set.	**1.1.2a** A boxplot summarizes a quantitative data set by partitioning its values into four groups, each consisting of the same number of data values. Boxplots are used to depict the spread of a distribution. **1.1.2b** A histogram summarizes a quantitative data set by partitioning its values into equal-width intervals and displaying bars whose heights indicate the frequency of values contained in each interval. Histograms are used to depict the shape of a distribution.
1.1.3 Analyze data distributions with respect to their centers.	**1.1.3a** The mean is the only point in the domain of a distribution where the sum of the deviations, or differences, between the mean and each point in the distribution is zero. **1.1.3b** The mean can be thought of as the center of mass of the data set. It is a weighted average that accounts for the number of data points that exists for every given value in the data set. **1.1.3c** Measures of center can be used to compare the typical values of the distributions. They provide useful information about whether one distribution is typically larger, smaller, or about the same as another distribution.

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
1.1.4 Analyze data distributions with respect to their symmetry or direction of skew.	**1.1.4a** For symmetric distributions, such as the normal distribution, the proportion of data in any range to the left of the mean is equal to the proportion of data in the corresponding range to the right of the mean. **1.1.4b** Skew describes the asymmetry of a distribution. The direction of skew is indicated by the longer tail of data values in an asymmetric distribution. **1.1.4c** When a distribution is skewed, its mean and median will differ. The farther apart the mean and median are in a distribution, the more skewed the distribution will appear.
1.1.5 Analyze data distributions with respect to their variability.	**1.1.5a** Measures of variability quantify the typical spread of a data distribution. They are used to describe how similar the values of a data set are to each other. A distribution with low variability will have data values that are clustered at the center, so the distribution is well characterized by its measures of center. A distribution with high variability will have data values that are spread out from the center, so the distribution is less well characterized by its measures of center. **1.1.5b** The interquartile range is the length of the interval that contains the middle 50% of the values in a distribution. **1.1.5c** The total variation of a distribution can be measured by the sum of the squared deviations from the mean. The variance of a distribution is the average of the squared deviations from the mean. **1.1.5d** The standard deviation is the square root of the variance. The standard deviation can be interpreted as a typical distance of the data values from the mean.
1.1.6 Model a data distribution with a normal distribution.	**1.1.6a** The normal distribution is a model of a data distribution defined by its mean and standard deviation. The normal distribution is bell-shaped and symmetric about the mean. In a normal distribution, the frequency of data values tapers off at one standard deviation above or below the mean. **1.1.6b** When a normal distribution is used to model a data distribution, approximately 68% of the data values fall within one standard deviation of the mean. Approximately 95% of the data values fall within two standard deviations of the mean. Over 99% of the data values fall within three standard deviations of the mean. **1.1.6c** For normally distributed data, the mean and median are the same number, and they correspond to the mode, which is the value in the distribution with the highest frequency.

Content Boundary: In this unit, students are introduced to the normal distribution as a model for some data distributions, similar to how linear functions can be used to model some two-variable data sets. The normal distribution is often used to answer probabilistic questions. Those types of questions should be reserved for the lessons of Key Concept 1.2: Chance events.

Cross Connection: Students likely come to this course with a basic understanding of how to calculate some summary statistics, but with limited conceptual understanding about their meaning and utility. A goal of this unit is to expand students' understanding of measures of center and spread. The focus of the unit should be on using these measures to analyze data distributions.

KEY CONCEPT 1.2: CHANCE EVENTS

Exploring patterns in random events to anticipate the likelihood of outcomes

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
1.2.1 Create or analyze a data display for a categorical data set.	**1.2.1a** Venn diagrams and contingency tables are common displays of categorical data and are useful for answering questions about probability. **1.2.1b** The intersection of two categories is the set of elements common to both categories. **1.2.1c** The union of two categories is the set of elements found by combining all elements of both categories. **1.2.1d** For categorical data, variability is determined by comparing relative frequencies of categories.
1.2.2 Determine the probability of an event.	**1.2.2a** The sample space is the set of all outcomes of an experiment or random trial. An event is a subset of the sample space. **1.2.2b** Probabilities are numbers between 0 and 1 where 0 means there is no possibility that an event can occur, and 1 means the event is certain to occur. The probability of an event occurring can be described numerically as a ratio of the number of favorable outcomes to the number of total outcomes in a sample space. **1.2.2c** A probability distribution is a function that associates a probability with each possible value or interval of values for a random variable. The sum of the probabilities over all possible values of the independent variable must be 1.
1.2.3 Calculate relative frequencies, joint frequencies, marginal frequencies, or conditional probabilities for a categorical data set.	**1.2.3a** Relative frequencies are the number of times an event occurs divided by the total number of observations. They can be used to estimate probabilities of future events occurring. **1.2.3b** Joint frequencies are events that co-occur for two or more variables. They are the frequencies displayed in cells in a two-way contingency table. **1.2.3c** Marginal frequencies are events that summarize the frequencies across all levels of one variable while holding the second variable constant. They are the row totals and column totals in a two-way contingency table. **1.2.3d** The conditional probability of B, given A has already occurred, is the proportion of times B occurs when restricted to events only in A.
1.2.4 Determine if two events are independent.	**1.2.4a** Two events, A and B, are independent if the occurrence of A does not affect the probability of B. **1.2.4b** Two events, A and B, are independent if the probability of A and B occurring together is the product of their probabilities.

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
1.2.5 Calculate the probability of a range of values of an independent variable, given a mean, standard deviation, and normal distribution.	**1.2.5a** The normal distribution can be used to model a probability distribution that is bell-shaped and symmetric about the mean. **1.2.5b** When a normal distribution is used as a model of a probability distribution, the probability of a data value occurring above the mean is 0.5 and the probability of a data value occurring below the mean is 0.5. **1.2.5c** When a normal distribution is used as a model of a probability distribution, the probability of data occurring within one standard deviation of the mean is approximately 0.68, the probability of data occurring within two standard deviations of the mean is approximately 0.95, and the probability of data occurring within three standard deviations of the mean is approximately 0.997. These proportions can be used to determine the probability of an event occurring in a population.

Content Boundary: Throughout the course framework, the terms *random variable* and *independent variable* are used interchangeably. These terms describe different aspects of the same variable. The term *random variable* describes the process by which the variable was sampled, and *independent variable* is used when the frequency or probability distribution of the variable is of interest.

Cross Connection: In this unit, students will explore how the normal distribution can be used to model a probability distribution. This is a slightly different application of the normal distribution than students used in the previous key concept. In those lessons, students were expected to answer questions about the percent of data, or expected percent of data, that occurred within certain ranges. In the lessons of this key concept, students are expected to answer questions about the probability that an event would occur within a given range.

Cross Connection: Students will likely have some understanding of probability and randomness from previous courses. It is important for students to understand that, mathematically, the term *random* means that the outcome of a single trial may not be known, although over repeated trials, the proportions of the different outcomes may be predictable.

KEY CONCEPT 1.3: INFERENCES FROM DATA

Using probability and statistics to make claims about a population

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
1.3.1 Distinguish between accuracy and precision as measures of statistical variability and statistical bias in measurements.	**1.3.1a** Accuracy is how close the measurements in a measurement process are to the true value being estimated. Accuracy is determined by comparing the center of a sample of measurements to the true value of the measure. **1.3.1b** Precision is how close the measurements in a measurement process are to one another. Precision is determined by examining the variability of a sample of measurements. **1.3.1c** Bias is the tendency of a measurement process to systematically overestimate or underestimate the true measure of a phenomenon. Bias is an indication of the inaccuracy of the measurement process.
1.3.2 Describe how the size of a sample impacts how well it represents the population from which it was drawn.	**1.3.2a** The law of large numbers states that the mean of the results obtained from a large number of trials will tend to become closer to the true value of the phenomenon being measured as more trials are performed. This means we can trust larger samples more than smaller ones. **1.3.2b** The law of large numbers assumes that there is no systematic error of measurement in the sample.
1.3.3 Design a method for gathering data that is appropriate for a given purpose.	**1.3.3a** An experiment is a method of gathering information about phenomena where the independent variable is manipulated by the researcher. **1.3.3b** An observational study is a method of gathering information about phenomena where the independent variable is not under the control of the researcher. **1.3.3c** A survey is a method of gathering information from a sample of people using a questionnaire.
1.3.4 Identify biases in sampling methods for experiments, observational studies, and surveys.	**1.3.4a** Experiments can be subject to systematic bias if the experiment does not sample from the population randomly and does not randomly assign sampling units to experimental and control conditions. **1.3.4b** Observational studies can be subject to sampling bias if the sampling unit being observed is not selected randomly. **1.3.4c** Surveys can be subject to bias from several factors, including sampling bias and response bias.

Content Boundary: A traditionally challenging concept is informally introduced in this key concept: the law of large numbers (Learning Objective 1.3.2). For this concept, students should not be expected to develop a complete understanding. A full understanding of the law of large numbers is beyond the scope of this course. If students go on to take a more advanced statistics course, such as AP Statistics, they will explore this concept more thoroughly.

Unit 2: Tools and Techniques of Geometric Measurement

Suggested Timing: Approximately 7 weeks

This unit introduces students to the basic objects of geometry and the tools used to explore these objects throughout the remainder of the course. The basic objects students investigate in this unit include lines, rays, segments, and angles. These figures serve as the building blocks of more complex objects that students explore in later units. Students continue to expand their understanding of measurement by developing techniques for quantifying and comparing the attributes of geometric objects. The tools they use to analyze objects include straightedges, compasses, rulers, protractors, dynamic geometry software, the coordinate plane, and right triangles. In addition, students use an informal understanding of transformations throughout the unit to justify whether two basic objects are congruent. They formalize transformations and define congruence and similarity through transformations in Unit 3. This unit culminates with an introduction to right triangle trigonometry, which integrates the tools and techniques of the unit into an investigation of new ways to express the relationship between angle measures and side lengths.

Throughout Units 2–4, specific learning objectives require students to prove geometric concepts. Students' proofs can be organized in a variety of formats, such as two-column tables, flowcharts, or paragraphs. The format of a student's proof is not as important as their ability to justify a mathematical claim or provide a counterexample disproving one. They should develop an understanding that a mathematical proof establishes the truth of a statement by combining previously developed truths into a logically consistent argument.

ENDURING UNDERSTANDINGS

Students will understand that ...

- A formal mathematical argument establishes new truths by logically combining previously known facts.
- Measuring features of geometric figures is the process of assigning numeric values to attributes of the figures, which allows the attributes to be compared.
- Pairs of lines in a plane that never intersect or that intersect at right angles have special geometric and algebraic properties.
- Right triangles are simple geometric shapes in which we can relate the measures of acute angles to ratios of their side lengths.

KEY CONCEPTS

- **2.1: Measurement in geometry –** Using lengths, angles, and distance to describe and compare shapes
- **2.2: Parallel and perpendicular lines –** Determining if and how lines intersect to analyze spatial relationships in the real world
- **2.3: Measurement in right triangles –** Using the relationships between the side lengths and angle measures of right triangles to create new measurements

KEY CONCEPT 2.1: MEASUREMENT IN GEOMETRY

Using lengths, angles, and distance to describe and compare shapes

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
2.1.1 Describe and correctly label a line, ray, or line segment.	**2.1.1a** For any two distinct points in a plane, there is only one line that contains them. **2.1.1b** A line is straight, has no width, extends infinitely in two directions, and contains infinitely many points. A line can be named by a single lowercase letter, or it can be named by any two distinct points that lie on the line. **2.1.1c** A ray is a portion of a line that has a single endpoint and extends infinitely in one direction. A ray can be named by its endpoint and any other point on the ray, with its endpoint listed first. **2.1.1d** A line segment is a portion of a line between and including two endpoints. A line segment can be named by its two endpoints.
2.1.2 Describe and correctly label an angle.	**2.1.2a** An angle is a geometric figure formed when two lines, line segments, or rays share an endpoint. The point common to both lines, line segments, or rays is called the vertex of the angle. **2.1.2b** An angle can be named by its vertex. An angle can also be named using its vertex and the names of a nonvertex point that lies on each of its sides. For such angle names, the point that indicates the vertex is the second of the three points.
2.1.3 Measure a line segment.	**2.1.3a** The length of a line segment is the distance between its endpoints. **2.1.3b** The length of a line segment is measured using a specified unit of measure. Units of measure can be formal or informal.
2.1.4 Measure an angle.	**2.1.4a** An angle can be measured by determining the amount of rotation one ray would make about the vertex of the angle to coincide with the other ray. The amount of rotation is measured as a fraction of the rotation needed to rotate a full circle. **2.1.4b** An angle can be measured with reference to a circle whose center is the vertex of the angle by determining the fraction of the circular arc between the intersection points of the rays and the circle. The length of the circular arc is measured as a fraction of the circle's circumference. **2.1.4c** An angle can be measured in units of radians, equaling the arc length spanned by the angle when its vertex coincides with the center of a unit circle.

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
2.1.5 Prove whether two or more line segments are congruent.	**2.1.5a** Two line segments are congruent if and only if one segment can be translated, rotated, or reflected to coincide with the other segment without changing the length of either line segment. **2.1.5b** Two line segments are congruent if and only if they have the equal lengths.
2.1.6 Prove whether two or more angles are congruent.	**2.1.6a** Two angles are congruent if and only if one angle can be translated, rotated, or reflected to coincide with the other angle without changing the measure of either angle. **2.1.6b** Two angles are congruent if and only if they have equal measures.
2.1.7 Construct a congruent copy of a line segment or an angle.	**2.1.7a** A synthetic geometric construction utilizes only a straightedge and a compass to accurately draw or copy a figure. **2.1.7b** A straightedge is a tool for connecting two distinct points with a line segment. **2.1.7c** A compass is a tool for copying distances between pairs of points.
2.1.8 Calculate the distance between two points.	**2.1.8a** The distance between two points in the plane is the length of the line segment connecting the points. **2.1.8b** The distance between two points in the coordinate plane can be determined by applying the Pythagorean theorem to a right triangle whose hypotenuse is a line segment formed by the two points and whose sides are parallel to each axis.
2.1.9 Solve problems involving segment lengths and/or angle measures.	**2.1.9a** Given line segment $\overline{AC}$ and a point, *B*, that lies on the segment between points *A* and *C*, the measure of segment $\overline{AC}$ is the sum of the measures of segments $\overline{AB}$ and $\overline{BC}$. **2.1.9b** Given $\angle AOC$ and ray $\overrightarrow{OB}$ that lies between $\overrightarrow{OA}$ and $\overrightarrow{OC}$, the measure of $\angle AOC$ is equal to the sum of the measures of $\angle AOB$ and $\angle BOC$. **2.1.9c** Two angles are called complementary if the sum of their measures is 90°. Two angles are complementary if they form a right angle when adjacent. **2.1.9d** Two angles are called supplementary if the sum of their measures is 180°. Two angles are supplementary if they form a straight angle when adjacent.

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
2.1.10 Solve problems involving a segment bisector or an angle bisector.	**2.1.10a** The midpoint of a line segment is the point located on the line segment equidistant from the endpoints. **2.1.10b** In the coordinate plane, the x- and y- coordinates of the midpoint of a line segment are the arithmetic means of the corresponding coordinates of the endpoints. **2.1.10c** A bisector of an angle is a line, ray, or line segment that contains the vertex of the angle and divides the angle into two congruent adjacent angles. **2.1.10d** Points that lie on the angle bisector are equidistant from the sides of the angle.

Content Boundary: When a mathematical statement includes the phrase “if and only if” to join two sentences, it means that the sentences are logically equivalent. That is, both sentences are true or both sentences are false. These sentences are sometimes referred to as “biconditional statements.” Students are expected to know that these statements are both true or both false, but it is not necessary for them to know the term “biconditional” for this course.

KEY CONCEPT 2.2: PARALLEL AND PERPENDICULAR LINES

Determining if and how lines intersect to analyze spatial relationships in the real world

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
2.2.1 Justify the relationship between the slopes of parallel or perpendicular lines in the coordinate plane using transformations.	**2.2.1a** The relationship between the slopes of parallel lines can be justified by comparing their slope triangles using translation. **2.2.1b** The relationship between the slopes of perpendicular lines can be justified by comparing their slope triangles using rotation by 90°.
2.2.2 Solve problems involving two or more parallel lines, rays, or line segments.	**2.2.2a** Two distinct lines, rays, or line segments in the coordinate plane are parallel if and only if they have the same slope or are both vertical. **2.2.2b** A transversal is a line that intersects a set of lines. Two lines, rays, or line segments intersected by a transversal will be parallel if and only if the same-side interior angles formed by the lines and the transversal are supplementary. **2.2.2c** Two lines intersected by a transversal will be parallel if and only if the corresponding angles, alternate interior angles, or alternate exterior angles formed by the lines and the transversal are congruent.
2.2.3 Construct a line, ray, or line segment parallel to another line, ray, or line segment that passes through a point not on the given line, ray, or line segment.	**2.2.3a** Given a line and a point not on the given line, there is exactly one line through the point that will be parallel to the given line. **2.2.3b** Two parallel lines, rays, or line segments in the coordinate plane will have equal slopes and contain no common points.
2.2.4 Solve problems involving the triangle sum theorem.	**2.2.4a** The sum of the interior angles of a triangle in a plane is 180°.
2.2.5 Solve problems involving two or more perpendicular lines, rays, or line segments.	**2.2.5a** A line, ray, or line segment is perpendicular to another line, ray, or line segment if and only if they form right angles at the point where the two figures intersect. **2.2.5b** A line, ray, or line segment is perpendicular to another line, ray, or line segment in the coordinate plane if and only if the two figures intersect and their slopes are opposite reciprocals of each other, or if one is vertical and other is horizontal.
2.2.6 Construct the perpendicular bisector of a line segment.	**2.2.6a** The perpendicular bisector of a line segment intersects the line segment at its midpoint and forms four right angles with the line segment. **2.2.6b** The perpendicular bisector of a line segment is determined by identifying two points in a plane that are equidistant from the endpoints of the line segment and constructing a line, ray, or line segment through those two points. **2.2.6c** Every point that lies on the perpendicular bisector of a line segment is equidistant from the endpoints of the line segment.

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
2.2.7 Construct a line, ray, or line segment perpendicular to another line, ray, or line segment.	**2.2.7a** A horizontal line, ray, or line segment in the coordinate plane is perpendicular to a vertical line, ray, or line segment if they intersect. **2.2.7b** Two perpendicular lines, rays, or line segments in the coordinate plane will intersect and have slopes that are opposite reciprocals of each other, or one will be vertical and the other will be horizontal. **2.2.7c** Applying the perpendicular bisector construction to a point on a line, ray, or line segment is sufficient to construct a line, ray, or line segment perpendicular to the given line, ray, or line segment.

Content Boundary: Some learning objectives in this key concept require students to create both synthetic and analytic arguments for geometric relationships. Pre-AP expects students to use tools and techniques of synthetic geometry to determine, justify, or explain relationships of figures studied in a plane without coordinates and to use tools and techniques of analytic geometry to determine, justify, or explain relationships of figures studied in the coordinate plane. Students are expected to develop proficiency in both realms and to move fluently between them. However, it is not necessary that they use the terms *synthetic* and *analytic*.

Cross Connection: In Pre-AP Algebra 1, students made extensive use of *slope triangles* – right triangles whose legs are parallel to the axes of the coordinate plane – to calculate the slope of a non-vertical and non-horizontal line. In this course, students connect their prior knowledge of slope triangles with their understanding of geometric transformations to gain new insights into the relationships between the slopes of parallel and perpendicular lines.

KEY CONCEPT 2.3: MEASUREMENT IN RIGHT TRIANGLES

Using the relationships between the side lengths and angle measures of right triangles to create new measurements

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
2.3.1 Prove whether two right triangles are similar using informal similarity transformations.	**2.3.1a** Two right triangles are similar if and only if one triangle can be translated, reflected, and/or rotated so it coincides with the other after dilating one triangle by a scale factor. **2.3.1b** Two right triangles are similar if and only if their corresponding angles have equal measures. **2.3.1c** Two right triangles are similar if and only if their corresponding side lengths are in proportion.
2.3.2 Determine the coordinates of a point on a line segment.	**2.3.2a** The coordinates of a point along a line segment in the coordinate plane that divides the line segment into a given ratio can be determined using similar triangles.
2.3.3 Prove the Pythagorean theorem using similar right triangles.	**2.3.3a** An altitude drawn from the right angle of a right triangle to the hypotenuse creates similar right triangles. **2.3.3b** When an altitude is constructed from the right angle to the hypotenuse of a right triangle, the proportions of the side lengths of the similar right triangles formed can be used to prove the Pythagorean theorem.
2.3.4 Associate the measures of an acute angle, $\angle A$, in a right triangle to ratios of the side lengths.	**2.3.4a** The sine of the measure of $\angle A$ is the ratio of the length of the side opposite the angle and the length of the hypotenuse. **2.3.4b** The cosine of the measure of $\angle A$ is the ratio of the length of the side adjacent to the angle and the length of the hypotenuse. **2.3.4c** The tangent of the measure of $\angle A$ is the ratio of the length of the side opposite the angle and the length of the side adjacent to the angle.
2.3.5 Explain why a trigonometric ratio depends only on an angle measure of a right triangle and not on the side lengths.	**2.3.5a** Trigonometric ratios are functions whose input is an acute angle measure and whose output is a ratio of two side lengths in a right triangle. **2.3.5b** The ratio of the lengths of two sides of a right triangle will equal the ratio of the lengths of the corresponding sides of a similar right triangle. Therefore, the ratios of the sides depend only on the angle measure.
2.3.6 Determine an acute angle measure in a right triangle, given a ratio of its side lengths, using an understanding of inverses.	**2.3.6a** For acute angles in a right triangle, the angle measure and the ratio of the lengths of any two specific sides have a one-to-one correspondence. **2.3.6b** Given a ratio of any two side lengths in a right triangle, it is possible to determine the acute angle measures of the right triangle.

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
2.3.7 Model contextual scenarios using right triangles.	**2.3.7a** Contextual scenarios that involve nonvertical and nonhorizontal segments or the distance between two points that do not lie on a vertical or horizontal line can be modeled by right triangles. **2.3.7b** Trigonometric ratios can be used to solve problems or model scenarios involving angles of elevation or depression.

Content Boundary: Formally defining inverse trigonometric functions is beyond the scope of this course. However, students should understand that because each acute angle in a right triangle is uniquely associated with a specific ratio of side lengths, then a ratio of side lengths can be used to determine a specific acute angle in a right triangle. That is, students should be expected to “go forward” by determining the sine, cosine, or tangent of an acute angle and to “go backward” by determining the acute angle whose sine, cosine, or tangent ratio is given. Students are expected to use a scientific calculator to determine an angle measure, given a trigonometric ratio.

Unit 3: Measurement in Congruent and Similar Figures

Suggested Timing: Approximately 7 weeks

Informal transformations are the way we, as humans, compare two objects to see if they are congruent. We turn, twist, and flip objects to see if one can lay exactly on the other without bending, stretching, or breaking either object. When they match, we say the objects are congruent. If they do not match, but they have the same shape and the same scaled measurements, we say the objects are similar. Transformations in geometry give us language to describe these turns, twists, flips, and scaling precisely and systematically. This unit formalizes the concept of congruence and similarity of planar objects by identifying the essential components of rigid motion and similarity transformations. Students are expected to become proficient with transformations that involve coordinates as well as with transformations that do not involve coordinates. Throughout the course, transformations are presented as functions. This connection further develops students' understanding of functions and connects the statistics and geometry units of the course. It also creates a bridge between Algebra 1 and Algebra 2 since concept of function permeates and links nearly all aspects of high school mathematics. Students develop further insights into congruence and similarity by exploring which transformations affect angle measures and distances between pairs of points and which do not. Students apply their understandings of transformations, congruence, and similarity to solve problems involving polygons and circles.

Throughout Units 2–4, specific learning objectives require students to prove geometric concepts. Students' proofs can be organized in a variety of formats, such as two-column tables, flowcharts, or paragraphs. The format of a student's proof is not as important as their ability to justify a mathematical claim or provide a counterexample disproving one. They should develop an understanding that a mathematical proof establishes the truth of a statement by combining previously developed truths into a logically consistent argument.

ENDURING UNDERSTANDINGS

Students will understand that …

- Transformations are functions that can affect the measurements of a geometric figure.
- Congruent figures have equal corresponding angle measures and equal distances between corresponding pairs of points.
- Similar figures have equal corresponding angle measures, and the distances between corresponding pairs of points are proportional.
- The geometry of a circle is completely determined by its radius.

KEY CONCEPTS

- **3.1: Transformations of points in a plane** – Defining transformations to describe the movement of points and shapes
- **3.2: Congruent and similar polygons** – Using transformations to compare figures with the same size or same shape
- **3.3: Measurement of lengths and angles in circles** – Using measurements in circles to make sense of round flat objects in the physical world

KEY CONCEPT 3.1: TRANSFORMATIONS OF POINTS IN A PLANE

Defining transformations to describe the movement of points and shapes

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
3.1.1 Perform transformations on points in a plane.	**3.1.1a** Transformations describe motions in the plane. Analyzing these transformations indicates if and how these motions affect lengths and angle measures of figures. Congruence and similarity are defined in terms of measurements that are preserved by transformations. **3.1.1b** A transformation is a function whose inputs and outputs are points in the plane. A set of all input points of a transformation is called a preimage; a set of all output points of the preimage is called an image. **3.1.1c** A rigid motion transformation preserves both the distance between pairs of points and the angle measures. A similarity transformation preserves angle measures but not necessarily distances between pairs of points.
3.1.2 Express transformations using function notation.	**3.1.2a** Given a transformation T and two points, A and B, the notation $T(A)=B$ means that the image of point A under transformation T is point B. The transformation is said to map point A to point B. **3.1.2b** Algebra can be used to express how a transformation affects the x- and y-coordinates of points. All transformations can be represented using function notation, but some transformations are difficult to define as algebraic expressions.
3.1.3 Prove that a rigid motion transformation maps an object to a congruent object.	**3.1.3a** A rigid motion transformation is a transformation that preserves distances between pairs of points as well as angle measures. **3.1.3b** A translation is a transformation that maps each point in the plane to an image that is a specified distance in a specified direction from the preimage. **3.1.3c** A reflection is a transformation that maps each point in the plane to its mirror image across a line called the axis of symmetry. **3.1.3d** A rotation is a transformation that maps each point in the plane to an image that is turned by a specified angle about a fixed point called the center of rotation.

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
3.1.4 Solve problems involving rigid motion transformations.	**3.1.4a** Applying one or more translations, rotations, and reflections maps an object to a congruent object. **3.1.4b** Any transformation that preserves distance between points and angle measures can be written as a sequence of translations, reflections, and/or rotations. **3.1.4c** If two figures are congruent, there must exist a sequence of one or more rigid motion transformations that maps one figure to the other.
3.1.5 Prove that a similarity transformation maps an object to a similar object.	**3.1.5a** A similarity transformation is a sequence of a dilation and/or one or more rigid motion transformations. **3.1.5b** A dilation from a fixed point, called the center, with a scale factor k is a transformation that maps each point in the plane to an image whose distance from the center is k times the distance between the center and the preimage, in the same direction as the preimage. **3.1.5c** Dilations of figures do not affect the angle measures of a figure.
3.1.6 Solve problems involving similarity transformations.	**3.1.6a** Dilating the plane by a scale factor k with center $(0, 0)$ will scale each coordinate by k. **3.1.6b** A dilation maps a line not passing through the center of the dilation to a parallel line and maps a line passing through the center of dilation to itself. **3.1.6c** The scale factor of a dilation can be determined by dividing a length from the image by its corresponding length in the preimage. **3.1.6d** The perimeter of the image of a figure is the perimeter of the preimage scaled by the same scale factor as the dilation.

Content Boundary: Students are expected to use algebra to express translations in the coordinate plane, reflections across the x-axis, the y-axis, and the line $y = x$, and rotations about the origin, clockwise or counterclockwise, by angles of 90° and 180°. Students are also expected to identify axes of symmetry and angles of rotation beyond those listed above. However, using algebra to express reflections across lines other than those listed, or rotations about angles other than 90° or 180° is beyond the scope of the course. It is most important that students understand that some transformations are difficult to express using algebra, but that function notation can be used to communicate the relationship between the inputs and outputs of any transformation.

KEY CONCEPT 3.2: CONGRUENT AND SIMILAR POLYGONS

Using transformations to compare figures with the same size or same shape

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
3.2.1 Prove that two triangles are congruent by comparing their side lengths and angle measures.	**3.2.1a** If the three sides and three angles of a triangle are congruent to the three sides and three angles of another triangle, then the two triangles are congruent. **3.2.1b** If two triangles are congruent, then all six corresponding parts of the triangles are also congruent.
3.2.2 Prove that two triangles are congruent by comparing specific combinations of side lengths and angle measures.	**3.2.2a** If three sides of one triangle are congruent to three sides of another triangle, then the triangles are congruent (SSS). **3.2.2b** If two sides of a triangle and the interior angle they form are congruent to two sides of another triangle and the interior angle they form, then the triangles are congruent (SAS). **3.2.2c** If two angles and the side adjacent to both angles of a triangle are congruent to two angles and the side adjacent to both angles in another triangle, then the triangles are congruent (ASA).
3.2.3 Prove that two triangles are similar.	**3.2.3a** Two triangles are similar if and only if they have three pairs of congruent angles. **3.2.3b** Two triangles are similar if and only if the lengths of their corresponding sides are in proportion. **3.2.3c** Two triangles are similar if and only if one can be mapped to coincide with the other after applying a similarity transformation.
3.2.4 Prove theorems about parallelograms.	**3.2.4a** Proofs about parallelograms are based on relationships among their sides, angles, and diagonals. **3.2.4b** A line segment between two opposite vertices in a parallelogram forms two congruent triangles that share a common side. **3.2.4c** For a parallelogram in the coordinate plane, the slopes of the sides and diagonals can be used to prove statements about the parallelogram.

Content Boundary: This key concept is traditionally the major focus of high school geometry courses. It is certainly valuable that students prove theorems about congruent and similar triangles and quadrilaterals. Students are expected to use a variety of formats to construct mathematical arguments including but not limited to two-column proofs and paragraph proofs. The format of a student's proof is not as important as their ability to justify or provide a counterexample to a mathematical claim.

KEY CONCEPT 3.3: MEASUREMENT OF LENGTHS AND ANGLES IN CIRCLES

Using measurements in circles to make sense of round flat objects in the physical world

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
3.3.1 Determine whether a particular point lies on a given circle.	**3.3.1a** A point in the coordinate plane lies on a circle if its coordinates satisfy the equation of a circle. **3.3.1b** All points that lie on a circle are equidistant from the center of the circle.
3.3.2 Translate between the geometric and algebraic representations of a circle.	**3.3.2a** A circle is the set of all points equidistant from a given point. **3.3.2b** In the coordinate plane, the graph of the equation $(x-h)^2+(y-k)^2=r^2$ is the set of all points located r units from the point (h, k). This is a circle with radius r and center (h, k).
3.3.3 Prove that any two circles are similar.	**3.3.3a** Every circle can be expressed as the image of any other circle under a similarity transformation.
3.3.4 Determine the measure of a central angle or the circular arc it intercepts.	**3.3.4a** A central angle is an angle whose vertex is the center of a circle and whose sides are, or contain, two radii of the circle. **3.3.4b** The measure of an arc is defined as the measure of the central angle that intercepts the arc.
3.3.5 Determine the measure of an inscribed angle or the circular arc it intercepts.	**3.3.5a** An inscribed angle is an angle whose vertex lies on a circle and whose sides contain chords of the circle. **3.3.5b** The measure of an inscribed angle is half the measure of the arc it intercepts. Equivalently, the measure of the intercepted arc is twice the measure of the inscribed angle. **3.3.5c** Inscribed angles that intercept the same arc have equal angle measures.
3.3.6 Determine the length of a circular arc.	**3.3.6a** The length of a circular arc depends on the measure of the central angle that intercepts the arc and the radius of the circle. **3.3.6b** The ratio of the length of a circular arc and the circumference of the circle is equal to the ratio of the measure of the central angle that intercepts the arc and the angle measure of a full circle.
3.3.7 Construct a line, ray, or line segment tangent to a circle.	**3.3.7a** A line, ray, or line segment tangent to a circle intersects the circle at exactly one point. **3.3.7b** A line, ray, or line segment tangent to a circle is perpendicular to a radius of the circle at the point of intersection. **3.3.7c** In the coordinate plane, the slope of the line, ray, or line segment tangent to the circle and the slope of the radius that intersects this tangent line, ray, or line segment will be opposite reciprocals, or one will be vertical and the other will be horizontal.

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
3.3.8 Solve a system of equations consisting of a linear equation and the equation of a circle.	**3.3.8a** The intersection of a line and a circle corresponds to an algebraic solution of the system of their corresponding equations. **3.3.8b** An algebraic solution to a system of equations is an ordered pair that makes all equations true simultaneously. The system may have zero, one, or two solutions.

Content Boundary: It is beyond the scope of the course for students to know that a unit circle has a radius of length 1, or to know the coordinates of points on the circle that correspond to special reference angles.

Cross Connection: The length of a circular arc, as defined through Learning Objective 3.3.6, explicitly connects students' prior knowledge of ratios to their current study of geometry. It is more important for students to understand that an arc length is proportional to the circumference of the circle than it is for them to memorize a formula relating arc length and central angle measure.

Unit 4: Measurement in Two and Three Dimensions

Suggested Timing: Approximately 7 weeks

This unit deepens students' understanding of measurement by expanding the concept of measurement to two dimensions through the areas of planar figures and to three dimensions through volumes of solid figures. One reason for studying area is that it often represents quantities that are otherwise difficult to compute. For example, the area under the graph of an object's speed corresponds to its total distance traveled. Therefore, techniques for calculating area can be adapted to find other quantities. Students likely have prior experience with calculating the areas of conventional figures and composites of those figures. Students may also have experience calculating the volumes of conventional solids. The unit introduces students to Cavalieri's principle, which relates the area of a figure to its cross-sectional lengths and the volume of a solid to its cross-sectional areas. The focus of the unit is on justifying area and volume formulas with which students are already familiar and using area and volume to model real-world physical scenarios.

Throughout Units 2-4, specific learning objectives require students to prove geometric concepts. Students' proofs can be organized in a variety of formats, such as two-column tables, flowcharts, or paragraphs. The format of a student's proof is not as important as their ability to justify a mathematical claim or provide a counterexample disproving one. They should develop an understanding that a mathematical proof establishes the truth of a statement by combining previously developed truths into a logically consistent argument.

ENDURING UNDERSTANDINGS

Students will understand that ...

- The area of a figure depends on its height and its cross-sectional widths.
- The volume of a solid depends on its height and its cross-sectional areas.
- The geometry of a sphere is completely determined by its radius.

KEY CONCEPTS

- **4.1: Area as a two-dimensional measurement –** Connecting one- and two-dimensional measurements to develop an understanding of area as a measurement of flat coverage
- **4.2: Volume as a three-dimensional measurement –** Connecting two- and three-dimensional measurements to develop an understanding of volume as a measurement of space occupied
- **4.3: Measurements of spheres –** Measuring areas and volumes of spheres to make sense of round objects in the physical world

KEY CONCEPT 4.1: AREA AS A TWO-DIMENSIONAL MEASUREMENT

Connecting one- and two-dimensional measurements to develop an understanding of area as a measurement of flat coverage

Learning Objectives *Students will be able to …*	**Essential Knowledge** *Students need to know that …*
4.1.1 Use Cavalieri's principle to solve problems involving the areas of figures.	**4.1.1a** If two figures have congruent bases and equal heights, and line segments in the interiors of those figures that are parallel to, and equal distances from the base are congruent, then the figures will have equal area.
4.1.2 Determine the area of a sector.	**4.1.2a** The area of a sector depends on the measure of the central angle that forms the sector and the radius of the circle. **4.1.2b** The ratio of the area of a sector and the area of the circle is equal to the ratio of the measure of the central angle that forms the sector and the angle measure of a full circle.
4.1.3 Determine the effect of a similarity transformation on the area of a figure.	**4.1.3a** The area of the image of a figure is the area of the preimage scaled by the square of the scale factor of the dilation.

Content Boundary: Prior to this course, students will have explored the area formulas for planar figures, such as triangles, quadrilaterals, other polygons, and circles. In this course, students deepen their understanding of area by connecting length measures within a figure to its area, and by using area to solve real-world problems.

Cross Connection: The area of a sector, as defined through Learning Objective 4.1.2, explicitly connects students' prior knowledge of ratios to their current study of geometry. As with the length of a circular arc in Unit 3, it is more important that students understand that the area of a sector is proportional to the area inside the circle than it is for them to memorize the related formula.

KEY CONCEPT 4.2: VOLUME AS A THREE-DIMENSIONAL MEASUREMENT

Connecting two- and three-dimensional measurements to develop an understanding of volume as a measurement of space occupied

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
4.2.1 Justify the volume formula for a right prism.	**4.2.1a** The cross section of a right prism is the polygon formed by the intersection of the solid with a plane parallel to its base. **4.2.1b** The volume of a right prism is equal to the product of the height of the solid and the area of its base.
4.2.2 Justify the volume formula for pyramids.	**4.2.2a** The cross section of a pyramid is the polygon formed by the intersection of the solid with a plane parallel to its base. **4.2.2b** The volume of a pyramid is equal to one-third of the volume of its associated prism. That is, the volume of a pyramid is one-third of the product of the height of the solid and the area of its base.
4.2.3 Justify the volume formula for a right cylinder.	**4.2.3a** The cross section of a right cylinder is the circle formed by the intersection of the solid with a plane parallel to its base. **4.2.3b** The volume of a right cylinder is equal to the product of the height of the solid and the area of its base.
4.2.4 Justify the volume formula for a cone.	**4.2.4a** The cross section of a cone is the circle formed by the intersection of the solid with a plane parallel to its base. **4.2.4b** The volume of a cone is equal to one-third of the volume of its associated cylinder. That is, the volume of a cone is one-third of the product of the height of the solid and the area of its base.
4.2.5 Use Cavalieri's principle to solve problems involving volumes of solids.	**4.2.5a** If two solid figures have congruent bases and equal heights, and cross sections that are parallel to and equal distances from each base are congruent, then the solids have equal volume.
4.2.6 Solve contextual problems involving volume of solid figures.	**4.2.6a** Physical objects in many real-world scenarios can be modeled by solid geometric figures such as prisms, pyramids, cylinders, and cones.

Cross Connection: The concept of Cavalieri's principle, which students use to solve problems involving volumes of solids, connects the area of a cross section of a solid to the volume of that solid. Understanding the relationship between the area of a cross section and the volume of a solid will help students who progress to AP Calculus make sense of why finding volumes of solids is an application of integration.

KEY CONCEPT 4.3: MEASUREMENTS OF SPHERES

Measuring areas and volumes of spheres to make sense of round objects in the physical world

Learning Objectives *Students will be able to ...*	**Essential Knowledge** *Students need to know that ...*
4.3.1 Define spheres in terms of distance.	**4.3.1a** A sphere is an object in three-dimensional space that is the set of all points equidistant from a given point, called its center.
4.3.2 Justify the surface area formula for a sphere.	**4.3.2a** The surface area of a sphere is given by the formula $SA = 4\pi r^2$, where r represents the length of the radius of the sphere.
4.3.3 Justify the volume formula for a solid sphere.	**4.3.3a** The volume of a solid sphere is given by the formula $V = \frac{4}{3}\pi r^3$, where r represents the length of the radius of the sphere.
4.3.4 Solve contextual problems using spheres.	**4.3.4a** Round physical objects in real-world scenarios can be modeled by spheres.

Content Boundary: It is likely that students have some familiarity with the surface area and volume formulas for a sphere. The focus of this key concept is for students to develop an informal understanding of the derivation of the surface area and volume formulas for spheres and to use spheres to model physical scenarios.

Pre-AP Geometry with Statistics Model Lessons

Model lessons in Pre-AP Geometry with Statistics are developed in collaboration with geometry and statistics educators across the country and are rooted in the course framework, shared principles, and areas of focus. Model lessons are carefully designed to illustrate on-grade-level instruction. Pre-AP strongly encourages teachers to internalize the lessons and then offer the supports, extensions, and adaptations necessary to help all students achieve the lesson goals.

The purpose of these model lessons is twofold:

- **Robust instructional support for teachers:** Pre-AP Geometry with Statistics model lessons are comprehensive lesson plans that, along with accompanying student resources, embody the Pre-AP approach to teaching and learning. Model lessons provide clear and substantial instructional guidance to support teachers as they engage students in the shared principles and areas of focus.
- **Key instructional strategies:** Commentary and analysis embedded in each lesson highlight not just what students and teachers do in the lesson, but also how and why they do it. This educative approach provides a way for teachers to gain unique insight into key instructional moves that are powerfully aligned with the Pre-AP approach to teaching and learning. In this way, each model lesson works to support teachers in the moment of use with students in their classroom.

Teachers have the option to use any or all model lessons alongside their own locally developed instructional resources. Model lessons target content areas that tend to be challenging for teachers and students. While the lessons are distributed throughout all four units, they are concentrated more heavily in the beginning of the course to support teachers and students in establishing a strong foundation in the Pre-AP approach to teaching and learning.

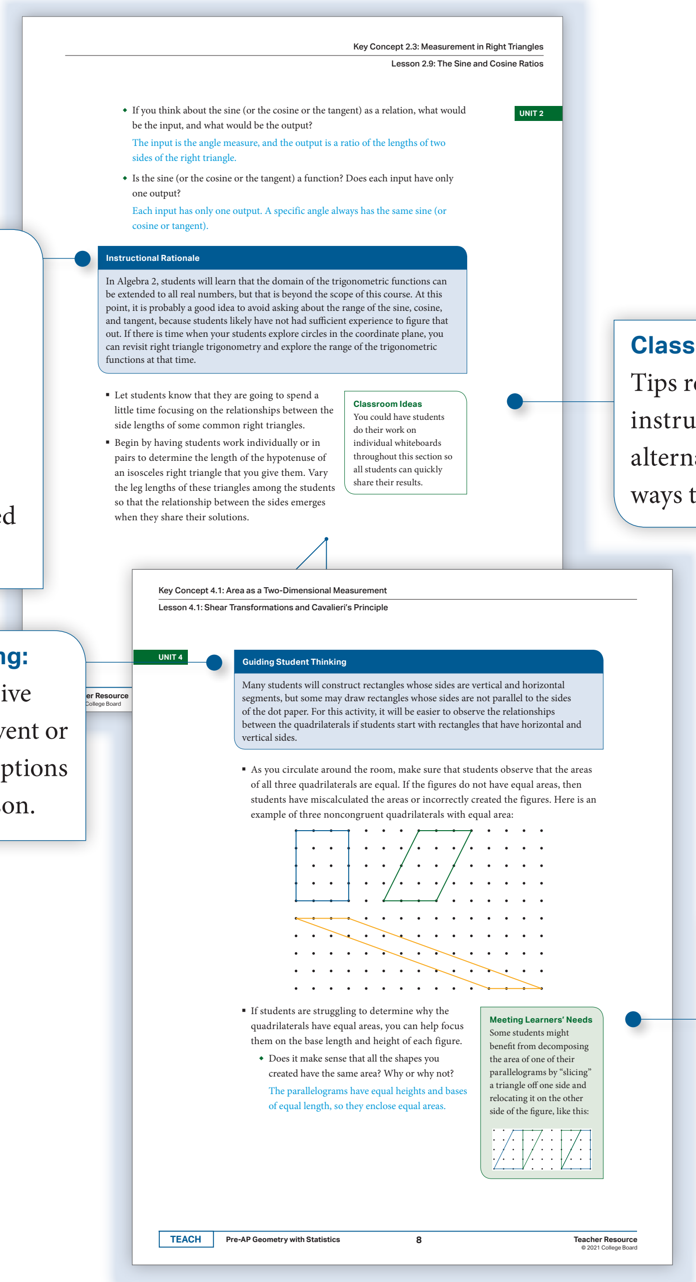
SUPPORT FEATURES IN MODEL LESSONS

The following support features recur throughout the Pre-AP Geometry with Statistics lessons, to promote teacher understanding of the lesson design and provide direct-to-teacher strategies for adapting lessons to meet their students' needs:

- Instructional Rationale
- Guiding Student Thinking
- Meeting Learners' Needs
- Classroom Ideas

Instructional Rationale: Insight into the strategic design and purpose of the instructional choices, flow, and scaffolding within the model lesson. Rationales often describe how a concept is continued later in the lesson or unit.

Guiding Student Thinking: Ways to facilitate productive student thinking and prevent or address student misconceptions in critical areas of the lesson.



Key Concept 2.3: Measurement in Right Triangles

Lesson 2.9: The Sine and Cosine Ratios

UNIT 2

- If you think about the sine (or the cosine or the tangent) as a relation, what would be the input, and what would be the output?
  The input is the angle measure, and the output is a ratio of the lengths of two sides of the right triangle.
- Is the sine (or the cosine or the tangent) a function? Does each input have only one output?
  Each input has only one output. A specific angle always has the same sine (or cosine or tangent).

Instructional Rationale

In Algebra 2, students will learn that the domain of the trigonometric functions can be extended to all real numbers, but that is beyond the scope of this course. At this point, it is probably a good idea to avoid asking about the range of the sine, cosine, and tangent, because students likely have not had sufficient experience to figure that out. If there is time when your students explore circles in the coordinate plane, you can revisit right triangle trigonometry and explore the range of the trigonometric functions at that time.

- Let students know that they are going to spend a little time focusing on the relationships between the side lengths of some common right triangles.
- Begin by having students work individually or in pairs to determine the length of the hypotenuse of an isosceles right triangle that you give them. Vary the leg lengths of these triangles among the students so that the relationship between the sides emerges when they share their solutions.

Classroom Ideas

You could have students do their work on individual whiteboards throughout this section so all students can quickly share their results.

Key Concept 4.1: Area as a Two-Dimensional Measurement

Lesson 4.1: Shear Transformations and Cavalieri's Principle

UNIT 4

Guiding Student Thinking

Many students will construct rectangles whose sides are vertical and horizontal segments, but some may draw rectangles whose sides are not parallel to the sides of the dot paper. For this activity, it will be easier to observe the relationships between the quadrilaterals if students start with rectangles that have horizontal and vertical sides.

- As you circulate around the room, make sure that students observe that the areas of all three quadrilaterals are equal. If the figures do not have equal areas, then students have miscalculated the areas or incorrectly created the figures. Here is an example of three noncongruent quadrilaterals with equal area:

- If students are struggling to determine why the quadrilaterals have equal areas, you can help focus them on the base length and height of each figure.
  - Does it make sense that all the shapes you created have the same area? Why or why not?
    The parallelograms have equal heights and bases of equal length, so they enclose equal areas.

Meeting Learners' Needs

Some students might benefit from decomposing the area of one of their parallelograms by "slicing" a triangle off one side and relocating it on the other side of the figure, like this:

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Classroom Ideas: Tips related to the logistics of the instruction, such as suggestions for alternative presentation methods or ways to alleviate pacing concerns.

Meeting Learners' Needs: Optional differentiation strategies to address diverse learning needs, such as ideas for just-in-time skill building during a lesson or ways to break a task into smaller tasks, if needed, to make it more accessible.

Pre-AP Geometry with Statistics Assessments for Learning

Pre-AP Geometry with Statistics assessments function as a component of the teaching and learning cycle. Progress is not measured by performance on any single assessment. Rather, Pre-AP Geometry with Statistics offers a place to practice, to grow, and to recognize that learning takes time. The assessments are updated and refreshed periodically.

LEARNING CHECKPOINTS

Based on the Pre-AP Geometry with Statistics Course Framework, the learning checkpoints require students to examine data, models, diagrams, and short texts—set in authentic contexts—in order to respond to a targeted set of questions that measure students' application of the key concepts and skills from the unit. All eight learning checkpoints are automatically scored, with results provided through feedback reports that contain explanations of all questions and answers as well as individual and class views for educators. Teachers also have access to assessment summaries on Pre-AP Classroom, which provide more insight into the question sets and targeted learning objectives for each assessment event.

The following tables provide a synopsis of key elements of the Pre-AP Geometry with Statistics learning checkpoints.

Format	Two learning checkpoints per unit Digitally administered with automated scoring and reporting Questions target both concepts and skills from the course framework
Time Allocated	Designed for one 45-minute class period per assessment
Number of Questions	10–12 questions per assessment ▪ 7–9 four-option multiple choice ▪ 3–5 technology-enhanced questions

Domains Assessed	
Learning Objectives	Learning objectives within each key concept from the course framework
Skills	Three skill categories aligned to the Pre-AP mathematics areas of focus are assessed regularly across all eight learning checkpoints: ▪ greater authenticity of applications and modeling ▪ engagement in mathematical argumentation ▪ connections among multiple representations

Question Styles	Question sets consist of two to three questions that focus on a single stimulus or group of related stimuli, such as diagrams, graphs, or tables. Questions embed mathematical concepts in real-world contexts. *Please see page 64 for a sample question set that illustrates the types of questions included in Pre-AP learning checkpoints and the Pre-AP final exam.*

PERFORMANCE TASKS

Each unit includes one performance-based assessment designed to evaluate the depth of student understanding of key concepts and skills that are not easily assessed in a multiple-choice format.

These tasks, developed for high school students across a broad range of readiness levels, are accessible while still providing sufficient challenge and the opportunity to practice the analytical skills that will be required in AP mathematics courses and for college and career readiness. Teachers participating in the official Pre-AP Program will receive access to online learning modules to support them in evaluating student work for each performance task.

Format	One performance task per unit Administered in print Educator scored using scoring guidelines
Time Allocated	Approximately 45 minutes or as indicated
Number of Questions	An open-response task with multiple parts

Domains Assessed	
Key Concepts	Key concepts and prioritized learning objectives from the course framework
Skills	Three skill categories aligned to the Pre-AP mathematics areas of focus: ▪ greater authenticity of applications and modeling ▪ engagement in mathematical argumentation ▪ connections among multiple representations

PRACTICE PERFORMANCE TASKS

Practice performance tasks in each unit provide students with the opportunity to practice applying skills and knowledge in a context similar to a performance task, but in a more scaffolded environment. These tasks include strategies for adapting instruction based on student performance and ideas for modifying or extending tasks based on students' needs.

Performance Assessments At-a-Glance

Unit	Performance Assessment	Title	Teacher Access	Student Access
Unit 1 Measurement in Data	Practice Performance Task	Staffing the Grocery Store	Teacher Resources: Units 1 & 2	Student Resources: Unit 1
	Practice Performance Task	Are Grades and Homework Connected?		Student Resources: Unit 1
	Performance Task	Designing a Study		Teacher-distributed handout
Unit 2 Tools and Techniques of Geometric Measurement	Practice Performance Task	The Flatiron Footprint	Teacher Resources: Units 1 & 2	Student Resources: Unit 2
	Performance Task	Prove Me Wrong		Teacher-distributed handout

FINAL EXAM

Pre-AP Geometry with Statistics includes a final exam featuring multiple-choice and technology-enhanced questions as well as an open-response question. The final exam is a summative assessment designed to measure students' success in learning and applying the knowledge and skills articulated in the Pre-AP Geometry with Statistics Course Framework. The final exam's development follows best practices such as multiple levels of review by educators and experts in the field for content accuracy, fairness, and sensitivity. The questions on the final exam have been pretested, and the resulting data are collected and analyzed to ensure that the final exam is fair and represents an appropriate range of the knowledge and skills of the course.

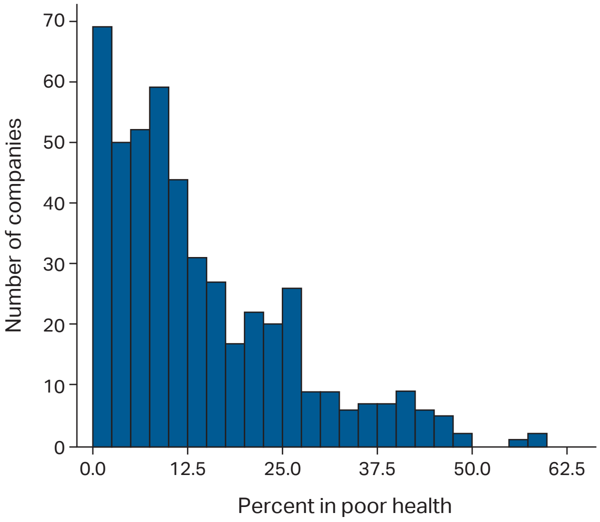
The final exam is designed to be delivered on a secure digital platform in a classroom setting. Educators have the option of administering the final exam in a single extended session or two shorter consecutive sessions to accommodate a range of final exam schedules.

Multiple-choice and technology-enhanced questions are delivered digitally and scored automatically with detailed score reports available to educators. This portion of the final exam is designed to build on the question styles and formats of the learning checkpoints; thus, in addition to their formative purpose, the learning checkpoints provide practice and familiarity with the final exam. The open-response question, modeled after the performance tasks, is delivered as part of the digital final exam but is designed to be scored separately by educators using scoring guidelines that are designed and vetted with the question.

SAMPLE ASSESSMENT QUESTIONS

The following questions are representative of what students and educators will encounter on the learning checkpoints and final exam.

1.



A historian was interested in researching the health of Virginia soldiers near the start of the American Civil War. The data collected for each company, or small group, of soldiers were the percentage of soldiers in each company that were in poor health. A histogram of the data is shown in the figure.

Which of the following statements is most consistent with the distribution of soldiers in poor health?

(A) The total number of companies that reported a percent of soldiers in poor health that was above the mean is equal to the total number of companies that reported a percent of soldiers in poor health below the mean.

(B) The mean and median values for percent of soldiers in poor health are similar.

(C) The variance and the interquartile range of values for percent of soldiers in poor health are similar.

(D) The median value for the percent of soldiers in poor health is less than the mean value.

Assessment Focus

In question 1, students must analyze the distribution of data to determine the relationship between the mean and median based on the skew in the distribution.

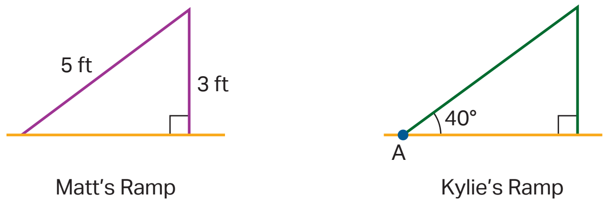
Correct Answer: D

Learning Objective:

1.1.5 Analyze data distributions with respect to their variability.

Area of Focus: Greater Authenticity of Applications and Modeling

2.



Two friends, Matt and Kylie, use different skateboard ramps at their local skate park, as represented in the diagram. They want to know which ramp is steeper. Matt says his ramp has a vertical support (shown) that is 3-feet tall and a ramp length of 5 feet. Kylie measured the angle that the bottom of her ramp made with the ground (angle A) and found it to be 40°.

Based on the known measurements, which ramp is steeper?

(A) Matt's ramp has a slope of $\frac{3}{5}$, which is greater than Kylie's ramp slope of $\tan(40°)$.

(B) Matt's ramp has a slope of $\frac{3}{4}$, which is greater than Kylie's ramp slope of $\sin(40°)$.

(C) Kylie's ramp has a slope of $\cos(40°)$, which is greater than Matt's ramp slope of $\frac{3}{5}$.

(D) Kylie's ramp has a slope of $\tan(40°)$, which is greater than Matt's ramp slope of $\frac{3}{4}$.

Assessment Focus

Question 2 requires students to apply the Pythagorean theorem and use the tangent of an angle to determine and compare the slopes of two different segments. Students use these slopes as evidence to support a claim.

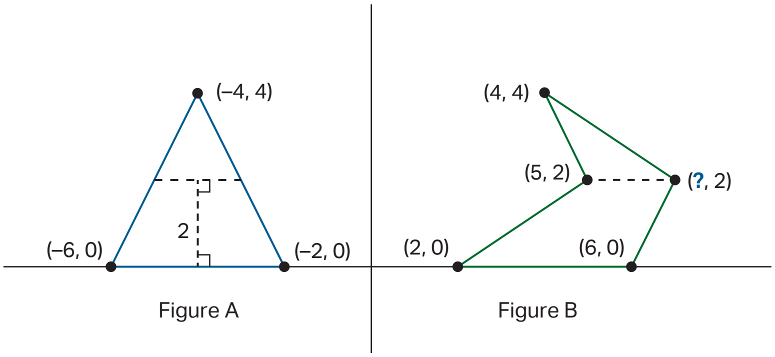
Correct Answer: D

Learning Objective:

2.3.4 Associate the measures of an acute angle, $\angle A$, in a right triangle to ratios of the side lengths.

Area of Focus: Engagement in Mathematical Argumentation

3. The following two figures have the same area.



In the diagram of Figure B, one x-coordinate is represented with a question mark (?). Sahir claims that this missing x-coordinate is 7. Which of the following reasons best supports his claim?

(A) The horizontal distance between points on the sides of Figure B is 2, therefore the missing x-coordinate is 2 more than 5.

(B) The horizontal distances at the same height between two points of Figure A and two points of Figure B are equal.

(C) The slope of the line segment whose endpoints are (6, 0) and (7, 2) is 2.

(D) If points on Figures A and B have the same y-coordinate, then the points will have the same horizontal distance from the y-axis.

Assessment Focus

Question 3 requires students to apply Cavalieri's principle to support a claim about a missing coordinate in a shape.

Correct Answer: B

Learning Objective:

4.1.1 Use Cavalieri's principle to solve problems involving the areas of figures.

Area of Focus: Engagement in Mathematical Argumentation

Pre-AP Geometry with Statistics Course Designation

Schools can earn an official Pre-AP Geometry with Statistics course designation by meeting the requirements summarized below. Pre-AP Course Audit Administrators and teachers will complete a Pre-AP Course Audit process to attest to these requirements. All schools offering courses that have received a Pre-AP Course Designation will be listed in the Pre-AP Course Ledger, in a process similar to that used for listing authorized AP courses.

PROGRAM REQUIREMENTS

- The school ensures that Pre-AP frameworks and assessments serve as the foundation for all sections of the course at the school. This means that the school must not establish any barriers (e.g., test scores, grades in prior coursework, teacher or counselor recommendation) to student access and participation in the Pre-AP Geometry with Statistics coursework.
- Teachers have read the most recent *Pre-AP Geometry with Statistics Course Guide*.
- Teachers administer each performance task and at least one of two learning checkpoints per unit.
- Teachers and at least one administrator per site complete a Pre-AP Summer Institute or the Online Foundational Module Series. Teachers complete at least one Online Performance Task Scoring Module.
- Teachers align instruction to the Pre-AP Geometry with Statistics Course Framework and ensure their course meets the curricular requirements summarized below.
- The school ensures that the resource requirements summarized below are met.

CURRICULAR REQUIREMENTS

- The course provides opportunities for students to develop understanding of the Pre-AP Geometry with Statistics key concepts and skills articulated in the course framework through the four units of study.
- The course provides opportunities for students to engage in the Pre-AP shared instructional principles.
    - close observation and analysis
    - evidence-based writing
    - higher-order questioning
    - academic conversation

- The course provides opportunities for students to engage in the three Pre-AP mathematics areas of focus. The areas of focus are:
  - greater authenticity of applications and modeling
  - engagement in mathematical argumentation
  - connections among multiple representations
- The instructional plan for the course includes opportunities for students to continue to practice and develop disciplinary skills.
- The instructional plan reflects time and instructional methods for engaging students in reflection and feedback based on their progress.
- The instructional plan reflects making responsive adjustments to instruction based on student performance.

RESOURCE REQUIREMENTS

- The school ensures that participating teachers and students are provided computer and internet access for completion of course and assessment requirements.
- Teachers should have consistent access to a video projector for sharing web-based instructional content and short web videos.

Accessing the Digital Materials

Pre-AP Classroom is the online application through which teachers and students can access Pre-AP instructional resources and assessments. The digital platform is similar to AP Classroom, the online system used for AP courses.

Pre-AP coordinators receive access to Pre-AP Classroom via an access code delivered after orders are processed. Teachers receive access after the Pre-AP Course Audit process has been completed.

Once teachers have created course sections, student can enroll in them via access code. When both teachers and students have access, teachers can share instructional resources with students, assign and score assessments, and complete online learning modules; students can view resources shared by the teacher, take assessments, and receive feedback reports to understand progress and growth.

Unit 1

Unit 1
Measurement in Data

Overview

SUGGESTED TIMING: APPROXIMATELY 7 WEEKS

This unit offers a sustained and focused examination of statistics and probability to support the development of students' quantitative literacy. Statistics and probability help us perform essential real-world tasks such as making informed choices, deciding between different policies, and weighing competing knowledge claims. While topics of statistics and probability are commonplace in high school geometry courses, students often have limited opportunities to engage in statistical and probabilistic reasoning and sense-making. To move students toward a sophisticated understanding of data, students are expected to think about data sets as distributions which are functions that associate data values with their frequency or their probability. This encourages students to connect their knowledge of functions to concepts of statistics and probability, creating a more complete understanding of mathematics.

Throughout the unit, students generate their own data through surveys, experiments, and simulations that investigate some aspect of the real world. They engage in statistical calculations and probabilistic reasoning as methods of analysis to make sense of data and draw inferences about populations. Incorporating statistics and probability in the same course as geometry allows students to experience two distinct forms of argumentation: geometrical reasoning as drawing conclusions with certainty about an ideal mathematical world, and probabilistic reasoning as drawing less-than-certain conclusions about the real world. The conclusions of a probability argument are presented as ranges that have varying degrees of certainty.

ENDURING UNDERSTANDINGS

This unit focuses on the following enduring understandings:

- Statistics are numbers that summarize large data sets by reducing their complexity to a few key values that model their center and spread.
- Distributions are functions whose displays are used to analyze data sets.
- Probabilistic reasoning allows us to anticipate patterns in data.
- The method by which data are collected influences what can be said about the population from which the data were drawn, and how certain those statements are.

KEY CONCEPTS

This unit focuses on the following key concepts:

- 1.1: The Shape of Data
- 1.2: Chance Events
- 1.3: Inferences from Data

UNIT RESOURCES

The tables below outline the resources provided by Pre-AP for this unit.

Lessons for Key Concept 1.1: The Shape of Data				
Lesson Title	**Learning Objectives Addressed**	**Essential Knowledge Addressed**	**Suggested Timing**	**Areas of Focus**
Lesson 1.1: A-Maze-ing Statistics	1.1.1, 1.1.2	1.1.1c, 1.1.1d, 1.1.2a	~90 minutes	Connections Among Multiple Representations, Greater Authenticity of Applications and Modeling
Lesson 1.2: Exploring Variables	1.1.2	1.1.2a, 1.1.2b	~135 minutes	Greater Authenticity of Applications and Modeling, Connections Among Multiple Representations

Lesson 1.3: Measures of Center	1.1.1, 1.1.3, 1.1.4, 1.1.6	1.1.1a, 1.1.1b, 1.1.3a, 1.1.3b, 1.1.3c, 1.1.4a, 1.1.4b, 1.1.4c, 1.1.6c	~90 minutes	Greater Authenticity of Applications and Modeling, Connections Among Multiple Representations
Lesson 1.4: Standard Deviation and Variance	1.1.1, 1.1.5	1.1.1b, 1.1.1d, 1.1.5a, 1.1.5b, 1.1.5c, 1.1.5d	~90 minutes	Greater Authenticity of Applications and Modeling, Connections Among Multiple Representations
Lesson 1.5: Distributions as Functions	1.1.2	1.1.2b	~90 minutes	Greater Authenticity of Applications and Modeling, Connections Among Multiple Representations
Lesson 1.6: The Normal Distribution	1.1.6	1.1.6a, 1.1.6b, 1.1.6c	~90 minutes	Greater Authenticity of Applications and Modeling, Connections Among Multiple Representations
All learning objectives and essential knowledge statements for this key concept are addressed with the provided materials.				

Learning Checkpoint 1: Key Concept 1.1 (~45 minutes)

This learning checkpoint assesses learning objectives and essential knowledge statements from Key Concept 1.1. For sample items and learning checkpoint details, visit Pre-AP Classroom.

UNIT 1

Practice Performance Task for Unit 1 (~45 minutes)
This practice performance task assesses learning objectives and essential knowledge statements addressed up to this point in the unit.

Lessons for Key Concept 1.2: Chance Events

Lesson Title	Learning Objectives Addressed	Essential Knowledge Addressed	Suggested Timing	Areas of Focus
Lesson 1.7: Introduction to Probability	1.2.2	1.2.2a, 1.2.2b, 1.2.2c	~135 minutes	Greater Authenticity of Applications and Modeling, Engagement in Mathematical Argumentation
Lesson 1.8: Venn Diagrams	1.2.1, 1.2.3	1.2.1a, 1.2.1b, 1.2.1c, 1.2.3b, 1.2.3d	~45 minutes	Greater Authenticity of Applications and Modeling, Engagement in Mathematical Argumentation
Lesson 1.9: Contingency Tables	1.2.1, 1.2.3	1.2.1a, 1.2.1b, 1.2.1c, 1.2.1d, 1.2.3a, 1.2.3b, 1.2.3c, 1.2.3d	~90 minutes	Greater Authenticity of Applications and Modeling, Connections Among Multiple Representations
Lesson 1.10: Independent Events	1.2.4	1.2.4a, 1.2.4b	~90 minutes	Engagement in Mathematical Argumentation
Lesson 1.11: Modeling Probability with the Normal Distribution	1.2.5	1.2.5a, 1.2.5b, 1.2.5c	~45 minutes	Greater Authenticity of Applications and Modeling

All learning objectives and essential knowledge statements for this key concept are addressed with the provided materials.

Practice Performance Task for Unit 1 (~45 minutes)
This practice performance task assesses learning objectives and essential knowledge statements addressed up to this point in the unit.

Lessons for Key Concept 1.3: Inferences from Data

Lesson Title	Learning Objectives Addressed	Essential Knowledge Addressed	Suggested Timing	Areas of Focus
Lesson 1.12: Accuracy and Precision	1.3.1	1.3.1a, 1.3.1b, 1.3.1c	~90 minutes	Engagement in Mathematical Argumentation

The following Key Concept 1.3 learning objectives and essential knowledge statements are addressed in the Pre-AP Performance Task for Unit 1.

- Learning Objectives: 1.3.2, 1.3.3, 1.3.4
- Essential Knowledge Statements: 1.3.2a, 1.3.2b, 1.3.3a, 1.3.3b, 1.3.3c, 1.3.4a, 1.3.4b, 1.3.4c

Learning Checkpoint 2: Key Concepts 1.2 and 1.3 (~45 minutes)
This learning checkpoint assesses learning objectives and essential knowledge statements from Key Concepts 1.2 and 1.3. For sample items and learning checkpoint details, visit Pre-AP Classroom.

Performance Task for Unit 1 (~180 minutes)
This performance task assesses learning objectives and essential knowledge statements from the entire unit.

UNIT 1

LESSON 1.1

A-Maze-ing Statistics

OVERVIEW

LESSON DESCRIPTION

Part 1: Designing the Maze Experiment
In this part of the lesson, students record the time it takes them to complete a maze. Then they develop a list of questions based on this experience and use their lists to design an experiment that relates to factors that affect completing mazes.

Part 2: Performing the Maze Experiment
In this part of the lesson, students complete their maze experiment. They collect data with the aim of answering the questions they generated in the first part of the lesson.

Part 3: Drawing Conclusions About the Maze Experiment
In this part of the lesson, students use prior knowledge of statistics and data displays to present and investigate the data set and draw conclusions about the data. They begin to answer some of the questions they posed in Part 1.

CONTENT FOCUS

This lesson provides students with an opportunity to engage in the process of designing, carrying out, and collecting data from an experiment. Students perform a maze-completion experiment and analyze the data they collect to answer questions they had previously generated. Through this activity, students activate their prior knowledge about measures of center (mean, median, and mode), measures of spread (range and interquartile range), and boxplots.

AREAS OF FOCUS

- Connections Among Multiple Representations
- Greater Authenticity of Applications and Modeling

SUGGESTED TIMING

~90 minutes

LESSON SEQUENCE

This lesson is part of a lesson sequence (~585 minutes) that includes Lessons 1.1 through 1.6.

HANDOUTS

Lesson

- 1.1.A: Warm-Up Maze
- 1.1.B: Maze A
- 1.1.C: Maze B
- 1.1.D: A-Maze-ing Statistics

MATERIALS

- access to **Desmos.com**
- calculator
- stopwatches or timers
- display timer

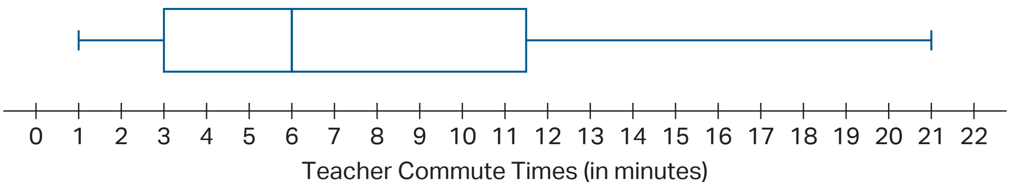
COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ Statistics are numbers that summarize large data sets by reducing their complexity to a few key values that model their center and spread.	
Learning Objectives	**Essential Knowledge**
1.1.1 Determine appropriate summary statistics for a quantitative data distribution.	**1.1.1c** The mean and the median summarize a data distribution by identifying a typical value, or center, of the distribution. The mean and median have the same units as the values in the data distribution. **1.1.1d** The standard deviation, interquartile range, and range summarize a data distribution by quantifying the variability, or spread, of the data set. The standard deviation, interquartile range, and range have the same units as the values in the data distribution.
1.1.2 Create a graphical representation of a quantitative data set.	**1.1.2a** A boxplot summarizes a quantitative data set by partitioning its values into four groups, each consisting of the same number of data values. Boxplots are used to depict the spread of a distribution.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

A random sample of 21 teachers from a local school district were surveyed about their commute times to work. Their responses, rounded to the nearest half minute, were recorded and displayed using the following boxplot. All responses for commute times were different.



(a) Identify the quartiles and the median commute times for the teachers surveyed.

(b) Based on the sample, must it be true that one of the teachers surveyed had a commute time equal to the median commute time? Justify your response.

(c) One student looked at the boxplot and remarked that more teachers had commute times between 11.5 minutes and 21 minutes than between 1 minute and 3 minutes. Do you agree or disagree? Explain your answer.

PART 1: DESIGNING THE MAZE EXPERIMENT

In this part of the lesson, students record the time it takes them to complete a maze. Then they develop a list of questions based on this experience and use their lists to design an experiment that relates to factors that affect completing mazes. The goal of this activity is to motivate students to develop an understanding the process of gathering data and the utility of statistics.

- Begin the lesson by letting students know that they will be completing a maze and timing themselves. This practice maze will introduce students to the experiment that follows. Distribute **Handout 1.1.A: Warm-Up Maze** facedown to each student.
- If possible, display a timer or stopwatch in a central location and ask students to begin working on the maze when you provide the signal. Because the maze does not indicate where to enter and where to exit, you can allow students to choose or you can specify that they should enter the maze at the top and exit the maze at the bottom. When students finish the maze, they should record the time, in seconds, it took them to complete the maze.
- Next, ask students to pair up with a classmate and jointly create a list of questions they have about the experience. Explain that they will investigate one of their questions. As you circulate around the room, you can use guiding questions such as the following to inspire their curiosity:
    - Would you be better or worse at completing a maze with your nondominant hand?
    - Would you be faster or slower at completing a maze with your nondominant hand?
    - Would the difference between the time needed to complete a maze with your dominant hand and the time needed to complete a maze with your nondominant hand be significant?
    - Do you think the type of maze would affect the time required to complete it?
- After students have brainstormed possible questions to answer, steer the conversation toward determining whether they would complete a maze faster if they used their dominant hand or their nondominant hand.
- Next, have pairs of students describe what kind of experiment they could carry out to try to answer this question. They should discuss whether each student should complete two mazes, one with each hand, or if they should randomly assign the hand used to each person. They should also discuss whether they should use one maze or two mazes to complete the experiment. After giving student pairs several minutes of discussion time, have each pair share their ideas with the class and engage in an academic conversation about the suggestions they make.

Guiding Student Thinking

The discussion about how to design the experiment may bring to light several potential sources of bias. If students debate whether the participants should self-select or be assigned the order in which they complete the mazes, you can explain how this could cause bias. Many experimental procedures call for quantities of interest to be randomized to minimize conscious or unconscious bias by the experimenter. In an experiment investigating the relationship between the maze completion time and hand dominance, we could randomize which maze students complete first, based on the assumption that there are two mazes with comparable difficulty, and control the hand dominance by choosing whether the student uses their dominant or nondominant hand first. Alternatively, we could randomize the order of the hand dominance and control the order of the mazes to be completed, again based on the assumption that there are two mazes with comparable difficulty. Or, we could also choose to randomize both which maze to complete first and which hand to use to complete the maze.

- Have the class come to a consensus about the design of the experiment. Then you can pose the following questions:
  - Why might we begin a statistics unit with an activity like this?
  - What does the word *statistics* mean to you?
- As students answer these questions, listen for them to say that the process of "doing statistics" often involves making calculations with numbers. They may also suggest that statistics are used to help answer questions using data.

Instructional Rationale

The questions that end this part of the lesson are intended to reveal some prior assumptions students have about statistics. It is likely that many students see statistics only as numbers or as calculations to perform with arbitrary data sets that have no connection to any actual phenomena. A major goal of this unit is for students to understand that statistics can both guide the development of experiments and surveys and help us answer questions after the experiment or survey has been completed.

PART 2: PERFORMING THE MAZE EXPERIMENT

In this part of the lesson, students complete the maze experiment they designed. They collect data with the goal of answering the questions they generated in the first part of the lesson.

- Students designed the experiment in Part 1, so you should provide **Handout 1.1.B: Maze A**, **Handout 1.1.C: Maze B**, and a timer to students. It is important for the data analysis that students make choices about how they will administer the mazes and how they will determine which hand to use first. They should document their choices, because these choices may be relevant when assessing the data and drawing conclusions.

- You can distribute **Handout 1.1.D: A-Maze-ing Statistics** for students to use to record their data. They will use this handout later in the lesson to record summary statistics for the data.

Guiding Student Thinking

Some students might be concerned that one maze is more difficult than the other. For example, if Maze A is easier and each student completes it with their dominant hand, students may wonder if the faster time is a result of the ease of the maze or the use of the dominant hand. This is a great opportunity to discuss with students how to properly design experiments that will provide them with unbiased results from which they can confidently draw conclusions.

- If students haven't already decided on a way to record the data, this would be a good time to help them with that process. If you have students work with a partner, then one person can complete the maze while the other one records the time.
- After the students have completed both mazes using the agreed-upon conditions, create a class set of data points for the times, in seconds, needed to complete each maze. Students can record these in **Handout 1.1.D: A-Maze-ing Statistics**.

Classroom Ideas

If you have access to a spreadsheet that can be projected, you can display it and have students enter their results directly into the spreadsheet. After all the data has been recorded, students can reference the shared spreadsheet to perform data analysis.

PART 3: DRAWING CONCLUSIONS ABOUT THE MAZE EXPERIMENT

In this part of the lesson, students use prior knowledge of statistics and data displays to present, investigate, and draw conclusions about their data set. They begin to answer some of the questions they posed in Part 1.

- Now that the data has been collected, students will begin the process of analyzing the data to draw conclusions. Their analysis should be focused on answering at least some of the questions they posed in Part 1.
- If you can, display the class data and ask students:
  - Now that we have all these data values, what do they tell us about the experiment?

- What are some tools or techniques you learned in prior courses that will help you summarize the data set?
- Given all these different statistics, which would be the most useful for this data?

- This is a chance for students to review things that they have learned in previous courses. As students cite specific statistics terms, invite them to also define these terms in their own words. Record the terms and their definitions in a central place for all students to review. Listen for terms such as mean, median, mode, quartiles, boxplot, maximum, minimum, bar graph, and five-number summary (minimum, lower quartile, median, upper quartile, and maximum).
- Ask students to share ways to find measures of center of the data. Measures of center are helpful because they summarize an entire data set using one number.

Meeting Learners' Needs

As students name different summary statistics, you can follow up by asking them to briefly explain what each statistic is and how to determine it. If students seem unsure about how to find a specific statistic, this would be a good opportunity to provide a brief refresher.

Instructional Rationale

If each student or pair of students has access to a computer, **desmos.com** provides an ideal workspace for data analysis, because this will be the program used throughout the course. This lesson serves as a good introduction to the statistics functions of the Desmos software. Students will need the data set, however, so you can either create a spreadsheet to share with them or you can have them copy it from the display.

- As students work individually or with a partner to create a boxplot for the data set, they can plot it using Desmos or create it by hand. After students complete their boxplots, make sure to display a correct boxplot so that students can check their work. Instructions for using Desmos in this activity appear in the table on the next page.
- Once students have created a boxplot for each data set, have them use Desmos to calculate a five-number summary. Then ask students to find the other summary statistics to complete

Meeting Learner's Needs

If a student has a physical limitation, their maze-completion time might be an outlier in the class data set. While we want to incorporate these values in the class analysis, students can be sensitive about their completion times, so it may be preferable to avoid calling attention to outliers in this situation.

Handout 1.1.D: A-Maze-ing Statistics. The relevant commands for Desmos appear in the following box.

- Students should make three summary statements on the handout based on the boxplots and the statistics they calculated. Their summary statements should be related to the questions from Part 1 that they wanted to answer.

Task	Desmos Command
Create a list using a data set.	A = [list of numbers separated by commas] *Lists of data values can be copied and pasted into Desmos.*
Display a boxplot of data set A.	boxplot(A) *In the boxplot line, "Offset" refers to the y-value of the center of the boxplot and "Height" refers to the height of the box.*
Calculate the mean of data set A.	mean(A)
Calculate the median of data set A.	median(A)
Calculate the lower quartile of data set A.	quartile(A,1)
Calculate the upper quartile of data set A.	quartile(A,3)
Calculate the maximum of data set A.	max(A)
Calculate the minimum of data set A.	min(A)

You can find the statistics and graphing commands by clicking on the keyboard icon at the bottom left of the command pane, clicking on the **functions** button, and choosing either the **Stats** or **Dist** menus.

Meeting Learners' Needs

If students struggle to recall how to create a boxplot or what its necessary components are, this would be a good opportunity to provide a brief refresher.

- After students have written their summary statements, have them share their statements with the class. Ask questions that encourage students to use evidence-based reasoning, such as "How do you know?" For example, if a student states, "Students are better at mazes using their dominant hand," help them write a more thoroughly supported statement such as, "The data indicates that students are better at completing mazes using their dominant hands because the mean time measured is 4 seconds faster than completing mazes using their nondominant hand."

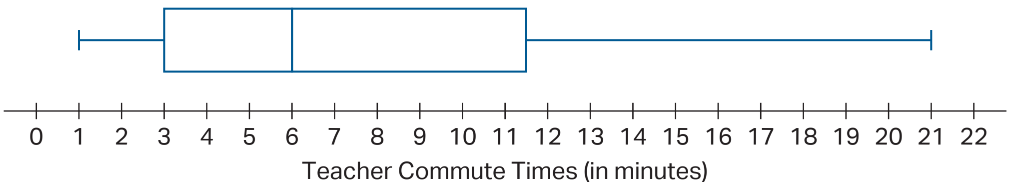
- You can conclude class by letting students know that they will be developing their understanding of statistics and data displays over the course of several weeks. This experiment was designed to get students thinking about how statistics are really used to answer questions of interest.

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

A random sample of 21 teachers from a local school district were surveyed about their commute times to work. Their responses, rounded to the nearest half minute, were recorded and displayed using the following boxplot. All responses for commute times were different.



(a) Identify the quartiles and the median commute times for the teachers surveyed.

Based on the boxplot, the first quartile is 3 minutes and the third quartile is 11.5 minutes. The median commute time is 6 minutes.

(b) Based on the sample, must it be true that one of the teachers surveyed had a commute time equal to the median commute time? Justify your response.

Yes, one teacher must have a commute time of 6 minutes. By definition, the median is the value in the middle of the data set when the data set is put in ascending order. Since there were 21 teachers surveyed, the 11th teacher's commute time must be the median because 10 teachers have commute times less than the 11th teacher's commute time and 10 teachers have commute times greater than the 11th teacher's commute time.

(c) One student looked at the boxplot and remarked that more teachers had commute times between 11.5 minutes and 21 minutes than between 1 minute and 3 minutes. Do you agree or disagree? Explain your answer.

I disagree with this statement. A boxplot displays ordered data in intervals of 25 percent, so 25% of teachers had commute times from 11.5 minutes through 21 minutes, which is the same percentage of teachers who had commute times from 1 minute up to 3 minutes.

Guiding Student Thinking

If students struggle with part (b) it could be because they do not understand that the median must be a value in a data set with an odd number of values. The median splits an ordered data set into two parts with equal numbers of data values. If a data set consists of 21 values, 10 data values will be above the median, 10 data values will be below the median, and one data value is the middle value between these two groups of values.

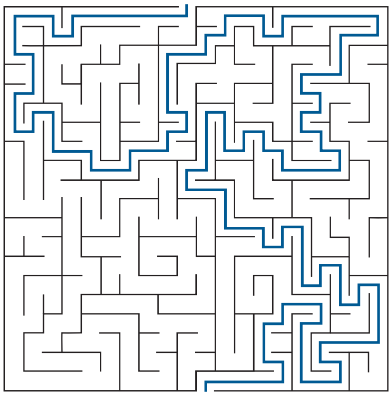
If students struggle with part (c) it could be because they believe that the length of the box or whisker corresponds with the number of data values it represents. Boxplots show the range of the values in each quartile, not the number of values in each quartile. Each interval contains 25% of the data.

HANDOUT ANSWERS & GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

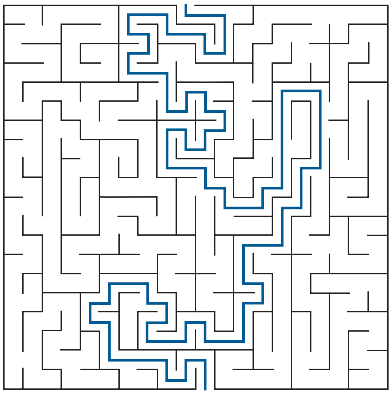
Handout 1.1.A: Warm-Up Maze

The solution to the maze is:



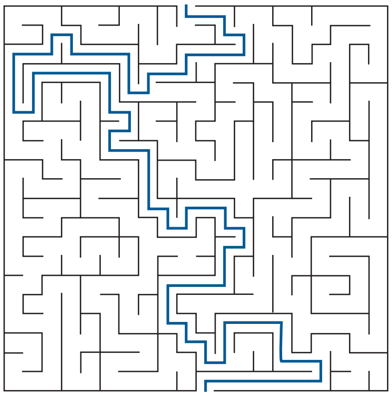
Handout 1.1.B: Maze A

The solution to the maze is:



Handout 1.1.C. Maze B

The solution to the maze is:



Handout 1.1.D: A-Maze-ing Statistics Handout

This handout is designed to give students space to record the summary statistics for the maze experiment. Answers will vary with the data collected.

UNIT 1

LESSON 1.2

Exploring Variables

OVERVIEW

LESSON DESCRIPTION

Part 1: Collecting Data

In this part of the lesson, students collect census-like data from the class to support a discussion about the different types of data: categorical and quantitative.

Part 2: Displaying Data

In this part of the lesson, students create different data displays that best represent the variables investigated in the survey in Part 1. Students learn that quantitative data can be represented by displays such as boxplots and histograms, while categorical data are best represented by displays like pie charts and bar graphs.

CONTENT FOCUS

The goal of this lesson is to build on students' understanding of data collection and data displays. In the lesson, students learn about the different types of data that can be collected—categorical and quantitative—and the displays that are best suited for each type of data. Several important terms are defined in the lesson, including *variable* and *distribution*, each of which has a different meaning in statistics than it does in algebra.

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Connections Among Multiple Representations

SUGGESTED TIMING

~135 minutes

LESSON SEQUENCE

This lesson is part of a lesson sequence (~585 minutes total) that includes Lessons 1.1 through 1.6.

MATERIALS

- a spreadsheet tool, such as Google Sheets or Microsoft Excel
- access to **Desmos.com**

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ Statistics are numbers that summarize large data sets by reducing their complexity to a few key values that model their center and spread. ▪ Distributions are functions whose displays are used to analyze data sets.	
Learning Objectives	**Essential Knowledge**
1.1.2 Create a graphical representation of a quantitative data set.	**1.1.2a** A boxplot summarizes a quantitative data set by partitioning its values into four groups, each consisting of the same number of data values. Boxplots are used to depict the spread of a distribution. **1.1.2b** A histogram summarizes a quantitative data set by partitioning its values into equal-width intervals and displaying bars whose heights indicate the frequency of values contained in each interval. Histograms are used to depict the shape of a distribution.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

A small video game company is planning their upcoming slate of games. To obtain some information about the playing habits of their gamers, they conducted a survey of 50 gamers. Nine of the responses are shown in the table below.

Age	Favorite Game Genre	Preference for Single Player or Multiplayer	Hours Played per Week	Preferred Gaming Platform
24	Action	Single Player	2	Console
15	Sports	Single Player	2	PC
18	Sports	Multiplayer	3	Console
15	Role-Playing Game	Multiplayer	2	PC
29	Role-Playing Game	Single Player	3	PC
13	Action	Single Player	1	Console
31	Puzzle	Single Player	7	Mobile

(a) Identify the variables for which a bar graph would be the most appropriate display for summarizing the data. Explain your choice.

(b) Identify the variables for which a histogram would be the most appropriate display for summarizing the data. Explain your choice.

PART 1: COLLECTING DATA

In this part of the lesson, students collect some census-like data from the class to prompt a discussion about the different types of data: categorical and quantitative.

- To begin the lesson, let students know that they will be engaging in a data collection activity and that they will use the results to construct different displays of the data.
- Have the class create a data set with the following information from each student:
    - Age (years)
    - Height (cm)
    - Foot length (cm)
    - Allergies (Y/N)
    - Birth month
    - How they travel to school
    - Length of travel time to school
    - Number of pets
    - What superpower would you choose: flying, invisibility, super strength, or telepathy?

Classroom Ideas

Some students may be sensitive about providing some of the information in the list. For example, students who are older or younger, or who are much shorter or taller than most of their classmates might feel particularly open to ridicule by answering those questions. You can choose any list of variables as long as there are some quantitative and some categorical ones. Treat this list as a suggestion for the kinds of data you could collect.

Instructional Rationale

This data set will be used for several lessons in Key Concept 1.1 and for Lesson 1.9 in Key Concept 1.2. Students often find it more motivating to answer questions with information they have personally collected. It would be valuable to save this data set in a spreadsheet so it can be easily accessed in later lessons.

- There are many ways to engage students in data collection. One way is to create a survey with the questions above using a Google Form or other online survey tool. Once students input their answers, you can create a shareable spreadsheet with all of the data for them to reference. Another way to quickly collect and record the data in an easy-to-use form would be to create a shareable spreadsheet for the class. Students would input their answers directly into the spreadsheet.

Classroom Ideas

To preserve some class time, you could have students answer the questions using an online survey tool outside of class a few days before you plan to teach this lesson. Then you would be able to begin the class with the discussion about the data types.

- As students are collecting their data, you can provide some definitions for them, starting with the word *data* itself:

  In statistics, the term data refers to a set of values that can be sorted by variables.

- To activate some prior knowledge about terminology, you can ask students to reflect on where they have heard the term *variable* before:

  - Where have you heard the term *variable* before? What did the term mean in that situation?

    In algebra class, we used the term *variable* to refer to quantities that can change. We usually used letters to represent variables.

- You can provide the following definition of *variable* for your students, clarifying that the way the word is used in statistics is somewhat different from the way it is used in algebra.

  A variable is any attribute or characteristic that can be measured.

- You can help students start to understand these terms by explaining that we use the term *data* to talk about all the information they gathered about age, height, foot length, allergies, birth month, school travel method, school travel time, number of pets, and preferred superpower together. We use the term *variable* to talk about the individual aspects that we measured about age, height, foot length, allergies, birth month, school travel method, school travel time, number of pets, and preferred superpower separately.

- Once students collect all the data, engage them in thinking about the differences in the kinds of data they collected today and the data they collected in Lesson 1.1. You can ask some questions like the ones below:

  - In the maze experiment in Lesson 1.1, each piece of data collected was a number. These kinds of variables are called *quantitative*. In the survey in this lesson, did we collect quantitative data? Did we collect another type of variable?

    Students will likely recognize that several of their responses will not be quantitative.

  - Variables that are not quantitative are called *categorical*. Which of the questions measure a categorical variable? Why are they not quantitative?

    The questions about the method for traveling to school, if you have an allergy, or birth month are categorical. They are not quantitative because they do not have a numerical answer.

  - Can you use a boxplot to display data for a quantitative variable? Can you use a boxplot to display data for a categorical variable?

    Data for quantitative variables can be displayed using boxplots, but data for categorical variables could not be meaningfully displayed using boxplots.

Guiding Student Thinking

Some students may recognize that many quantitative variables can be turned into categorical variables by creating categories to sort the data into. For example, height can be sorted into short, medium, and tall, and age can be sorted into categories like youth, adolescent, young adult, etc. Some variables, like grade level, can be quantitative if the number of the grade is reported but can also be categorical if the name of the grade is reported.

- You can let students know that they will explore data displays for quantitative variables and categorical variables in the next part of the lesson.

PART 2: DISPLAYING DATA

Now that students have a better understanding of the types of variables found in data sets, this next part of the lesson gives them opportunities to practice creating data displays to present the variables from the survey in Part 1. Students should utilize their knowledge from Part 1 to recognize that quantitative variables can be represented using boxplots and histograms, while categorical data are best represented by pie charts and bar graphs.

- To begin this part of the lesson, make sure that students can identify each variable from the survey in Part 1 as categorical or quantitative:

  Age (years): Quantitative

  Height (cm): Quantitative

  Foot length (cm): Quantitative

  Allergies (Y/N): Categorical

  Birth month: Categorical

  How they travel to school: Categorical

  Length of travel time to school: Quantitative

  Number of pets: Quantitative

  Preferred superpower: Categorical

Meeting Learners' Needs

If students are struggling to differentiate between quantitative and categorical variables, you may want to have them brainstorm some additional examples to help solidify the appropriate use of those terms. Quantitative examples could include the number of hours spent studying or an average quiz score. Categorical examples could include different high school mascots or their favorite movies. Some students might find it helpful to connect the root *quant-* in *quantitative* to the word *quantity*, which means number. Likewise, the word *categorical* can be linked to the word *category*.

- Before students get started on creating data displays, you can define the term *distribution* as it relates to data:

  The distribution of a variable is a description of the values of data (categorical or quantitative) and the frequency of the values.

Instructional Rationale

It is important to introduce the term *distribution* early in the unit, as it will be used throughout the statistics and probability lessons and is the basis for understanding how data can be modeled with functions. It is not necessary that students master what a data distribution is at this point in the unit, because they will develop this understanding over the course of the unit.

- Next, ask students some questions to get them thinking about the data displays that they know:
  - What kinds of data displays do you know? Name as many as you can.

    Students may mention boxplots, scatterplots, dot plots, bar graphs, and pie charts.
  - Which of these data displays do you think would be best for displaying quantitative variables? Which of these data displays do you think would be best for displaying categorical variables? Explain your reasoning.

    Categorical variables can be displayed using pie charts or bar graphs and quantitative variables can be displayed using boxplots. Dot plots are useful for determining the frequencies of data values, so they can be used for categorical variables or for quantitative variables that have been broken into categories.
- To help students see the advantages and disadvantages of different data displays, have them compare two different displays of the same data for the same categorical variable. Divide the class in half and have one group of students create a pie chart and the other group make a bar graph for the "How they travel to school" variable. Students should use a digital tool such as Google Sheets or Microsoft Excel. Students will need to see the data for this variable displayed, or you can share the data set digitally or on paper.

Classroom Ideas

This part of the lesson is particularly effective if you can have two students pair up so they can practice using spreadsheets to create data displays. The students can support each other as they work through unfamiliar commands.

- Once students complete the bar graphs and pie charts, ask them to discuss with a partner the advantages and disadvantages of each type of data display. Have a few students share out their observations about the data displays.

  Students may say that a bar graph provides frequencies of data values and you can easily find out the sample size, while a pie chart is an efficient visual representation of each response relative to the other responses as well as to all responses.

- Next, ask students to think about the data for the "Age" variable. Some students might like to see the data for this variable displayed in a central location. Ask students to consider what kind of data display could be used for the "Age" variable.
- If students suggest that the "Age" variable could be displayed as a boxplot, have them create a boxplot with the data. As with the data in Lesson 1.1, students can create a boxplot using Desmos. You can use this as an opportunity to review with students the five-number summary and how to identify each of these numbers in the boxplot. If students do not suggest using a boxplot, you may need to suggest it for them.

Instructional Rationale

Having students think about what a boxplot reveals and what it obscures will get them thinking about another type of display for quantitative data: histograms. Students are likely already familiar with bar graphs, but they may be confused about the difference between bar graphs and histograms. Through this lesson and the upcoming ones, students develop an understanding that histograms are similar to bar graphs, but are used for continuous quantitative variables. Histograms are an important data display that students will use throughout the unit.

- Once students complete the boxplot, have them discuss in pairs the advantages and disadvantages of a boxplot as a data display. Encourage students to also think about what information a boxplot does and does not show.

  Listen for students to mention that a boxplot allows us to easily find the maximum, minimum, and median data values (50% of the respondents are younger than 16 or older than 17, for example) but does not give us any information about the frequency (the number of students that were surveyed).

Guiding Student Thinking

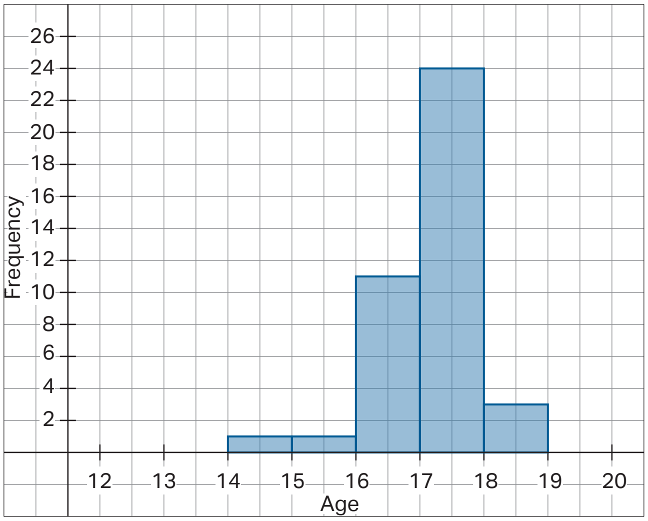
Students might mention that another issue with a boxplot is that it doesn't take into account that a student who is 16 years and 11 months old is actually closer to 17 than 16, but is considered "16." In reality, students are not really 15 years old; they are between 15 and 16 years old. This may naturally lead students to think about quantitative data as either continuous or discrete. This sets the stage for the development of the histogram, which has bins that are ranges of possible numerical values.

- At this point, you can challenge students to think about whether or not the "Age" variable could be displayed in a way that shows the frequency of the ages. Give students a minute to generate some ideas about this.
- You can let students know that since a bar graph shows frequency, quantitative data can be displayed using a bar graph that uses an age range as a category. For example, students reporting that they are 15 years old are really older than 15.0 but younger than 16.0, so they are in the range $15 \le x < 16$.
- As students think about how to create a bar graph for the "Age" variable, the main idea for them to understand is that quantitative data can be reduced to categorical data by creating nonoverlapping ranges called *bins*. For the "Age" variable, the bin size is 1 year.
- Since the bins represent a range of the quantitative data, it is customary to plot them along the horizontal axis. Then the frequencies of the ages (in bins) will be plotted along the vertical axis. In this orientation, the bars will be vertical.

Guiding Student Thinking

Some students may want to plot the age bins along the vertical axis and the frequencies along the horizontal axis. This is not the usual convention, but the data could be graphed that way. In Lesson 1.5, students will start to think of a data distribution as a function and use the normal curve as a model for a data set with certain characteristics. To ensure that the new understanding of function connects to and extends students' existing understanding of function, it will be advantageous for them to consider the data values as the independent variable, plotted along the x-axis, and the frequency of the data values as the dependent variable, plotted along the y-axis.

- To construct a histogram of the "Age" variable, count the frequency of ages that fall into each age range and draw a bar at that height for each bin. Below is an example of an age histogram constructed using Desmos. The Desmos commands are shown in the table that follows.
- It is the usual convention to count the data that corresponds to the left endpoint in a bin, in this case including exactly 15 years old in the bin from 15 to 16 years old, and not to count the data that corresponds to the right endpoint, in this case 16 years old, in the bin. This is represented by the inequality $15 \leq x < 16$.



Task	Desmos Command
Create a list using a data set.	A = [list of numbers separated by comma] *Lists of data values can be copied and pasted into Desmos.*
Display a histogram of data set A.	histogram(A) *In the histogram line you can choose the bin alignment to be center or left (default is center) and you can determine the bin width (default is 1).*

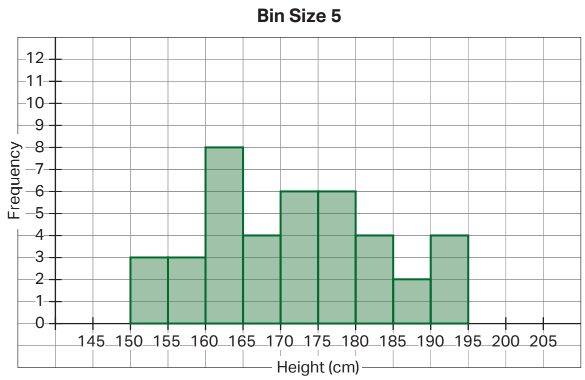
You can find the statistics and graphing commands by clicking on the keyboard icon at the bottom left of the command pane, clicking on the **functions** button, and selecting either the **Stats** or **Dist** menus.

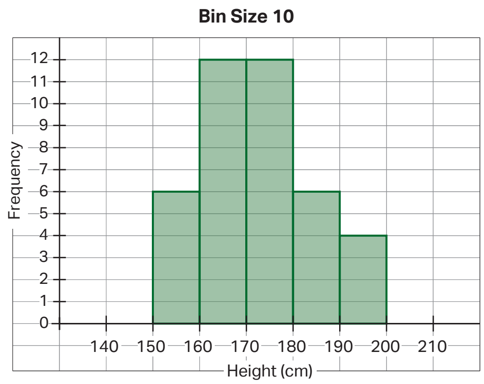
Guiding Student Thinking

Because students will have created both bar graphs and histograms for variables in this data set, they may struggle with the differences between them. You can generate a quick class discussion by creating a comparison chart similar to this one reproduced below from the website **keydifferences.com/difference-between-histogram-and-bar-graph.html**.

Basis for comparison	Histogram	Bar Graph
Meaning	A histogram is a graphical representation that displays data using bars to show the frequencies of numerical data in nonoverlapping intervals.	A bar graph is a pictorial representation of data that uses bars to compare different categories of data.
Indicates	Distribution of continuous variables	Comparison of discrete variables
Presents	Quantitative data	Categorical data
Elements	Elements are grouped together, so that they are considered as ranges.	Elements are taken as individual entities.
Can bars be reordered?	No	Yes
Width of bars	Need not be the same	Same

- Encourage students to use Desmos to change the bin size to see how the data display changes. Below are two displays with different bin widths for the Height variable.





- Have students work in pairs to create appropriate data displays for other variables: allergies (Y/N), height (cm), and how students travel to school.
- As students work in pairs, circulate around the room and ask questions such as:
  - What kind of variables are these, categorical or quantitative?

    Height is a quantitative variable, while allergies and how students travel to school are categorical variables.
  - What types of data displays are appropriate for each kind of variable?

    Bar graphs and pie charts are appropriate for categorical data, while boxplots and histograms are appropriate for quantitative data.
  - What bin size, or range, would you choose for the height and why?

    Answers will vary depending on what students want to display.

Guiding Student Thinking

Students should experiment with different bin sizes. There is rarely a single best bin size for a given data set. Often the choice of bin size depends on what the person working with the data wants to illuminate. Bin sizes that are too small are no better than a dot plot. Bin sizes that are too wide will not show any meaningful differences between the frequencies of different intervals.

- To conclude the lesson, ask students to consider the similarities and differences between boxplots and histograms. You can have students list the features of each type of plot and then list the information that can and cannot be determined from each plot type. It is important for students to recognize that one kind of data display is not always better than another. Different data displays provide different types of information. Histograms are useful for identifying and comparing the frequencies of different responses and for visualizing the shape of the distribution. Boxplots are useful for seeing the median value and the variability of a data set.

Instructional Rationale

The purpose of this debrief is to get students to start thinking about the general shape of the data in terms of its symmetry and its spread, and about how those features are represented in a histogram or boxplot. Creating displays is only part of the data analysis. Students need opportunities to examine what the displays tell them about the data used to make them. At this point in the course, students may not have all the proper statistical vocabulary necessary for comprehensively describing what they observe, so it is acceptable for them to use colloquial language instead.

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

A small video game company is planning their upcoming slate of games. To obtain some information about the playing habits of their gamers, they conducted a survey of 50 gamers. Nine of the responses are shown in the table below.

Age	Favorite Game Genre	Preference for Single Player or Multiplayer	Hours Played per Week	Preferred Gaming Platform
24	Action	Single Player	2	Console
15	Sports	Single Player	2	PC
18	Sports	Multiplayer	3	Console
15	Role-Playing Game	Multiplayer	2	PC
29	Role-Playing Game	Single Player	3	PC
13	Action	Single Player	1	Console
31	Puzzle	Single Player	7	Mobile

(a) Identify the variables for which a bar graph would be the most appropriate display for summarizing the data. Explain your choice.

A bar graph would be the most appropriate display for summarizing data of a categorical variable. The variables Favorite Game Genre, Preference for Single or Multiplayer games, and Preferred Gaming Platform have data values that are nonnumerical.

(b) Identify the variables for which a histogram would be the most appropriate display for summarizing the data. Explain your choice.

A histogram would be the most appropriate display for summarizing data of a quantitative variable. The variables Age and Hours Played per Week have responses that are numerical values.

UNIT 1

LESSON 1.3

Measures of Center

OVERVIEW

LESSON DESCRIPTION

Part 1: Defining Mean as Center of Balance
In this part of the lesson, students engage in an activity to develop a conceptual understanding of the mean as the center of balance of a data set. They begin to understand the mean as the value from which the deviations of the data values sum to zero.

Part 2: Determining the Center of a Data Display
In this part of the lesson, students use what they learned in Part 1 to determine and justify the center of balance of a data set. They use a Desmos interactive to show that the sum of the deviations from the mean is zero.

Part 3: Exploring Shape and Center
In this part of the lesson, students engage in an activity to help them understand how the shape of a graph of a data set and its measures of center are affected by the inclusion of some extreme values to the data set.

CONTENT FOCUS

Students have a great deal of experience calculating the mean of a data set from prior coursework. In this lesson, students explore the mean as the center of balance of a data set. This lesson introduces students to the concept of deviations from the mean and how to use them to verify that the mean is the only value from which the deviations balance out. Students also explore the similarities and differences between boxplots and histograms.

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Connections Among Multiple Representations

SUGGESTED TIMING

~90 minutes

LESSON SEQUENCE

This lesson is part of a lesson sequence (~585 minutes) that includes Lessons 1.1 through 1.6.

HANDOUTS

Lesson

- 1.3.A: Changing Shape and Center

Practice

- 1.3.B: Connecting Histograms and Boxplots

MATERIALS

- rulers
- coins of the same denomination
- access to **Desmos.com**

COURSE FRAMEWORK CONNECTIONS

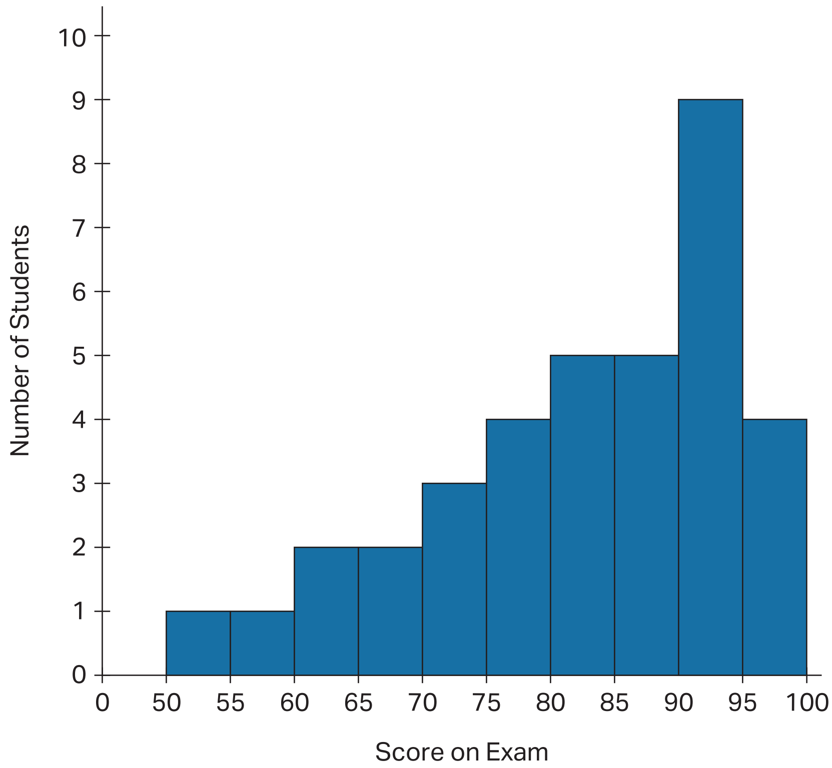
Enduring Understandings	
▪ Statistics are numbers that summarize large data sets by reducing their complexity to a few key values that model their center and spread. ▪ Distributions are functions whose displays are used to analyze data sets.	
Learning Objectives	**Essential Knowledge**
1.1.1 Determine appropriate summary statistics for a quantitative data distribution.	**1.1.1a** A data distribution is a function whose input is each value in a data set and whose output is the corresponding frequency of that value. **1.1.1b** Summary statistics describe the important features of data distributions including identifying a typical value, also called the center of the data, and describing the clustering of the data around the typical value, also called the spread of the data.
1.1.3 Analyze data distributions with respect to their centers.	**1.1.3a** The mean is the only point in the domain of a distribution where the sum of the deviations, or differences, between the mean and each point in the distribution is zero. **1.1.3b** The mean can be thought of as the center of mass of the data set. It is a weighted average that accounts for the number of data points that exists for every given value in the data set. **1.1.3c** Measures of center can be used to compare the typical values of the distributions. They provide useful information about whether one distribution is typically larger, smaller, or about the same as another distribution.

1.1.4 Analyze data distributions with respect to their symmetry or direction of skew.	**1.1.4a** For symmetric distributions, such as the normal distribution, the proportion of data in any range to the left of the mean is equal to the proportion of data in the corresponding range to the right of the mean. **1.1.4b** Skew describes the asymmetry of a distribution. The direction of skew is indicated by the longer tail of data values in an asymmetric distribution. **1.1.4c** When a distribution is skewed, its mean and median will differ. The farther apart the mean and median are in a distribution, the more skewed the distribution will appear.
1.1.6 Model a data distribution with a normal distribution.	**1.1.6c** For normally distributed data, the mean and median are the same number, and they correspond to the mode, which is the value in the distribution with the highest frequency.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

The histogram below displays the scores on a recent exam for a class of 37 students. Scores on this exam ranged from 51 to 98 points out of 100 possible points.



(a) Approximate the median exam score. Justify your response.

(b) Use the shape of the histogram of test grades for the class to determine if the students performed well or performed poorly on the test.

(c) Based on your answers to parts (a) and (b), do you believe the mean is greater than or less than the median? Explain and give an estimate for the mean.

UNIT 1

PART 1: DEFINING MEAN AS CENTER OF BALANCE

Students probably have some procedural knowledge of the mean of a data set from their prior coursework. In this part of the lesson, students engage in an activity to help them develop a conceptual understanding of the mean as the center of balance of a data set. Through the activity they should begin to understand that the mean is the value from which the deviations of the data values sum to zero. Without this understanding, it will be challenging for students to understand standard deviation in Lesson 1.4.

- Begin by posing a question to the class:
  - Suppose I put two pennies at different locations on the ruler. Do you think you could figure out how to balance the ruler on one finger so the ruler remains level?
- Allow students some time to decide whether or not this is possible. In general, students will realize that the balance point between the two pennies will be the point exactly midway between the two pennies. You can use this realization to preview the concept of the midpoint that students will explore in Unit 2.

Meeting Learner's Needs

Some students have difficulty visualizing how the ruler will balance. You can help them understand the task by demonstrating how to balance the ruler on your finger with its flat surface parallel to the floor, instead of on the tip of your finger.

- Now pose a slightly different question to students:
  - Suppose I put three pennies on a ruler at different locations. Do you think you could figure out how to balance the ruler on one finger so the ruler remains level?
- Allow students some time to decide whether or not this is possible. Since this is a more complex situation than balancing just two pennies, there might be some disagreements among the students. Some students will recognize that if a third penny is placed at the point exactly in between the two other pennies, then the balance point is the location of the third penny. Challenge students to think about placing the third penny at a point not exactly between the two other pennies.
- Let students experiment with different locations of the three pennies on the ruler. It is valuable to provide students with rulers and pennies so they can try to figure this out on their own. You can suggest that students use a pen or pencil under the ruler as a fulcrum on a desk or table, instead of having to balance the ruler on their finger.
- If some students struggle to get started with the problem, you can make it more concrete for the class by choosing locations on the ruler to place the three pennies. To make the calculations that follow relatively easy, you can choose whole number values.

- If students suggest finding the mean or the median of the three numbers that correspond to the pennies' locations on the ruler, have the class calculate both statistics. As long as the mean and the median are different, the median will not be a balance point for the pennies on the ruler. However, it would be valuable to try to balance the ruler on the median to show students that the median is not a balance point when the mean and median differ. Then you can demonstrate that the location of the mean is the balance point for the pennies on the ruler.
- At this point, it is important to explore some numerical relationships between the locations of the pennies on the ruler and the mean of the locations. All the following calculations assume that the pennies are located on the tick marks indicating 2, 3, and 10 inches, but they could be in any location. The concept behind the calculations will be the same regardless of the pennies' locations.
- Ask students some questions to get them thinking deeply about the relationship between the mean and the locations of the pennies.
    - We put the pennies at 2 inches, 3 inches, and 10 inches on the ruler. What is the mean of the locations? What does the mean represent?

      The mean is inch 5. The mean is the balance point of the three pennies.

    - How far away from the mean is each of the pennies? Pay attention to whether each penny's location is greater than or less than the mean.

      The penny at inch 10 is 5 inches greater than the mean. The penny at inch 2 is 3 inches less than the mean. The penny at inch 3 is 2 inches less than the mean.

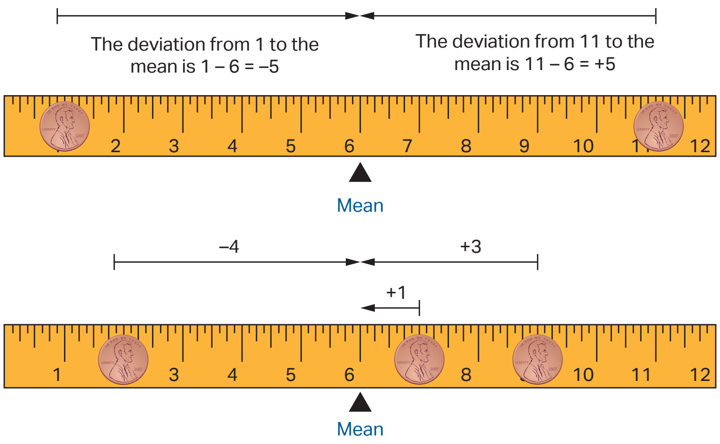
    - Add the differences between each penny's location and the mean, taking into account whether the differences are positive or negative relative to the mean. What do you get? Does that make sense to you? Why or why not?

      The sum of the differences is 0. This makes sense because the mean is the location where the pennies balance so the differences on each side of the mean should balance each other out.

Guiding Student Thinking

Students often neglect to include the units for the mean. It is important that students understand that the mean has the same units as the other values in the data set. Encourage students to use correct units when discussing the mean. Using the correct units will help them understand that the mean is a single value that is summarizing all the values in the data set.

- It may be difficult for student to articulate why it makes sense that the sum of the differences between each penny's location and the mean should be zero. It is all right if they struggle with that concept at this point. There will be more opportunities for students to reach this understanding.
- You can let students know that in statistics, the differences between the mean and the data values are called *deviations*. A deviation tells us how far away a data value is from the mean and also whether the data value is greater than or less than the mean.



- As a brief formative assessment, you can ask students some questions such as the following:
  - Suppose that two pennies were placed on the centimeter ruler at 3 cm and one penny was placed at 9 cm. Where would the balance point be?

    The mean is 5 cm, so the balance point is at 5 cm.
  - Describe the deviations of the locations from the mean. Is the sum of the deviations what we expect?

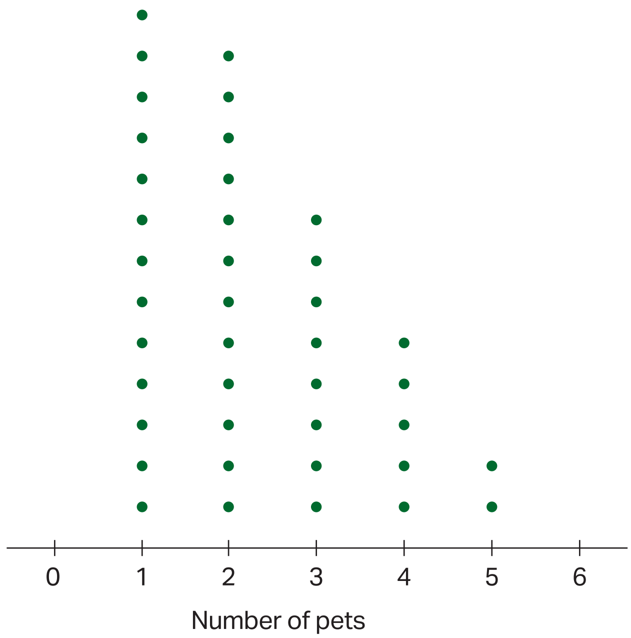
    The sum is 0. Each of the pennies at centimeter 3 has a deviation of -2 centimeters from 5 cm and the penny at centimeter 9 has a deviation of $+4$ centimeters from 5 cm. The sum of the three values is $-2-2+4=0$.
  - Choose five locations for five pennies, determine the balance point, and check the sum of the deviations to make sure it is zero.

    Student selections for the five locations and their resulting mean will vary, but the sum of the deviations should be equal to 0.

PART 2: DETERMINING THE CENTER OF A DATA DISPLAY

In this part of the lesson, students use what they learned in Part 1 to determine and justify the center of balance of a data set. They use a Desmos interactive to show that the sum of the deviations from the mean is zero.

- To begin this part of the lesson, let students know that they will be examining some of the data they collected in Lesson 1.2 and exploring how the value of the mean balances the deviations of the data set from which it was calculated. For this activity, they will need the "Number of pets" variable.
- Let students work with partners to construct a dot plot of the data. A sample dot plot is shown at right. They can use Desmos to construct the dot plot rather than draw it by hand. Commands for using Desmos to create a dot plot appear in the table below.



Task	Desmos Command
Create a list using a data set.	A = [list of numbers separated by comma] *Lists of data values can be copied and pasted into Desmos.*
Display a dot plot of data set A.	dotplot(A)

- As you circulate around the room, listen to what students observe about the dot plot, especially if they make references to the mean, median, mode, and range. Be sure to follow up by asking them to explain how those measures are calculated and what they represent.
- As you prepare to debrief with the class about their observations, make sure that all pairs of students have calculated the mean of the data set.

UNIT 1

- Ask the students some questions like the ones below to get them thinking about how the mean of the data is the balance point of the data set.
  - What is the mean of the data set and what does that value describe?

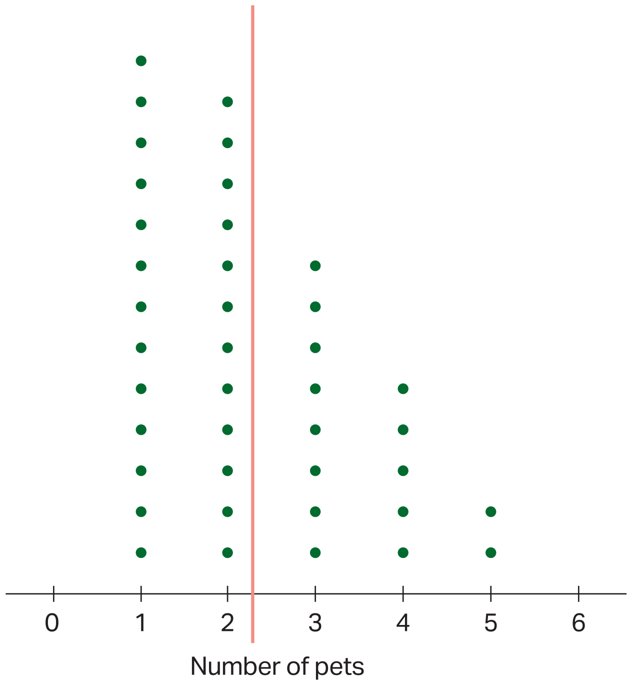
    The mean will vary depending on the data. (The sample data set has a mean of 2.275 pets.) Students will likely say that the mean is the "average" of the data set or the value where the deviations of each data value from it sum to zero.
  - Does it make sense that a home can have 2.275 pets? Why or why not?

    The mean does not have to be an actual data point, but is a measure of the center of a data set. It is a single value that summarizes the entire data set.

Meeting Learners' Needs

For students who are ready for an additional challenge, this data set could be an opportunity to explore how to calculate a weighted mean, either by hand or with a calculator tool. That is, instead of adding all the data values individually, students can multiply each data value by its frequency, sum the weighted values, and divide the sum by the number of data values. That calculation could reduce data entry errors.

- If you have not displayed the dot plot in a central location yet, you can do it now. Make sure that students agree that this is the same dot plot they graphed on their own by hand or with Desmos. As you graph the vertical line for the mean, have students do the same on their own graphs. The mean of 2.275 pets is shown below as a red line.



- Ask students to reflect on the similarities and differences between this dot plot and their observations from the penny experiment, especially in terms of what the mean represents for each data set. Give students a few moments to discuss their reflections with a partner.
  - How is the dot plot similar to and different from your work with a ruler and pennies?

    There are more values in the pet data set, but you could think about the dots as stacks of pennies and the x-axis as the ruler.
  - Do you think that the mean of the pet data balances out the deviations the same way the mean of the penny locations balanced the pennies? Why or why not?

    The mean will balance the deviations of the pet data, just like it did with the penny location data. The deviations tell us the difference of data values from the mean, so the mean balances the data values.
  - Determine the sum of the deviations of the values greater than the mean and the deviations less than the mean. Then make sure that the sum of the deviations is zero.

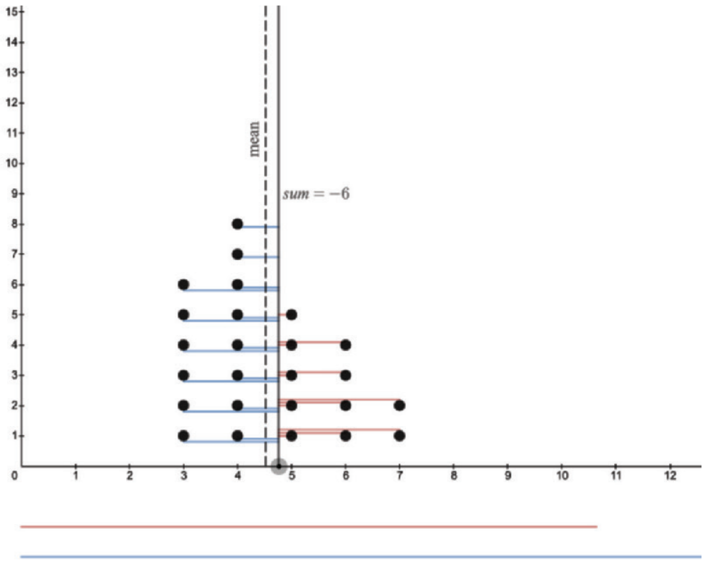
    There are 13 data values of 1, and each has a deviation of $1 - 2.275 = -1.275$ from the mean. There are 12 data values of 2, and each has a deviation of $2 - 2.275 = -0.275$ from the mean. The sum of the deviations of values less than the mean is $(13)(-1.275) + (12)(-0.275) = -16.575 + -3.3 = -19.875$.

    There are 8 data values of 3, and each of those has a deviation of $3 - 2.275 = 0.725$ from the mean. There are 5 data values of 4, and has a deviation of $4 - 2.275 = 1.725$ from the mean. There are 2 data values of 5, and each has a deviation of $5 - 2.275 = 2.725$ from the mean. The sum of the deviations of values greater than the mean is $(8)(0.725) + (5)(1.725) + (2)(2.725) = 5.8 + 8.625 + 5.45 = 19.875$.

    Because the sum of the deviations of values less than the mean is -19.875 and the sum of the deviations of values greater than the mean is 19.875, the sum of the deviations is 0.
  - Could there be more than one mean for a data set? Why or why not?

    Only one value balances a data set. Any other location would have deviations on each side that would have a nonzero sum.
- Encourage students to summarize what they've learned. It is important that students recognize that the mean is the only value for which the sum of the deviations, or differences, between it and each value in the distribution is zero.

- You can provide students with a visualization of how the mean balances out the deviations by pointing them to **preap.org/dotplot-mean**. This interactive allows students to drag a vertical line through the dot plot of a data set. It shows the sum of the deviations as horizontal lines below the x-axis, so students can see the value (the mean) that has equal deviations on the left and right.



PART 3: EXPLORING SHAPE AND CENTER

In this part of the lesson, students engage in an activity to help them understand how the shape of a graph and its measures of center are affected by the addition of some extreme values to the data set.

- You can begin this part of the lesson by asking students a question to get them thinking about how the shape and center of a data set might change when just a few values change.
  - Suppose we took a data set and replaced some of the lesser values with an equal number of greater values. How do you think the means of the data sets before and after this replacement would compare to each other?

    The mean of the new data set should be greater than the mean of the original data set because the sum of the data values in the new set would be greater than in the original set while the number of data values in both sets would be the same.

- How do you think the medians of the data sets before and after the replacement would differ?

  The median of the new data set would probably be greater than the median of the original data set, but maybe not by much because the center of the data might not have shifted very much. The change, if any, would depend on the values in the original data set.

- How do you think histograms of the original and new data sets would differ?

  The histogram of the new data set would have more and/or taller bars on the right side of the display than a histogram of the original data set.

- How do you think boxplots of the original and new data sets would differ?

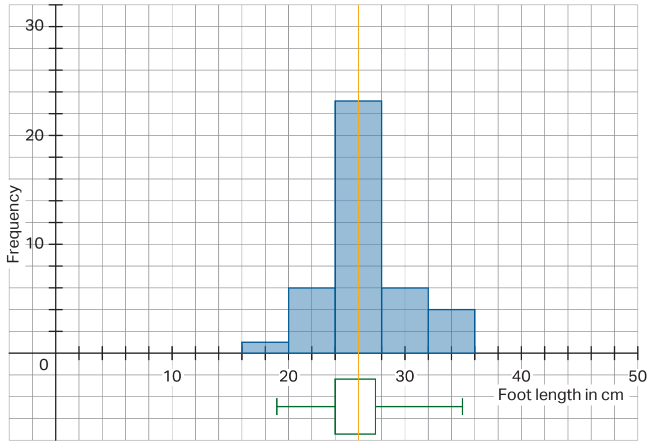
  The boxplot of the new data set might have a longer right whisker than the boxplot of the original data set.

- Distribute **Handout 1.3.A: Changing Shape and Center** to pairs of students. Students should work through the handout to determine what happens when a data set is changed.

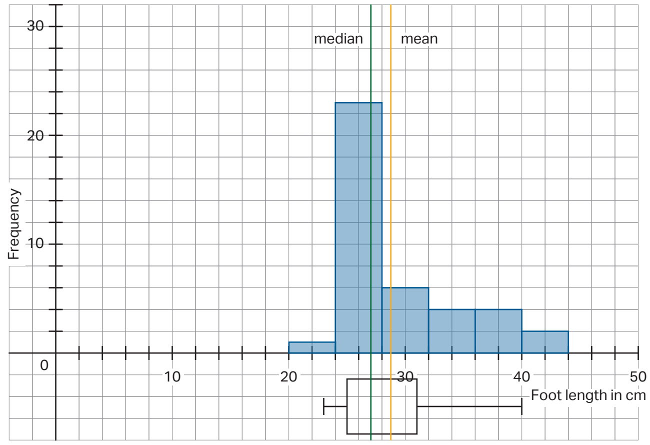
Instructional Rationale

The main goal of the handout is for students to explore the relationship between the symmetry or skew of a graph and the measures of central tendency.

- You can determine whether you want students to use Desmos or another graphing utility or spreadsheet tool to construct the displays and calculate the summary statistics. Regardless, they should transfer the displays to the handout for deeper analysis.
- For question 1(a), students should graph a histogram as shown on the next page and identify that the histogram is relatively symmetric. For question 1(b), students should graph a boxplot as shown on the next page and observe that the range of the histogram and boxplot are roughly equal and that the interquartile range roughly corresponds with the tallest bar(s) of the histogram. For question 1(c), students should determine that the mean is 26.05 and the median is 26, and then draw a vertical line at $x = 26$.



- For question 2(a), students should graph a histogram as shown below and identify that the histogram is not symmetric. For question 2(b), students should graph a boxplot as shown below and observe that the range of the histogram and boxplot are roughly equal and that the interquartile range roughly corresponds with the tallest bar(s) of the histogram. For question 2(c), students should determine that the mean is 28.7 and the median is 27, and then draw and label vertical lines at $x = 28.7$ and $x = 27$.



- For question 2(d), students should observe that, unlike with the first data set, the mean and median have noticably different values, and that the second histogram seems to have a longer tail on the right side of the display.
- To summarize the foot-length problem and get students to think about appropriate choices for measures of center, ask them some questions like the ones that follow:
  - What happened to the shape of the histogram when the six lowest data values were replaced?

    The second histogram has a tail, while the first histogram was symmetric.

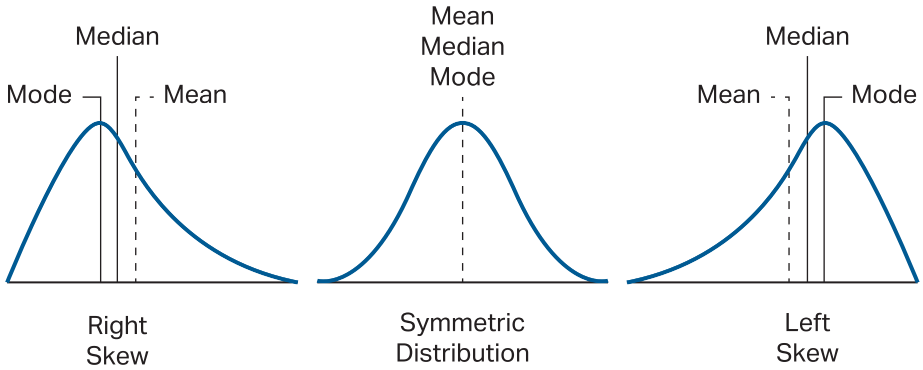
- For the first data set, would the median or the mean be the better measure of center to summarize the data set? For the second data set, would the median or the mean be the better measure of center? Explain why in each case.

  For the first data set, either the median or the mean would be acceptable since they are nearly equal. For the second data set, the better measure of center is the median because it is less affected by the new data from students with large feet.

- What does the tail in the graph of the second data set tell us about the relationship between the original data values and the new data values that replaced them? Why is the graph no longer symmetric?

  The tail indicates that the new data values were very different from the original data values they replaced. The graph is no longer symmetric because the new values that were added to the data set that were less typical than the data in the original data set.

- For question 3, students should label the data displays as shown below. They will likely need help explaining why the three measures of center differ in each display.



- Help clarify for students that, by convention, we describe the direction of skew by identifying which tail is longer. So, the right skew display has a long right tail and the left skew display has a long left tail.
- You can explain to students that for skewed distributions, the median is often (but not always) a better summary statistic for describing the center of the distribution. The median is much more resistant to extreme values, meaning that additional data values at or beyond the range of the data set will not cause the median to change very much. However, the mean is sensitive to extreme values, and additional data values at or beyond the edges of the data set will cause the mean to change quite a bit.

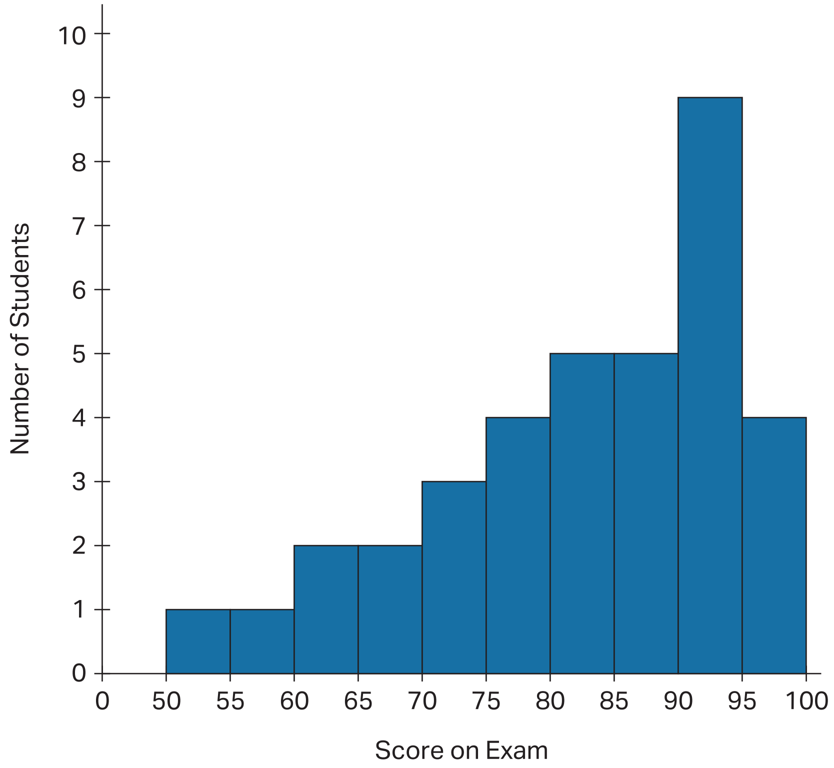
Handout 1.3.B: Connecting Histograms and Boxplots provides students with opportunities to match data displays without knowing the original data sets from which they were derived. This will help students understand how the data displays are related, and also what each display reveals and obscures. The handout also includes some sentence expansion exercises to help students develop skills in writing about statistics. You can choose to do Handout 1.3.B during class or out of class. Whatever your choice, it is important that students have an opportunity to engage in an academic conversation with classmates about their answers.

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

The histogram below displays the scores on a recent exam for a class of 37 students. Scores on this exam ranged from 51 to 98 points out of 100 possible points.



(a) Approximate the median exam score. Justify your response.

The median is the exam score for which half of the scores (18 students) are below it and half of the scores (18 students) are above it. In the first 7 bins, there are 18 students scores. Therefore, the median is the first exam score in the bin that is greater than or equal to 85 but less than 90. Since we do not know if all the students received different scores or not, we know the median is in the interval from 85 up to, but not including, 90, but we do not know its exact value.

(b) Use the shape of the histogram of test grades for the class to determine if the students performed well or performed poorly on the test.

The shape of the graph is skewed left because the asymmetric display has a tail that extends to the left. Therefore, more students did well on the exam and fewer did poorly.

(c) Based on your answers to parts (a) and (b), do you believe the mean is larger or smaller than the median? Explain and give an estimate for the mean.

For skewed distributions, the mean is closer to the tail than the median is. Therefore, the mean would be less than the median. An estimate for the mean is approximately 81 percent.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 1.3.A: Connecting Histograms and Boxplots

See lesson for solutions.

Handout 1.3.B: Connecting Histograms and Boxplots

For Part A of this handout, students should match the histograms, boxplots, measures of center, and description of skew or symmetry. The matching items are:

H1, B3, M2, D2

H2, B1, M3, D1

H3, B2, M1, D3

Part B of this handout is a sentence expansion activity. Students might be familiar with this kind of exercise from other Pre-AP courses. Students will expand each sentence stem three times, once for each of the conjunctions *because*, *but*, and *so*. The *because* statement should explain why something is true, the *but* statement should express a contrast, and the *so* statement should indicate cause and effect.

These are possible sentence expansions. Students may have many different ways to complete each sentence so that it expresses a true statement and is consistent with the conjunction it includes.

1. (a) For skewed data displays, the median is often a better estimate of the center of the distribution than the mean because the median is not significantly affected by extreme values.

(b) For skewed data displays, the median is often a better estimate of the center of the distribution than the mean, but <u>the mean is still the balance point of the distribution.</u>

(c) For skewed data displays, the median is often a better estimate of the center of the distribution than the mean, so <u>the median can be used as a typical value of the distribution.</u>

2. (a) A data display appears skewed to the right because <u>there are extreme values in the right tail of the display.</u>

(b) A data display appears skewed to the right, but <u>most of the data will be located on the left side of the display.</u>

(c) A data display appears skewed to the right, so <u>the median will be a better measure of center than the mean.</u>

3. (a) The mean is the center of balance in a data set because <u>the sum of the all deviations from each data value to the mean is zero.</u>

(b) The mean is the center of balance in a data set, but <u>the mean does not have to be a value in the data set.</u>

(c) The mean is the center of balance in a data set, so <u>the absolute values of the total deviations on each side of the mean are equal.</u>

UNIT 1

LESSON 1.4

Standard Deviation and Variance

OVERVIEW

LESSON DESCRIPTION

Part 1: Investigating Delivery Times
In this part of the lesson, students compare data about the delivery times of two food delivery companies to determine which company is more reliable. Working through this problem leads students to understand that a single statistic to summarize a data set is not always sufficient for drawing conclusions. Often, the variability of the data is an important consideration.

Part 2: Calculating Standard Deviation
Prior to this lesson, students described the spread, or variability, of a data set using the range or the interquartile range. In this part of the lesson, students explore the concepts behind the standard deviation, a measure of variability. The standard deviation is a statistic that quantitatively describes how close the data are to the mean.

Part 3: Solving the Delivery Problem
In this part of the lesson, students use what they learned about standard deviation and variance to complete the food delivery problem.

This problem has been adapted from the Taxicab Problem by the Smarter Balanced Assessment Consortium. The original problem is located here: ***https://portal.smarterbalanced.org/library/en/mathematics-content-specifications.pdf****.*

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Connections Among Multiple Representations

SUGGESTED TIMING

~90 minutes

LESSON SEQUENCE

This lesson is part of a lesson sequence (~585 minutes) that includes Lessons 1.1 through 1.6.

HANDOUTS

Lesson

- 1.4: The Delivery Problem

MATERIALS

- calculators or computers
- access to **Desmos.com**

CONTENT FOCUS

This lesson introduces students to the concept of standard deviation. One goal of the lesson is for students to understand that a measure of center, such as the mean, is only one type of summary statistic, and it might not convey all the relevant or important information about a data set. Two data sets could have the same mean, even though one data set could have all values clustered near the mean while the other has values spread relatively far from the mean. The standard deviation, like the interquartile range, is a measure of the variability, or spread, of the data. What makes the standard deviation different from the interquartile range is that its calculation accounts for the distance of each data value from the mean.

The convention for calculating standard deviation is to divide the sum of the squared deviations by one less than the number of data values, rather than by the number of data values. Therefore, most textbooks show the formula for standard deviation with a denominator of $n-1$. Using $n-1$ instead of n is called "Bessel's correction," and the many reasons for using it are beyond the scope of this course. One reason is that most data sets are only a sample of a population of interest, so dividing by $n-1$ yields a better estimate of the true value of the standard deviation for the population. A goal of this lesson is for students to understand that the standard deviation is a measure of the average distance of each data point from the mean of the data. To that end, Pre-AP treats all data sets as if they were populations, not samples, so dividing the sum of the squared deviations by n is appropriate. In future statistics courses, Bessel's correction will be fully developed.

COURSE FRAMEWORK CONNECTIONS

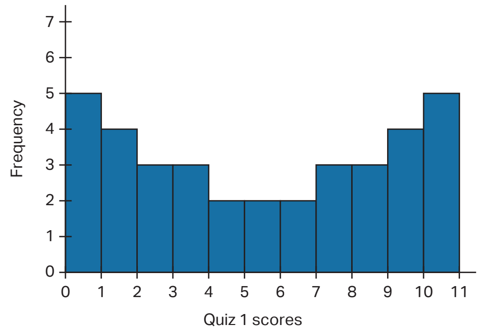
Enduring Understandings	
▪ Statistics are numbers that summarize large data sets by reducing their complexity to a few key values that model their center and spread.	
Learning Objectives	**Essential Knowledge**
1.1.1 Determine appropriate summary statistics for a quantitative data distribution.	**1.1.1b** Summary statistics describe the important features of data distributions including identifying a typical value, also called the center of the data, and describing the clustering of the data around the typical value, also called the spread of the data. **1.1.1d** The standard deviation, interquartile range, and range summarize a data distribution by quantifying the variability, or spread, of the data set. The standard deviation, interquartile range, and range have the same units as the values in the data distribution.
1.1.5 Analyze data distributions with respect to their variability.	**1.1.5a** Measures of variability quantify the typical spread of a data distribution. They are used to describe how similar the values of a data set are to each other. A distribution with low variability will have data values that are clustered at the center, so the distribution is well characterized by its measures of center. A distribution with high variability will have data values that are spread out from the center, so the distribution is less well characterized by its measures of center. **1.1.5b** The interquartile range is the length of the interval that contains the middle 50% of the values in a distribution.

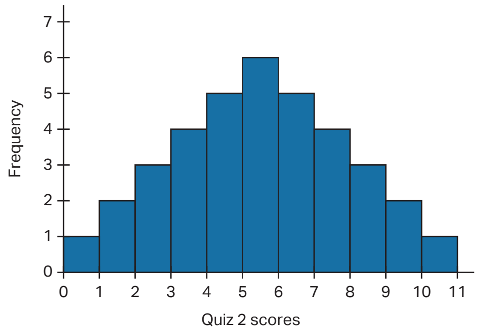
	1.1.5c The total variation of a distribution can be measured by the sum of the squared deviations from the mean. The variance of a distribution is the average of the squared deviations from the mean. **1.1.5d** The standard deviation is the square root of the variance. The standard deviation can be interpreted as a typical distance of the data values from the mean.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

A teacher recorded the scores on two different quizzes from her statistics class. The histograms below display the quiz scores, out of 10 points, for each of the 36 students.





(a) Which quiz, Quiz 1 or Quiz 2, has a larger standard deviation? Explain.

(b) The students took two more quizzes. Quiz 3 was difficult for the students, with most students scoring low, and Quiz 4 was easy for the students, with most students scoring high. Each quiz had 10 possible total points. Describe how the standard deviations for both quizzes might be similar or different, and draw possible histograms for each quiz.

(c) For their fifth quiz, every student received the same score. What is the standard deviation for Quiz 5? Explain.

UNIT 1

PART 1: INVESTIGATING DELIVERY TIMES

In this part of the lesson, students compare data about the delivery times of two food delivery companies to determine which company is more reliable. Working through this problem leads students to understand that a single statistic to summarize a data set is not always sufficient for drawing conclusions. Often, the variability of the data is an important consideration.

- To begin, let students know that they are going to analyze some data about wait times (the differences between scheduled and actual delivery times) to determine which of two food delivery services is more reliable. You do not have provide any more information than this right now. It is best to allow students some time to closely observe the problem details and start working.
- Distribute **Handout 1.4: The Delivery Problem** to students. Let them know that they should only work on Part 1 of the problem right now. Students can work in pairs or groups of three to start the problem.

Instructional Rationale

The reason this particular scenario is good for exploring standard deviation is that although Company A has a lower mean (and median) wait time, Company B has a smaller standard deviation in the wait times. This indicates that Company B is more consistent than Company A. Students will explore standard deviation in the second part of the lesson.

- As you circulate around the room while students work, observe where students are making progress and where they are struggling. Ask them some guiding questions to move them along, such as:
  - How can you create a display of this data? What tools do you have that you can use? What measures of center can you find? What units do you expect to use? How can you determine how much faster (or slower) one time is than another?

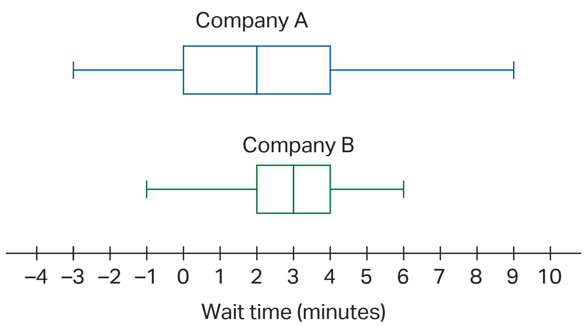
Meeting Learners' Needs

There are several ways in which you could differentiate instruction for this scenario. For example, you could have groups analyze only one of the two delivery companies.

- Students should have about 30 minutes to work on Part 1, which should be enough time to prepare arguments for each delivery company. Then gather the class back together for a whole-group discussion.
- For the discussion portion of this lesson, students should share their arguments for each company by writing them on the board or a personal white board, typing them in a shared document, or creating one or two slides they can present to the class. If you have a document camera, you could have students project their written work from the handout in a central location.
- During the discussion, listen for the methods and criteria students used to make their assessments, such as comparing boxplots or comparing the means and medians. Continue to probe students' decisions and arguments by asking them to explain what they did and what evidence they have to support their claims.

Guiding Student Thinking

Students might informally argue about the variability of the data sets by identifying that Company B has a smaller range and a smaller interquartile range than Company A, as indicated by the boxplot below. You can ask these students to explain the meaning of the variability in context. That is, students should be able to describe how the variability of the wait times affects how long a customer will wait. In other words, with which company is a customer more likely to have to wait a long time for their delivery?



UNIT 1

PART 2: CALCULATING STANDARD DEVIATION

Prior to this lesson, students described the spread, or variability, of a data set using the range or the interquartile range. In this part of the lesson, students explore the concepts behind the standard deviation, a measure of variability. The standard deviation is a statistic that quantitatively describes how close the data are to the mean.

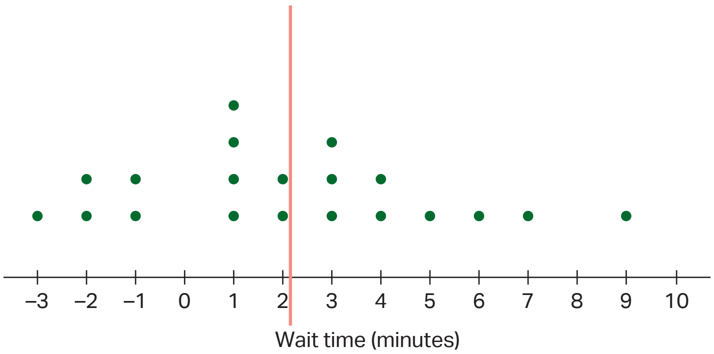
- Begin this part of the lesson by asking students to think about how important it is to know the variability of wait times.
    - Based on the data we have about the delivery times for Company A and Company B, is it enough to know just their mean or median wait times to compare them? Why or why not?

      Listen for students to say that even though Company A has a wider range of wait times than Company B, it would be helpful to know how the wait times are clustered in the data set to compare the companies.
- Let students know that they are going to explore a new statistic that is a measure of the variability of data.

Instructional Rationale

The sequence of questions below only attends to the delivery times for Company A. The intention is to support students through the process of calculating the standard deviation with Company A and then have them work through the same process on their own for Company B.

- To motivate students' initial thinking about deviations from the mean, display the information for Company A as a dot plot with the mean, 2.15 minutes, shown as a vertical red line.



- Ask students some questions to get them to connect the deviations from the mean (from Lesson 1.3) to this problem. Some questions you could use are:
  - Examine the dot plot of wait times for Company A. What do you notice? Would you say that the wait times are typically close to the mean wait time or typically far away from the mean wait time? How can you tell?

    Student answers will likely vary here. Some students will feel that the delivery times are close to the mean while others will disagree. Because the notion of closeness is relative, this kind of academic conversation can highlight why a quantitative measure of closeness, or spread, is helpful in analyzing data.
  - In the previous lesson, we added up the deviations from the mean to make sure that the mean was the center of balance. Do you think we could use the deviations from the mean to figure out a measure of the average distance from the mean?

    Student answers will vary but should generally agree that deviations from the mean provide a reasonable first calculation in quantifying closeness to the mean.

Guiding Student Thinking

You can expect students to struggle with the second question. It is valuable for students to be given some time to brainstorm how to quantify "closeness" to the mean. Since students understand that the mean is the sum of the values divided by the number of values in the set, they might suggest finding the mean of the deviations, or the mean of the distances from the mean (indicated by the red line in the display on the previous page). At this point, you do not have to correct any of their suggestions.

- Now let students determine the deviations from the mean for all the wait times from Company A. The calculations are shown in the following table.

Company A Wait Times	Wait Time – Mean	Deviation
-3	$-3-2.15$	-5.15
1	$1-2.15$	-1.15
2	$2-2.15$	-0.15
4	$4-2.15$	1.85
-1	$-1-2.15$	-3.15

continues

Company A Wait Times	Wait Time – Mean	Deviation
-2	$-2-2.15$	-4.15
5	$5-2.15$	2.85
4	$4-2.15$	1.85
1	$1-2.15$	-1.15
-2	$-2-2.15$	-4.15
2	$2-2.15$	-0.15
9	$9-2.15$	6.85
3	$3-2.15$	0.85
1	$1-2.15$	-1.15
-1	$-1-2.15$	-3.15
3	$3-2.15$	0.85
1	$1-2.15$	-1.15
7	$7-2.15$	4.85
6	$6-2.15$	3.85
3	$3-2.15$	0.85

- If students suggest finding the mean of the deviations, let them try it. They should see that since the sum of the deviations from the mean is zero, the mean will also be zero. Then you can ask students some questions like the ones below to get them to uncover the problem with finding the mean of the deviations:
  - If you add all the deviations together and divide this sum by the number of data points, what is the solution? Does it make sense that the total is zero? Why or why not?

    Because the mean is the center of balance, the sum of the deviations should be 0.

- However, we still need to find an "average" of the deviations from the mean. Why is the sum equal to zero and how might we deal with it?

  The different signs of the deviations are a problem because the negative and positive deviations cancel each other out. Students might suggest they need to "get rid of the negatives," which can be done by either taking the absolute value or by squaring each of the deviations.

Instructional Rationale

Students might be familiar with the mean absolute deviation from previous courses. For that calculation, students take the absolute value of the deviations and calculate the mean of the deviations. The reasons why statisticians prefer the standard deviation as a measure of variation are beyond the scope of this course. One reason is that squaring the nonzero deviations makes those that are between -1 and 1 smaller and those less than -1 or greater than 1 larger, and it provides a better perspective on the variability to magnify greater deviations and reduce lesser deviations.

- You may need to suggest to students that they square the deviations instead of simply taking their absolute values. This may seem strange to them because they are applying a type of function they have not yet used in statistics. You can help connect this process to their previous experience by pointing out that when they use the Pythagorean theorem to relate the lengths of the sides of a triangle, they use a similar process of squaring the distances. Students will eventually have to determine the square root, which they will do a few steps later in this process.
- Once you've convinced students that they should square the deviations, have them calculate the squared deviations. They should also determine the sum of the squared differences. The calculations are shown in the table on the next page.

Meeting Learners' Needs

When you explore the distance formula later in the course, you can link that process to this one. That will help students make sense of both procedures, since they both involve calculating distance. Many of the preferred mathematical methods for computing distance involve squaring then taking the square root.

Company A Wait Times	Wait Time – Mean	Deviation	Squared Deviation
−3	$-3-2.15$	−5.15	26.5225
1	$1-2.15$	−1.15	1.3225
2	$2-2.15$	−0.15	0.0225
4	$4-2.15$	1.85	3.4225
−1	$-1-2.15$	−3.15	9.9225
−2	$-2-2.15$	−4.15	17.2225
5	$5-2.15$	2.85	8.1225
4	$4-2.15$	1.85	3.4225
1	$1-2.15$	−1.15	1.3225
−2	$-2-2.15$	−4.15	17.2225
2	$2-2.15$	−0.15	0.0225
9	$9-2.15$	6.85	46.9225
3	$3-2.15$	0.85	0.7225
1	$1-2.15$	−1.15	1.3225
−1	$-1-2.15$	−3.15	9.9225
3	$3-2.15$	0.85	0.7225
1	$1-2.15$	−1.15	1.3225
7	$7-2.15$	4.85	23.5225
6	$6-2.15$	3.85	14.8225
3	$3-2.15$	0.85	0.7225
			Total = 188.55

- Before having students perform any further calculations, it is important to have them pause for a moment and think about what they have calculated, especially in terms of the units of the numbers they determined. You can ask questions like the ones below to help them reflect:
  - We determined the mean of the data and then determined the deviation of each data value from the mean. What are the units of the mean and the units of the deviations from the mean?

    The units for the mean are minutes. The units for the deviations from the mean are also minutes.
  - Does it make sense that the units for the mean and the deviations from the mean are the same? Why or why not?

    It makes sense that the units for the mean and the deviations from the mean are the same. The deviations from the mean describe how far away each data value is from the mean. Since the mean and each data value is in minutes, their difference is also in minutes.
  - What was the next step that we took? Why did we take that step?

    We squared each of the deviations from the mean. We did this step partly to make all the values positive.
  - What are the units for the squared deviations? Are they the same or different than the units for the deviations? Does that make sense to you?

    The units for the square deviations are different from the units for the deviations. Because we squared the deviations, which were measured in minutes, the units for the squared deviations should be in minutes^2 (or square minutes). It makes sense that the units are different because we transformed the data set.
- Now, have students find the mean of the squared deviations. Since there are 20 data points, the mean of the squared deviations is $\frac{188.55}{20} = 9.4275$.
- Again, pause for a moment and have students reflect on what they've determined. Direct students' attention to the units of the mean they calculated. You can ask some questions like the ones here:
  - We calculated a mean. What is it the mean of?

    The number we calculated is the mean of the squared deviations.
  - What are the units of this number? How do you know?

    The units of the mean of the squared deviations is minutes^2 (or square minutes). This is because the mean of a data set has the same units as the data values in the set.

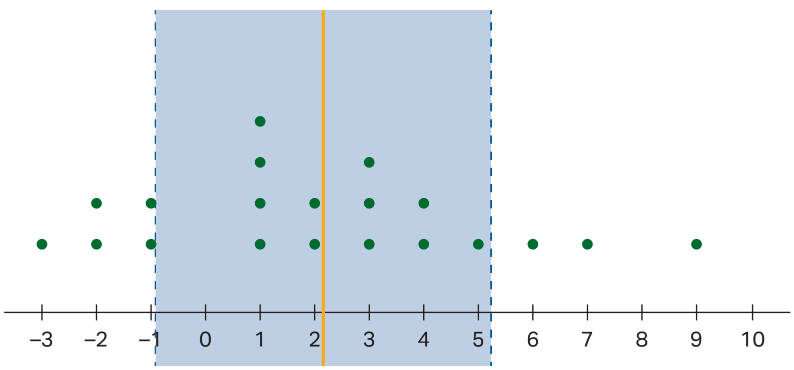
UNIT 1

- Does the number we calculated tell us the average deviation of the wait times from the mean? Why or why not?

  The number we calculated is not the average deviation of the delivery times from the mean. The units are not correct. An average deviation should have the same units as the data values.

- Let students know that the number they determined is called the *variance*. Because the variance is in square units, students will need to perform an additional operation to scale the units from square minutes back to minutes. You can point out that the word *variance* and the words *variation* and *variability* all share the same root word.
- To find the value called the standard deviation, we take the square root of the variance. In this case the standard deviation is $\sqrt{9.4275} \approx 3.07$ minutes.
- At this point, you can provide students with this definition of standard deviation:

  The standard deviation is a measure of the typical distance from the mean of data values in a distribution.
- Let students know that the standard deviation is often abbreviated with the symbol σ, the Greek letter sigma. In general, the standard deviation can be used in conjunction with the mean to provide an estimate of the range of the center of the data.
- To conclude, help students summarize the standard deviation in context. Since the mean is 2.15 minutes and the standard deviation is 3.07 minutes, the student council should expect that most deliveries will be within about 2.15 minutes plus or minus 3.07 minutes of the scheduled delivery time. In other words, the average delivery is made some time between 1 minute early and 5 minutes late.
- To show students a visual representation of the standard deviation, you can display the dotplot shown here. The graph has a shaded band for one standard deviation above and below the mean. You can have students observe the graph and notice that most of the delivery times (all except 8 of them) are located within this range around the mean.



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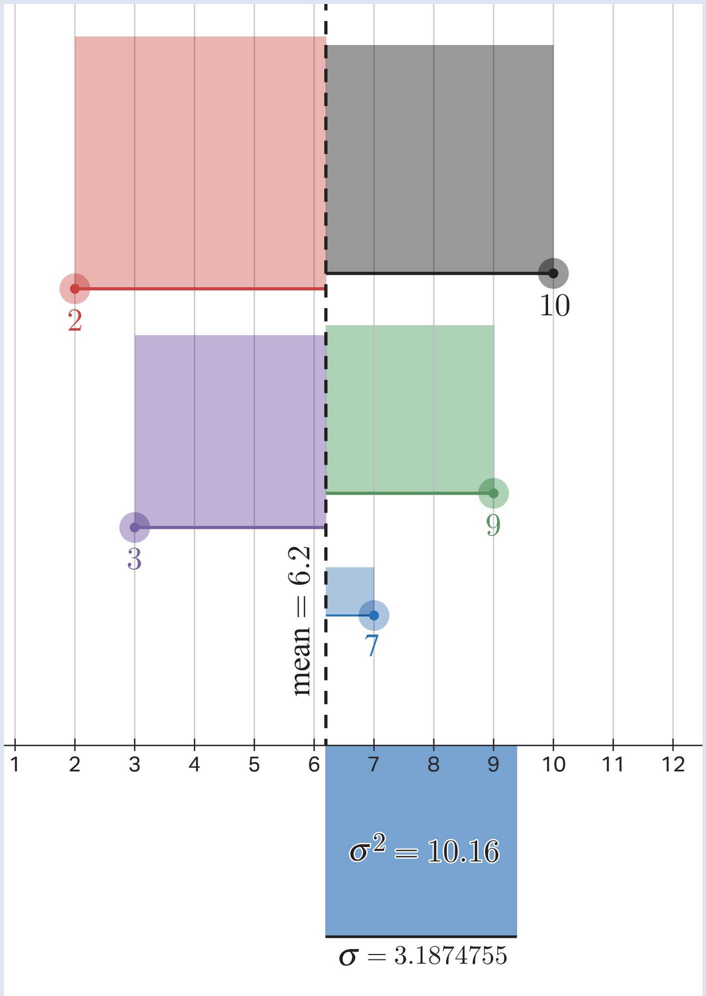
Guiding Student Thinking

It is valuable for students to engage with the Desmos interactive **preap.org/variance-stdev**, which provides an excellent visual depiction of the relationship between variance and standard deviation. As shown in the image below, the deviations are represented as horizontal line segments that connect each of the data points to a dashed line that represents their mean. The points are draggable, and the calculations are dynamic so the calculated statistics will change as the points change. The areas of the squares built on each line segment are a visual representation of the squared deviations. Below the horizontal axis, another square is shown, the variance, which can be interpreted as the mean of the squared deviations. The length of a side of this square—the square root of the variance—represents the standard deviation. This interactive is best used as part of a whole-class demonstration, but students could also benefit from having access to it on their own so they can experiment and make sense of the calculations.



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- You should have students repeat this process for Company B. The variance is about 3.05 minutes^2 and the standard deviation is approximately 1.75 minutes.
- You can show students how to use Desmos to calculate the variance and standard deviation for a data set. The commands are shown below.

Task	Desmos Command
Create a list using a data set.	A = [list of numbers separated by comma] *Lists of data values can be copied and pasted into Desmos.*
Calculate the variance of data set A.	varp(A)
Calculate the standard deviation of data set A.	stdevp(A)

You can find the statistics and graphing commands by clicking on the keyboard icon at the bottom left of the command pane, clicking on the **functions** button, and selecting either the **Stats** or **Dist** menus.

Note that the commands for variance and standard deviation include a *p* on the end of the command, which stands for *population*. The calculations performed by the *varp* and *stdevp* commands will mirror the ones in the lesson.

PART 3: SOLVING THE DELIVERY PROBLEM

In this part of the lesson, students use what they learned about standard deviation and variance to complete the food delivery problem.

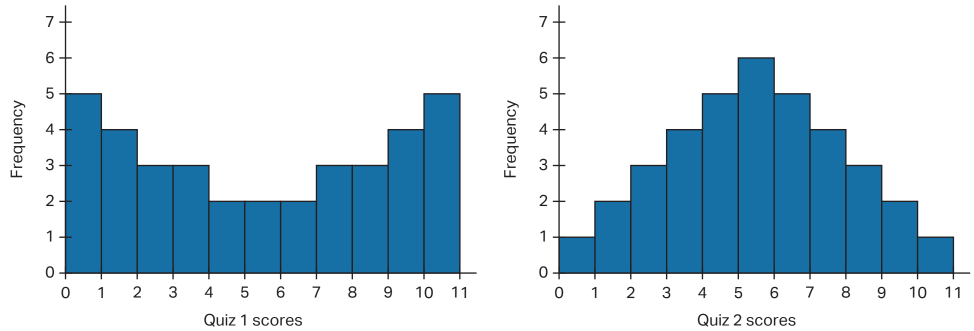
- Have students return to **Handout 1.4: The Delivery Problem** and revise the arguments they created for Company A and Company B in Part 1. Make sure they use standard deviation or variance in their revisions.
- Now ask students to turn to Part 2 and write a brief report for the student council moderator explaining which company they would recommend, citing evidence to support their recommendation. You can expect many groups to argue that Company B is a better choice. Even though the mean and median wait time is greater than those for Company A, the standard deviation is lower, meaning that it is less likely that deliveries will be very late.

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

A teacher recorded the scores on two different quizzes from her statistics class. The histograms below display the quiz scores, out of 10 points, for each of the 36 students.



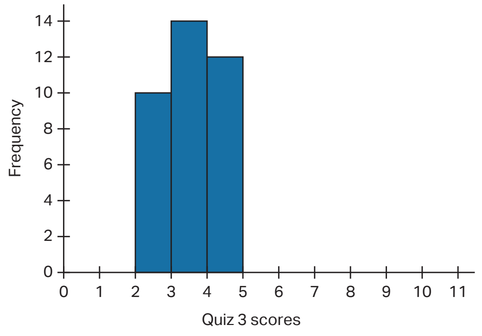
(a) Which quiz, Quiz 1 or Quiz 2, has a larger standard deviation? Explain.

The standard deviation measures the typical distance from the mean. While both quizzes have the same mean of 5 points (since the distributions are symmetric around 5 points), Quiz 1 has a larger standard deviation because more of its scores are farther from 5 points than for Quiz 2.

(b) The students took two more quizzes. Quiz 3 was difficult for the students, with most students scoring low, and Quiz 4 was easy for the students, with most students scoring high. Each quiz had 10 possible total points. Describe how the standard deviations for both quizzes might be similar or different, and draw possible histograms for each quiz.

The mean for the easy quiz would be greater than the mean for the hard quiz, because most students scored high on the easy quiz and low on the hard quiz. The standard deviations for both sets of data should be about the same, because for each quiz, most students' scores were close to each other. A possible histogram of each is shown on the next page.

UNIT 1



(c) For their fifth quiz, every students received the same score. What is the standard deviation for Quiz 5? Explain.

Since every student had the same score, the standard deviation would be 0. The mean quiz score would be whatever score every student scored and all deviations from the mean quiz score would be 0, so the standard deviation is also 0.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 1.4: The Delivery Problem

See lesson for solutions.

LESSON 1.5

Distributions as Functions

OVERVIEW

LESSON DESCRIPTION

Part 1: Representing Data Sets with Functions
This part of the lesson reminds students about the definition of function they learned in Algebra 1. It also deepens their understanding of what a function is by introducing the idea that a data distribution is a function whose inputs are data values and whose outputs are the frequencies of those data values.

Part 2: Understanding Histograms as Functions
This part of the lesson extends students' understanding of data distributions as functions by having them consider a different kind of function to represent binned data. For these functions, the inputs are data values and the outputs are the total numbers of data values contained in intervals, or bins, that include those input values. Understanding this type of function is vital for students as they explore how to use the normal curve as a model of data distributions in Lesson 1.6.

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Connections Among Multiple Representations

SUGGESTED TIMING

~90 minutes

LESSON SEQUENCE

This lesson is part of a lesson sequence (~585 minutes) that includes Lessons 1.1 through 1.6.

HANDOUTS

Lesson

- 1.5.A: Data Distributions as Functions

Practice

- 1.5.B: Binning Data

CONTENT FOCUS

One major mathematical goal of this lesson is to expand students' understanding of the function concept. In this lesson, students explore a function for which there is no algebraic formula. The concept of function that students generally use in middle school and early high school curricula is usually limited to two-variable equations. These equations are usually expressed with "$y =$ " followed by an algebraic expression that includes x variables. The Pre-AP Algebra 1 course introduced $f(x)$ notation to emphasize that functions map an input to an output. While the input–output aspects of functions were discussed in Algebra 1, the output was usually determined through calculations performed on an input value. By contrast, in this lesson, students explore how both frequency distributions and histograms exhibit the defining characteristic of a function—that each input (an individual data value) is associated with only one output (the frequency of the data value or the number of data values in a bin).

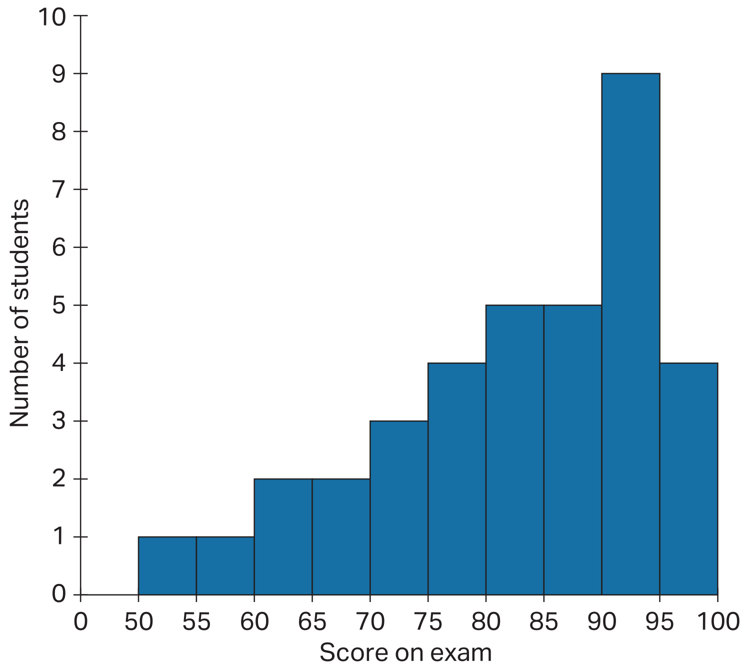
COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ Statistics are numbers that summarize large data sets by reducing their complexity to a few key values that model their center and spread. ▪ Distributions are functions whose displays are used to analyze data sets.	
Learning Objectives	**Essential Knowledge**
1.1.2 Create a graphical representation of a quantitative data set.	**1.1.2b** A histogram summarizes a quantitative data set by partitioning its values into equal-width intervals and displaying bars whose heights indicate the frequency of values contained in each interval. Histograms are used to depict the shape of a distribution.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

The histogram below displays the scores for a class of 37 students on a recent exam. Scores on this exam ranged from 51 to 98 points out of 100. Let the function f map an exam score to the height of the bar whose interval contains that score.



(a) What is the value of $f(83)$? Explain your answer.

(b) Can you tell from the histogram whether any students actually received a score of 83 points on the exam? Why or why not?

(c) Is the value of $f(95)$ equal to 9 or 4? Explain the reason for your answer.

UNIT 1

PART 1: REPRESENTING DATA SETS WITH FUNCTIONS

This part of the lesson prompts students to recall the definition of function they learned in Algebra 1. It also deepens their understanding of functions by introducing a data distribution as a function whose inputs are data values and whose outputs are the frequencies of those data values.

Instructional Rationale

Expanding students' definition of function to include data distributions helps to establish a continuity between this course and previous and future courses so students see that statistics and probability are part of the story of mathematics. Understanding a data distribution as a type of function is an important first step for students to understand why the normal distribution can be used as a model for data distributions. This development of students' understanding of the function concept is the first of two that occur in this course. The second occurs in Units 2 and 3 when students consider geometric transformations as functions that map the plane to itself.

- Begin the lesson by asking students to do a quickwrite about the prompt below:
    - Write down everything you think about when you hear the word *function*, especially as it relates to mathematics.
- Provide students with only a minute or two to write down their ideas. You can let students know that they can record their ideas as a list, as drawings, or using mathematical symbols. They should not be expected to write complete sentences here, as the idea of a quickwrite is to generate as many ideas as possible in a short period of time.
- Ask some students to share the ideas they included in their quickwrite. Record these ideas in a central location, and encourage students to build on each other's thoughts.
- Listen for students to mention the features of functions listed below. If they do not offer all three of these features, you may have to prompt them to recall them.

    1. A function is a relation between two sets such that each input has only one output.
    2. The domain of a function is the set of all possible inputs for the function. The range of a function is the set of all outputs for the function that result from the set of inputs.

Classroom Ideas

You could use a quickwrite to serve as the "think" step of a think-pair-share activity for this prompt. This would provide an opportunity for students to engage in an academic conversation about function concepts.

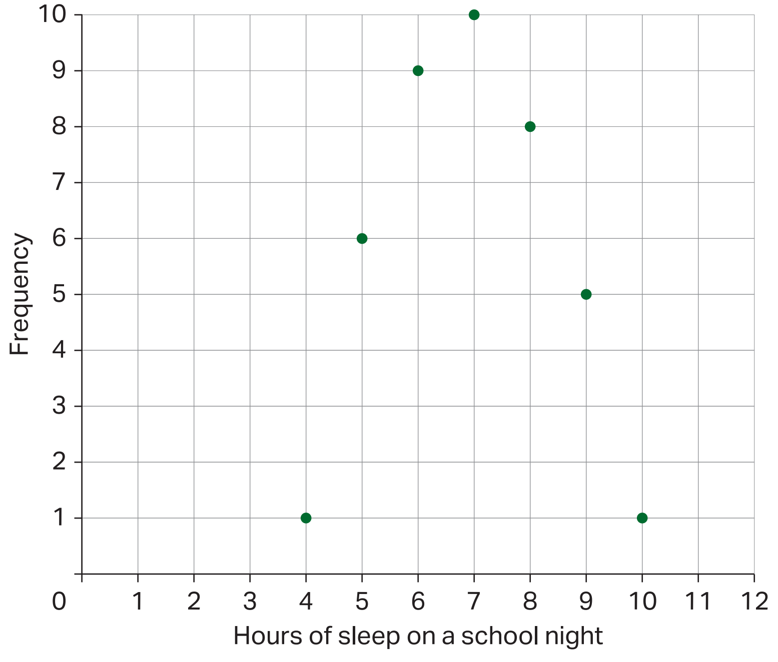
3. The notation "$f(x)$" is read as "f of x," where "f" is the name of a function, "x" stands for an input value in the domain of the function, and "$f(x)$" represents the output value in the range of the function that corresponds to the input value x.

- Students may mention other things about functions, including specific classes of functions such as linear, quadratic, and exponential, or different representations of functions such as graphical, numerical, algebraic, or verbal.
- Once students have completed a brief review of the characteristics of a function, they are ready to engage in the main task for this part of the lesson.
- Display the table of values below, which represents the results of a survey of teenagers who were asked about the number of hours they sleep each school night. Then you can ask students some questions to focus their attention on important features of the table.

Number of Hours Slept on a School Night	Frequency
4	1
5	6
6	9
7	10
8	8
9	5
10	1

  - What do you notice about the data in the table? What do you wonder about the data in the table?
  - How could we create a graph of the data set with the information from the table?

- Allow students some time to think about how the data can be graphed and what the graph would look like. Then give them time to work individually or with partners to create a graph. You can expect student pairs to generate academic conversations as they make sense of the data in the table. Students may suggest plotting the number of hours of sleep along the horizontal axis and the corresponding frequencies along the vertical axis, as shown below:



- As you circulate around the room to observe students as they generate their graphs, ask them some questions like the ones below to encourage further analysis of the data distribution.
  - Could the relationship between the number of hours slept and the number of students who reported those hours (the frequency) be described as a function? Why or why not?
  - Could the graph you generated from the data be described as a function? Why or why not?
  - If you could describe this data as a function, which set of values could be considered the domain of the function and which set of values could be considered the range of the function?
- After students have had a few minutes to generate their graphs and discuss the questions with their partners, have them share their thoughts with the whole class. Listen for students to state that the data could be represented by a function in

which the input is the number of hours slept on a school night and the output is the frequency that corresponds to the number of people who answered for each specific input (the number of hours slept).

- This share out and discussion should conclude with students understanding that any set of data can be described as a function where the inputs (domain) are the values of the phenomenon being measured, and the outputs (range) are the frequency of each measurement for that phenomenon.
- To check for students' understanding about how to use function notation in this context, ask them to answer some questions like the ones below:
    - Suppose that f is a function that maps the number of hours students said they slept on a school night to the frequency of that response. How would you interpret $f(5)=6$?

      In this context, $f(5)=6$ means that 6 students reported sleeping 5 hours on a school night.
    - How would you interpret $f(9)=5$?

      In this context, $f(9)=5$ means that 5 students reported sleeping 9 hours on a school night.
    - What is the value of $f(7)$ and what does it mean?

      The value of $f(7)$ is 10 and it means that 10 students reported sleeping 7 hours on a school night.
    - What is the contextual domain (the set of inputs) of this function?

      The contextual domain of this function is {4, 5, 6, 7, 8, 9, 10} because these were the only responses.
    - What is the contextual range (the set of outputs) of this function?

      The contextual range of this function is {1, 5, 6, 8, 9, 10} because these were the frequencies of the responses.

Meeting Learners' Needs

This is a good opportunity to help students who suggest that the data cannot be represented by a function because the frequency value of 1 is repeated. These students are likely confusing the direction of the association between the input and the output values that is necessary for a relation to be a function. You can help them recognize that each of the input values has only one corresponding output value.

Guiding Student Thinking

Some students might suggest that the domain of the function is all possible hours that could be slept on a school night, such as 0 to 12. The same reasoning could be argued about the range. It is valuable to acknowledge that the data in the table may be only part of a much larger data set with more precise hours of sleep per night, such as 5.5 hours. Considering the possible inputs and outputs—not just the actual inputs and outputs—of a function that models a distribution is an important first step in understanding how to use a continuous function, such as the normal curve, to model discrete data.

- Since these frequency functions are somewhat different from the functions students have explored in previous courses, you should take a moment to check students' understanding about the similarities and differences between the different classes of functions:
  - How is this kind of function (the frequency function) the same as other functions we have studied?

    As with linear, quadratic, or exponential functions, each input of the frequency function is associated with only one output.
  - How is this kind of function (the frequency function) different from other functions we have studied?

    There is no algebraic formula that associates an input to an output, as in linear, quadratic, or exponential functions.
- You might want to give students an opportunity to practice using function notation with data distributions. **Handout 1.5.A: Data Distributions as Functions** has some sample problems that you could use.

PART 2: UNDERSTANDING HISTOGRAMS AS FUNCTIONS

This part of the lesson extends students' understanding of data distributions as functions by having them consider a different kind of function to represent binned data. For these functions, the inputs are data values and the outputs are the total numbers of data values contained in intervals, or bins, that include the input values. Understanding this type of function is vital for students as they explore how to use the normal curve as a model of data distributions in Lesson 1.6.

- Begin this part of the lesson by reviewing students' answers to **Handout 1.5.A: Data Distributions as Functions**, problems 3(e), 3(f), and 3(g). In those parts of the problem, students were tasked with briefly exploring how to "bin" the travel times into 5-minute intervals and determine the frequency of the travel times within each 5-minute interval. That process should remind students of the method used to create a histogram.
- Have several students share their answers to problems 3(f) and 3(g). You can use this as an opportunity to check for understanding about the binning process and for constructing a histogram. This is also a good time to see what students understand about different kinds of frequency functions.

Classroom Ideas
If you collected data from students in Lesson 1.2, you can have students use that data to complete problem 1 on Handout 1.5.B.

- Now, you can have students work with a different data set. Use this data set about the heights in inches of 40 teenagers:

  {71, 75, 76, 69, 60, 65, 70, 61, 69, 65, 68, 70, 71, 75, 60, 64, 67, 64, 63, 63, 69, 65, 68, 73, 62, 66, 62, 70, 72, 65, 68, 62, 67, 72, 70, 67, 64, 75, 73, 65}

- Present the task below to students from **Handout 1.5.B: Binning Data**.

1. Suppose you are curious about the distribution of heights among teenagers. Use the data below—the heights of 40 teenagers measured to the nearest inch—to create two different histograms. Use different height ranges for the bins in each histogram.

   71, 75, 76, 69, 60, 65, 70, 61, 69, 65, 68, 70, 71, 75, 60, 64, 67, 64, 63, 63,

   69, 65, 68, 73, 62, 66, 62, 70, 72, 65, 68, 62, 67, 72, 70, 67, 64, 75, 73, 65

Handout 1.5.B

- Provide students with some time to work with a partner to construct the histograms. They can make any reasonable histogram using the data, but they should make two different histograms with the data.

Instructional Rationale

It is important for students to make multiple histograms of the same data set, even if they have very wide or very narrow bins, because they need to understand that different frequency functions can be defined from the same data set. The frequency function depends on the bin size. The questions at the end of **Handout 1.5.B: Binning Data** give students an opportunity to think about how the output values of a frequency function change depending on how bin intervals are defined.

- Make sure to debrief questions 1(a), 1(b), and 1(c). During this debrief, several pairs of students should share their histograms and observations with the class. You should choose specific histograms for students to share so that a diverse range of histograms generated by the class from the same data set is presented.
- Allow a few minutes for the class to engage in an academic conversation about the advantages and disadvantages of using different bin widths. Through this conversation, you can expect students to identify that narrow bin widths show almost the same information as dot plots, while wide bin widths combine so much information that they may not be helpful in answering questions about the variation in the distribution.
- Make sure to debrief questions 1(d), 1(e), and 1(f) with students. These questions are intended to help students understand that different frequency functions can be created from the same distribution.

Guiding Student Thinking

While students create their histograms and think about the functions defined by their binned data, they may ask about data values that fall on the boundary between two intervals. It is important to define the intervals so that the boundaries do not overlap, otherwise the binning would not be a function. In keeping with the convention established for AP Statistics, histogram bins are defined to **include** the left endpoint of the interval and **exclude** the right endpoint of the interval.

- Ask students some questions like the ones that follow to make sure they understand how to use a function to represent the relationship between the data value and the height of the bar of the histogram that includes that data value.
  - How are functions f and g from problem 1 like other functions we've studied?

    Just like linear, quadratic, or exponential functions, each input is associated with only one output.

- How are functions f and g from problem 1 similar to each other and how are they different from each other?

  Both functions have inputs that are data values and outputs that are the frequencies of those data values in specified intervals. They are different because the intervals (and interval widths) defined for each histogram are different. Therefore, the same input could have two different outputs, as represented by the different heights of the bars in their respective histograms.

- Are functions f and g from problem 1 the only two functions that can be defined with this data set? Explain why you think that way.

  No, you can define a new function with this data set that differs from f and g by selecting a different bin width.

As a wrap-up for this lesson, have students complete problem 2 on Handout 1.5.B. This problem has students answer questions about a data set whose histogram and associated frequency function are given. Students then create a new histogram using the existing histogram and create a new frequency function for the re-binned data.

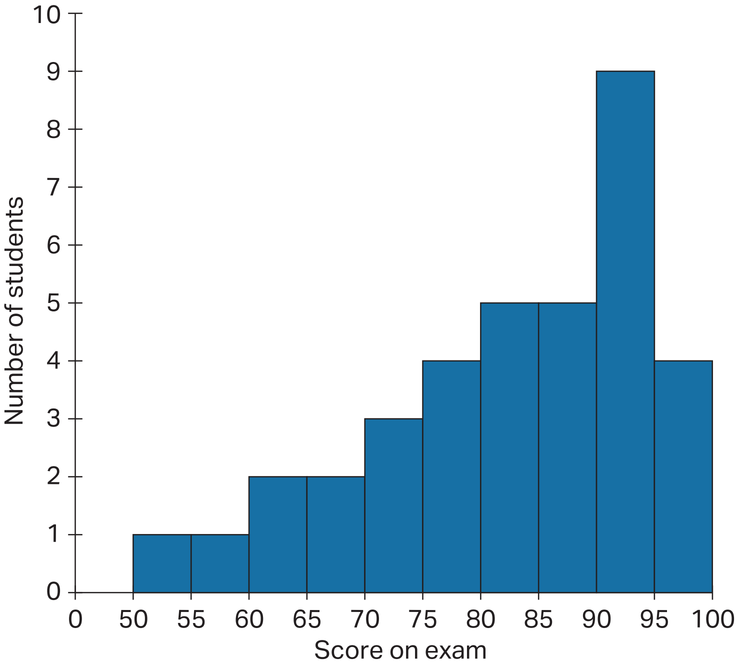
UNIT 1

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

The histogram below displays the scores for a class of 37 students on a recent exam. Scores on this exam ranged from 51 to 98 points out of 100. Let the function f map an exam score to the height of the bar whose interval contains that score.



(a) What is the value of $f(83)$? Explain your answer.

The value of $f(83)$ is 5 because the bar that contains a score of 83 points is 5 units tall.

(b) Can you tell from the histogram whether any students actually received a score of 83 points on the exam? Why or why not?

The histogram does not show whether any students actually received a score of 83 points. It only indicates that there were five students who received scores from 80 up to, but not including, 85 points.

(c) Is the value of $f(95)$ equal to 9 or 4? Explain the reason for your answer.

The value of $f(95)$ is 4 because the convention is for the left endpoint, but not the right endpoint, to be included in each interval. Therefore, a score of 95 points is in the interval from 95 to 100 points.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 1.5.A: Data Distributions as Functions

1. (a) The expression $f(\text{car}) = 2$ means that there were two responses of "car" as the method of travel.

   (b) The expression $f(\text{bus}) = 35$ means that there were 35 responses of "bus" as the method of travel.

   (c) The expression $f(\text{walk}) = 4$ means that there were four responses of "walk" as the method of travel.

Guiding Student Thinking

Students may interpret these expressions in a slightly different way by stating, for example, that "two people take a car to school." It is worthwhile to encourage students to improve the clarity and completeness of their statement by stating, for example, "Of the people who responded to the survey, two said they take a car to school." The context of the survey respondents might be implied in the student's answer, but it is more appropriate to fully incorporate the context when interpreting each function expression.

2. (a) The domain of the distribution is the set of all possible values a student could answer for the number of semesters they had completed. Some students might be taking the class in their first semester, while others could be taking it in a later semester. A reasonable domain, therefore, would be 0, 1, 2, and 3. The domain could include greater whole number values as well. The range of the distribution is the number of students for each of the possible answers. For example, if 15 students had completed no previous semesters, 30 students had completed one semester, 25 students had completed two semesters, and 10 students had completed three semesters, the range would be 10, 15, 25, and 30. The range could include values from 0 to 80, inclusive.

   (b) I disagree with the student's argument. A relation is a function if each input value is mapped to only one output value. In this case, the inputs are the number of semesters that a student has already completed and the outputs are the number of students that responded with that input value. For example, the number of students who responded with 0 for the number of completed semesters will total to a single value, so the input of 0 has exactly one output.

(c) (i) The expression is a correct use of function notation because the input is a value for what is being measured and the output is the number of those measurements. In this case, students were asked for their favorite car color, so the input is a color and the output is the number of students who responded with that color.

(ii) The expression is incorrect because the inputs and outputs have been switched.

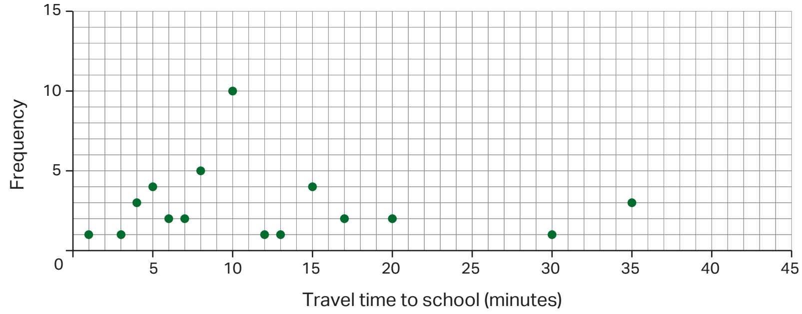
(iii) The expression is incorrect because it does not appropriately use either the input or output values.

3. (a) A frequency table for this data is:

Value	Frequency
1	1
2	0
3	1
4	3
5	4
6	2
7	2
8	5
9	0
10	10
11	0
12	1
13	1
14	0
15	4
16	0
17	2

Value	Frequency
18	0
19	0
20	2
21	0
22	0
23	0
24	0
25	0
26	0
27	0
28	0
29	0
30	1
31	0
32	0
33	0
34	0
35	3

(b) A graph of f should look like this:



One similarity between the graph of the function and the dot plot is that the height of the dots in the plot corresponds to the location of the point in the graph for each travel time. One difference between the graph of the function and the dot plot is that there is a single dot for each travel time that represents the total frequency instead of a stack of dots whose height is the frequency.

(c) $f(5) = 4$; it means that there were 4 people who responded that it took them 5 minutes to travel to school.

(d) The statement "Two students reported that it takes them 20 minutes to travel to school" would be expressed in function notation as $f(20) = 2$.

(e) The grouped frequency table would be:

Range of Travel Times to School	Frequency
$0 \le$ time < 5	5
$5 \le$ time < 10	13
$10 \le$ time < 15	12
$15 \le$ time < 20	6
$20 \le$ time < 25	2
$25 \le$ time < 30	0
$30 \le$ time < 35	1
$35 \le$ time < 40	3

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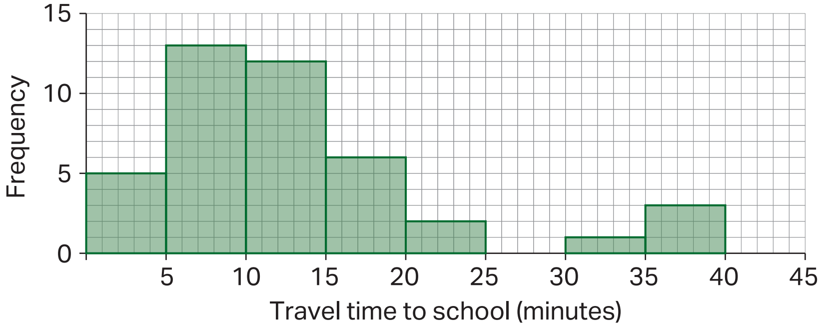
(f) The grouped frequency data could be considered a function because each input is a time and the output would be the frequency of all times in a defined interval that includes that time. Since each individual travel time can be in only one interval, and there is only one total frequency for each interval of travel times, the grouped frequency data could be considered a function.

(g) This graph looks like a histogram.



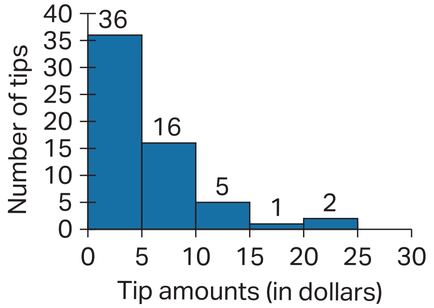
Handout 1.5.B: Binning Data

Answer for questions 1(a) and 1(b) will depend on the bin widths that students use to create the histograms. See lesson for details about the answers for 1(c), 1(d), 1(e), and 1(f).

2. (a) David earned 60 total tips this week.

(b) $f(4)=16$; $f(18)=0$

(c) A new histogram with bin widths of 5 is:



(d) $g(4)=36$; $g(18)=1$

(e) The values of $f(18)$ and $g(18)$ are different because in the first histogram, there were no tip amounts in the interval from \$17.50 up to, but not including, \$20. So, $f(18) = 0$. In the second histogram, the bin widths are wider so the interval that contains a tip amount of \$18 now includes all tip amounts from \$15 up to, but not including, \$20. Since there is one tip between \$15 and \$20, $g(18) = 1$.

LESSON 1.6

The Normal Distribution

OVERVIEW

LESSON DESCRIPTION

Part 1: Predicting Coin Flips

In this part of the lesson, students perform an experiment in which they flip coins to explore how patterns can emerge over many trials of a random event. This activity previews some concepts of probability.

Part 2: Modeling a Data Distribution with the Normal Curve

In this part of the lesson, students learn how to model a data distribution using a normal curve. For data distributions that are symmetric about the mean and whose data values are consistent with the empirical rule, a normal curve may be an appropriate model for the distribution. Using a normal curve to model a data distribution allows us to make generalizations and predictions about the population from which the data was drawn.

CONTENT FOCUS

The mathematics of this lesson is closely connected to the mathematics introduced in Lesson 1.5. In that lesson, students explored how to represent a data distribution with a function in which the inputs are data values and the outputs are the total numbers of data values contained in intervals, or bins, that include the input values. The expanded understanding of a data distribution as a function helps students understand how to use the normal distribution as a model for data distributions whose histograms are relatively symmetric and bell-shaped. Almost no data sets are perfectly normally distributed. However,

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Connections Among Multiple Representations

SUGGESTED TIMING

~90 minutes

LESSON SEQUENCE

This lesson is part of a lesson sequence (~585 minutes) that includes Lessons 1.1 through 1.6.

HANDOUTS

Lesson

- 1.6.A: A Note-Taking Guide for the Video

Practice

- 1.6.B: Practice with the Normal Distribution

MATERIALS

- coins or access to coin flipping simulations
- access to **Desmos.com** or other graphing utility
- internet access to the video “The Galton Board” (2:30), available at **youtube.com/watch?v=UCmPmkHqHXk**

when a distribution can be modeled by a normal curve, we can use the model to make generalizations and predictions about the scenario from which the data was drawn. Students should start to understand that using a normal curve to model a data distribution is similar to using a linear function to model the relationship among the data in a scatterplot.

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ Statistics are numbers that summarize large data sets by reducing their complexity to a few key values that model their center and spread. ▪ Distributions are functions whose displays are used to analyze data sets.	
Learning Objectives	**Essential Knowledge**
1.1.6 Model a data distribution with a normal distribution.	**1.1.6a** The normal distribution is a model of a data distribution defined by its mean and standard deviation. The normal distribution is bell-shaped and symmetric about the mean. In a normal distribution, the frequency of data values tapers off at one standard deviation above or below the mean. **1.1.6b** When a normal distribution is used to model a data distribution, approximately 68% of the data values fall within one standard deviation of the mean. Approximately 95% of the data values fall within two standard deviations of the mean. Over 99% of the data values fall within three standard deviations of the mean. **1.1.6c** For normally distributed data, the mean and median are the same number, and they correspond to the mode, which is the value in the distribution with the highest frequency.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

A cell phone company says that a fully charged battery lasts 14 hours. In reality, each battery behaves a little differently. For a large quantity of fully charged batteries, the mean length of time that batteries last is 14 hours, with a standard deviation of 1 hour. Assume that the lengths of time that battery charges last can be modeled by a normal distribution.

(a) What percent of fully charged cell phone batteries would be expected to last longer than 14 hours?

(b) What percent of fully charged cell phone batteries would be expected to last between 12 and 14 hours?

(c) What percent of fully charged cell phone batteries would be expected to last less than 15 hours?

(d) Would it be surprising if a fully charged cell phone battery lasted more than 17 hours? Explain why or why not.

PART 1: PREDICTING COIN FLIPS

In this part of the lesson, students perform an experiment in which they flip coins to explore how patterns can emerge over many trials of a random event. This part of the lesson previews some concepts of probability.

- To begin this part of the lesson, ask students this question to get them thinking about what it means for an occurrence to be *random*:

  Suppose that you were going to flip a coin four times and record the results. What do you think is more likely to happen: getting four heads in a row, or getting two heads and two tails in any order? Why do you think that?

- Give students a minute or two to think about the question, and then have a few students share their answers. You could encourage an academic conversation by having one student explain their answer, and then having a second student state whether they agree or disagree. The second student can then paraphrase the reasoning of the first student if they agree, and present an answer and reason of their own if they disagree. Many students will decide that they are more likely to get two heads and two tails in any order. Students may reason that since there is a 50% chance of getting heads, you would expect to get heads about half the time. It is less likely, though not impossible, to flip the coin four times and get a head on each flip.
- Let students know that they will be flipping a coin four times and tracking the results. Each student should perform 10 trials of four coin flips. You can have students record their results in a table of values like the one on the following page.

Classroom Ideas

You could have students perform the coin flips outside of class and bring the data to class with them. Students could use an online coin-flip simulator to generate the coin flips instead of physically performing the coin flips. However, it is important to perform the coin flips either physically or with a simulation. Students who do not actually perform the coin flips and simply write down 10 sequences of Hs and Ts will not choose Hs and Ts randomly and will end up with biased data.

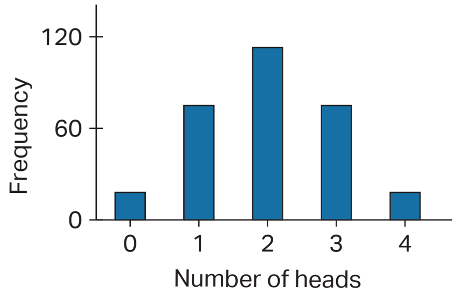
Trial Number	Results (All Four Coin Flips)	Number of Heads
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		

- You might need to remind students that they performed these trials to see which outcome occurs more frequently: 4 heads out of four flips or exactly 2 heads out of four flips, in any order. That is why the table above counts only the number of heads.

Guiding Student Thinking

Some students might object and insist that you should keep count of the number of tails as well. You can let them count the number of tails if they want to. It is a good opportunity to let them figure out that because there are only two choices for each individual coin flip, they only need to count the number of heads (or tails) for a trial of four flips, because from those data they can calculate the number of tails (or heads). That is, if four coins are flipped and you got 3 heads, then you must have gotten 1 tail. The coin flips are an example of a *binomial distribution*. However, a more detailed understanding of binomial distributions is beyond the scope of the course.

- Now, have students determine the class's frequency for groups of four flips of 0 heads, 1 head, 2 heads, 3 heads, and 4 heads. Then construct a bar graph (or dot plot) with the results. The graph should look roughly symmetric:



Classroom Ideas

One way to quickly gather data would be for students to enter the frequencies for groups of 0 heads, 1 head, 2 heads, 3 heads, and 4 heads in a shared spreadsheet that would instantly total the frequencies using a summation formula.

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- Once you have a graph of the results, ask students to closely observe and analyze it. Give them an opportunity to notice and wonder about the graph.
  - Write down what you notice about the graph. What features or characteristics do you see?

    The graph is roughly symmetric. The most common outcome was 2 heads. The number of times the class got only 1 heads out of four flips was roughly the same as the number of times we got 3 heads out of four flips. The number of times we got 0 heads was roughly equal to the number of times we got 4 heads.
- Now, ask a question to have them predict what a graph of multiple trials of six coin flips would look like.
  - Suppose we performed almost the same experiment, but we flipped a coin six times instead of four for each trial. What do you think the graph of the results would look like?

    The graph would be roughly symmetric. The most common outcome would be 3 heads. The frequency of 2 heads would be about the same as the frequency of 4 heads; the frequency of 1 heads would be about the same as the frequency of 5 heads; the frequency of 0 heads would be about the same as the frequency of 6 heads.

Instructional Rationale

The question about predicting the graph for trials of six coin flips can be an opportunity to check for student understanding about how the outcomes of multiple trials can be predicted even if the outcome of a single trial cannot be predicted. Also, it is better to ask for the graph of trials of an even number of coin flips rather than an odd number of coin flips. With an even number of coin flips, there are an odd number of possible outcomes. That means there will be a single most common outcome (an equal number of heads and tails), and consequently, a single maximum value in the resulting graphical display, which will make the conclusions a little more straightforward for students.

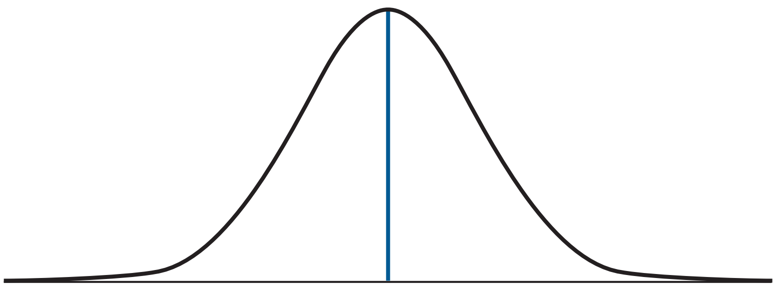
- Students should not be expected to determine values for the numbers of heads for the six coin flip trials, just to recognize that the graph would be symmetric and that the number of expected outcomes would taper off at the extremes. Once students have made some predictions about the outcomes of six coin flips and have recognized that the experimental results should fall into a symmetric graph, they are ready to move on to the next part of the lesson.

PART 2: MODELING A DATA DISTRIBUTION WITH THE NORMAL CURVE

In this part of the lesson, students learn how to model a data distribution using a normal curve. For data distributions that are symmetric about the mean and whose data values are consistent with the empirical rule, a normal curve may be an appropriate model for the distribution. Using a normal curve to model a data distribution allows us to make generalizations and predictions about the population from which the data was drawn.

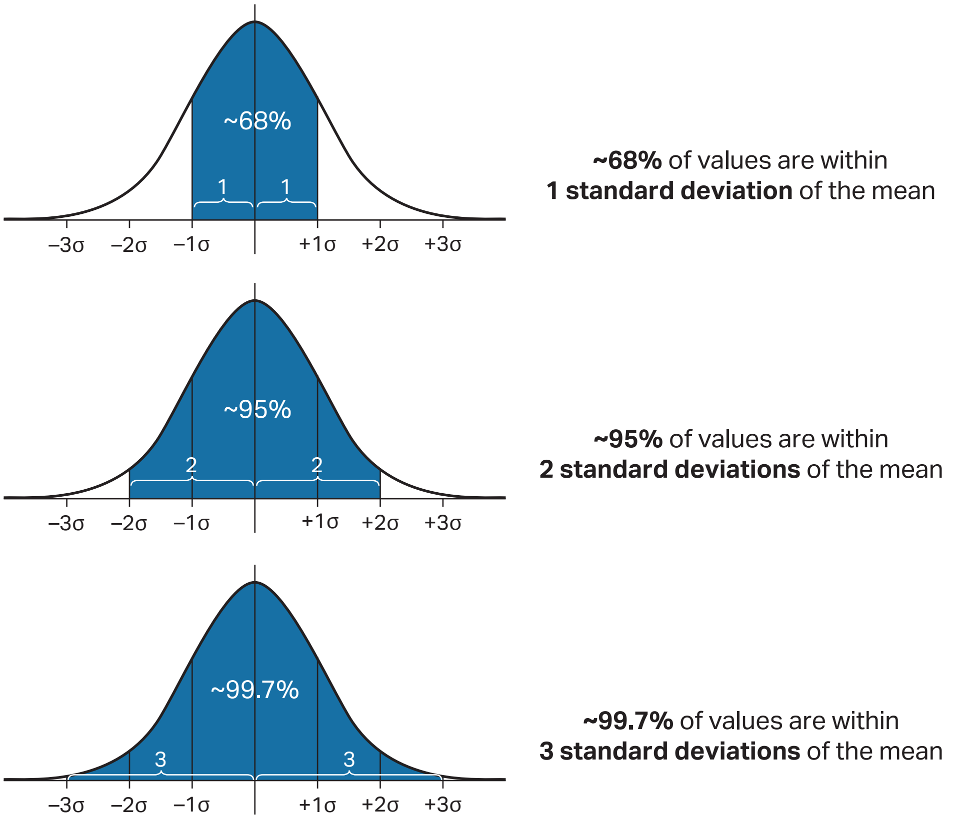
- The observation students made at the end of Part 1—that some experimental outcomes fall into a symmetric pattern that tapers off quickly toward the extremes—demonstrates an important mathematical concept: the normal distribution.
- Show students the first 2 minutes and 30 seconds of this video about the Galton Board: **youtube.com/watch?v=UCmPmkHqHXk**. After 2 minutes and 30 seconds, the host discusses several concepts, including probability and Pascal's triangle, that are not relevant for an introduction to the normal distribution.
- Let students know that when a histogram of a data distribution has a bell shape, the distribution can often be modeled by a *normal curve*, which is a bell-shaped continuous function that looks like this:

> **Meeting Learners' Needs**
> Some students will benefit from having an organizer to track their notes and questions through the video. A simple version of a graphic organizer is provided on **Handout 1.6.A: A Note-Taking Guide for the Video**.

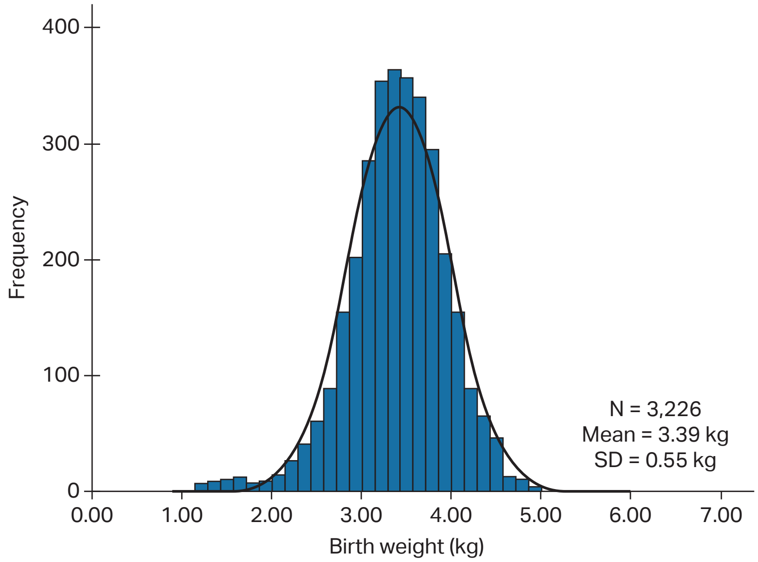


UNIT 1

- The properties of a normal curve make it a useful model for data distributions. In an approximately normal distribution, the percentages of data that falls within intervals determined by the mean and the standard deviation (σ) are:
  - approximately 68% of the data values are within one standard deviation of the mean;
  - approximately 95% of the data values are within two standard deviations of the mean;
  - and approximately 99.7% of the data values are within three standard deviations of the mean.



- You can let students know that many naturally occurring phenomena fall into an approximately normal distribution, including heights of people in a population.
- Display the histogram on the following page, which shows the distribution of baby birth weights in the United Kingdom. Then ask some questions about the distribution:



O'Cathain A., Walters S.J., Nicholl J.P., Thomas K.J., and Kirkham M. (2002). "Use of evidence based leaflets to promote informed choice in maternity care: randomized controlled trial in everyday practice." *British Medical Journal* 324: 643-646.

- What do you notice about the histogram and its relationship to the normal curve drawn on it?

  Students may notice many different aspects of the histogram, including that the bars in the center are taller than the normal curve or that there are more newborns with low birth weight than would be expected by the normal model.

- In the data shown, the mean weight of a newborn baby is 3.39 kilograms (about 7.47 pounds) and the standard deviation is 0.55 kilograms (about 1.21 pounds). Since there are 3,226 newborns represented and the data are approximately normal, about how many newborns from the study would you expect to have birth weights within one standard deviation of the mean?

  Since the data are approximately normal, we would expect 68% of the birth weights to be within one standard deviation of the mean. For these data, that would suggest that about 2,193 newborns would have birth weights between 2.84 kg and 3.94 kg (between 6.26 lbs and 8.68 lbs).

- About how many newborns would you expect to have birth weights within two standard deviations of the mean?

  Since the data are approximately normal, we would expect 95% of the birth weights to be within two standard deviations of the mean. For these data, that would suggest that about 3,065 newborns would have birth weights between 2.29 kg and 4.49 kg (between 5.05 lbs and 9.90 lbs).

- About how many newborns would you expect to have birth weights within three standard deviations of the mean?

  Since the data are approximately normal, we would expect 99.7% of the birth weights to be within three standard deviations of the mean. For these data, that would mean that around 3,216 newborns have birth weights between 1.74 kg and 5.04 kg (between 3.84 lbs and 11.11 lbs).

- Let students know that the percentages of the data corresponding to one, two, and three standard deviations in the normal curve are collectively referred to as the "68-95-99.7 rule," or the *empirical rule*. An advantage of using the normal curve as a model for a data set is that we can use it predict the likelihood of specific data values or intervals of values occurring.

  - If these data are an indication of birth weights in general, would you expect many newborns to have a birth weight of 5.5 kg (12 pounds)? Why or why not?

    It would be very unlikely for a newborn to weigh that much because the percent of the data occurring more than three standard deviations above the mean is very small.

Instructional Rationale

It is not necessary for students to calculate an exact probability at this point. For now, it is sufficient that they can argue that it would be unlikely for data values three standard deviations above or below the mean to occur.

- Students will likely need more practice with the normal curve. There are some problems on **Handout 1.6.B: Practice with the Normal Curve** that you can use to help students develop this concept.

Guiding Student Thinking

You can help students connect the concepts of modeling data using a function by reminding them how a scatterplot with a roughly linear shape can be modeled by a linear function whose parameters are the slope and vertical axis intercept. Similarly, data distributions with a symmetric bell shape can often be modeled using the normal curve, whose parameters are the mean and standard deviation of the data set.

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

A cell phone company says that a fully charged battery lasts 14 hours. In reality, each battery behaves a little differently. For a large quantity of fully charged batteries, the mean life of a battery is 14 hours, with a standard deviation of 1 hour. Assume that the battery life can be modeled by a normal distribution.

(a) What percent of fully charged cell phone batteries would be expected to last longer than 14 hours?

Since the mean is 14 hours and the distribution is approximately normal, 50% of fully charged cell phone batteries would be expected to last longer than 14 hours.

(b) What percent of fully charged cell phone batteries would be expected to last between 12 and 14 hours?

Since the mean is 14 hours and the standard deviation is 1 hour, approximately 95% of fully charged cell phone batteries would be expected to last between 12 and 16 hours. Since the distribution is symmetric, 47.5% of fully charged cell phone batteries would be expected to last between 12 and 14 hours.

(c) What percent of fully charged cell phone batteries would be expected to last less than 15 hours?

Since the mean is 14 hours and the standard deviation is 1 hour, approximately 68% of fully charged cell phone batteries would be expected to last between 13 and 15 hours. Therefore, 34% of fully charged cell phone batteries would be expected to last between 14 and 15 hours. Combining that with the fact that 50% of fully charged cell phone batteries would be expected to last fewer than 14 hours, 84% of cell phone batteries would be expected to last fewer than 15 hours.

(d) Would it be surprising if a fully charged cell phone battery lasted more than 17 hours? Explain why or why not.

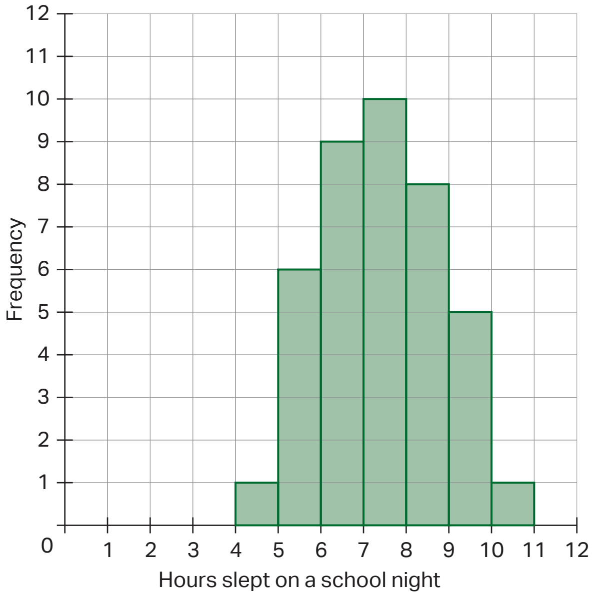
Approximately 99.7% of fully charged cell phone batteries would be expected to last between 11 and 17 hours, meaning 0.15% of all cell phone batteries would be expected to last longer than 17 hours. It would be very surprising if a fully charged cell phone battery lasted longer than 17 hours.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 1.6.B: Practice with the Normal Curve

1. (a) A histogram for the data using a bin width of 1 is shown below:



The histogram is roughly symmetric with the mode in the center of the distribution.

(b) The mean of the data is 6.925 hours of sleep. The median of the data is 7 hours of sleep. The standard deviation is approximately 1.403 hours. The mean and median are almost equal and in the center of the distribution, as would be expected of a symmetric distribution.

(c) There are 27 students reporting sleep hours within one standard deviation of the mean. That is 67.5% of the data. That value is slightly less than, but very close to what would be expected in a normal distribution model (68%).

(d) There are 38 students reporting sleep hours within two standard deviations of the mean. This is 95% of the data. That value is exactly what would be expected in a normal distribution model.

(e) There are 40 students reporting sleep hours within three standard deviations of the mean. This is 100% of the data. That value is slightly greater than, but extremely close to what would be expected in a normal distribution model (99.7%).

(f) The National Sleep Foundation recommends that teenagers get between 8 and 10 hours of sleep per night. According to the data, only 35% of the teenagers in the survey reported that they get that amount of sleep.

2. (a) Since the mean is 70 inches and the distribution is approximately normal, 50% of adult men in the United States are shorter than 70 inches.

   (b) Since the mean is 70 inches and the standard deviation is 4 inches, approximately 95% of adult men in the United States are between 62 and 78 inches tall. Since the distribution is symmetric, 47.5% of adult men in the United States are between 70 and 78 inches tall.

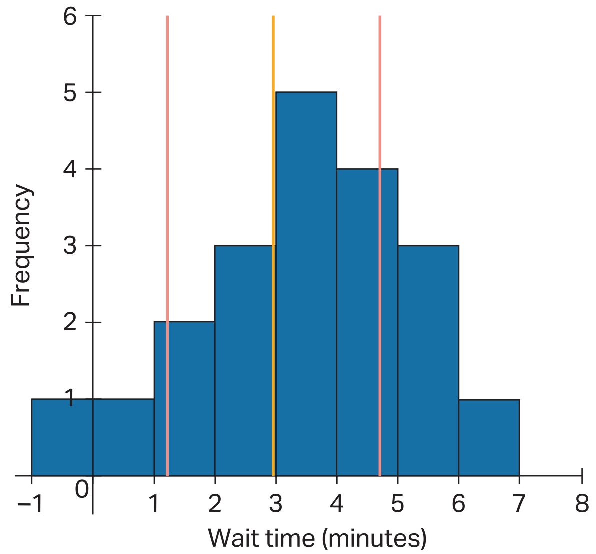
   (c) Since the mean is 70 inches and the standard deviation is 4 inches, approximately 68% of adult men in the United States are between 66 and 74 inches tall. Therefore, 34% of adult American men are between 70 and 74 inches tall. Combining that percentage with the 50% of adult men in the United States that are less than 70 inches tall, 84% of all men in the United States are shorter than 74 inches tall.

   (d) Approximately 99.7% of adult men in the United States are between 58 and 82 inches tall, meaning 0.15% of adult men in the United States are taller than 82 inches. Therefore, it would be surprising for a man in the United States to be taller than 82 inches.

Meeting Learners' Needs

Some students might want more information about teenagers and their sleep habits. You can share this resource with them: **sleepfoundation.org/articles/teens-and-sleep**.

3. (a) A histogram of the data using increments of 1 minute would look like this:



(b) The mean of the wait times is 2.95 minutes. The standard deviation of the wait times is 1.75 minutes. There are 12 data points (60%) within one standard deviation of the mean; there are 19 data points (95%) within two standard deviations of the mean; and there are 20 data points (100%) within three standard deviations of the mean. Even though the percent of data within one standard deviation of the mean is slightly lower than the expected 68%, these data could be modeled by a normal curve because the overall shape is symmetric and the percentages of data values are roughly consistent with the empirical rule.

(c) It is very unlikely to have to wait longer than 8 minutes for a scheduled delivery. A wait time of 8.2 minutes is three standard deviations above the mean. If these wait times are representative of the typical wait times, then it is very unlikely for a delivery to arrive 8 minutes late or later.

PRACTICE PERFORMANCE TASK

Staffing the Grocery Store

OVERVIEW

DESCRIPTION

In this practice performance task, students describe a data distribution, analyze how two different representations of the same distribution can reveal different aspects of it, and conclude that the empirical rule is not an if and only if statement.

CONTENT FOCUS

This task is designed to assess students' understanding of different measures of center and how measures of center are related to the shape of a distribution. This task also reinforces students' understanding of the empirical rule. It is intended to be used at the completion of Key Concept 1.1 in Unit 1. In the task, students construct a histogram, describe a distribution, and compare two distributions. This task also requires students to investigate whether the empirical rule by itself can be used to determine if a distribution is approximately normally distributed.

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Engagement in Mathematical Argumentation
- Connections Among Multiple Representations

SUGGESTED TIMING

~45 minutes

HANDOUT

Unit 1 Practice Performance Task: Staffing the Grocery Store

MATERIALS

- scientific calculator or graphing utility
- graph paper (optional)

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ Statistics are numbers that summarize large data sets by reducing their complexity to a few key values that model their center and spread. ▪ Distributions are functions whose displays are used to analyze data sets.	
Learning Objectives	**Essential Knowledge**
1.1.1 Determine appropriate summary statistics for a quantitative data distribution.	**1.1.1a** A data distribution is a function whose input is each value in a data set and whose output is the corresponding frequency of that value. **1.1.1b** Summary statistics describe the important features of data distributions including identifying a typical value, also called the center of the data, and describing the clustering of the data around the typical value, also called the spread of the data. **1.1.1c** The mean and the median summarize a data distribution by identifying a typical value, or center, of the distribution. The mean and median have the same units as the values in the data distribution. **1.1.1d** The standard deviation, interquartile range, and range summarize a data distribution by quantifying the variability, or spread, of the data set. The standard deviation, interquartile range, and range have the same units as the values in the data distribution.

Learning Objectives	Essential Knowledge
1.1.2 Create a graphical representation of a quantitative data set.	**1.1.2a** A boxplot summarizes a quantitative data set by partitioning its values into four groups, each consisting of the same number of data values. Boxplots are used to depict the spread of a distribution. **1.1.2b** A histogram summarizes a quantitative data set by partitioning its values into equal-width intervals and displaying bars whose heights indicate the frequency of values contained in each interval. Histograms are used to depict the shape of a distribution.
1.1.3 Analyze data distributions with respect to their centers.	**1.1.3a** The mean is the only point in the domain of a distribution where the sum of the deviations, or differences, between the mean and each point in the distribution is zero. **1.1.3b** The mean can be thought of as the center of mass of the data set. It is a weighted average that accounts for the number of data points that exists for every given value in the data set. **1.1.3c** Measures of center can be used to compare the typical values of the distributions. They provide useful information about whether one distribution is typically larger, smaller, or about the same as another distribution.

Learning Objectives	Essential Knowledge
1.1.4 Analyze data distributions with respect to their symmetry or direction of skew.	**1.1.4a** For symmetric distributions, such as the normal distribution, the proportion of data in any range to the left of the mean is equal to the proportion of data in the corresponding range to the right of the mean. **1.1.4b** Skew describes the asymmetry of a distribution. The direction of skew is indicated by the longer tail of data values in an asymmetric distribution. **1.1.4c** When a distribution is skewed, its mean and median will differ. The farther apart the mean and median are in a distribution, the more skewed the distribution will appear.
1.1.5 Analyze data distributions with respect to their variability.	**1.1.5a** Measures of variability quantify the typical spread of a data distribution. They are used to describe how similar the values of a data set are to each other. A distribution with low variability will have data values that are clustered at the center, so the distribution is well characterized by its measures of center. A distribution with high variability will have data values that are spread out from the center, so the distribution is less well characterized by its measures of center. **1.1.5b** The interquartile range is the length of the interval that contains the middle 50% of the values in a distribution. **1.1.5c** The total variation of a distribution can be measured by the sum of the squared deviations from the mean. The variance of a distribution is the average of the squared deviations from the mean. **1.1.5d** The standard deviation is the square root of the variance. The standard deviation can be interpreted as a typical distance of the data values from the mean.

Learning Objectives	Essential Knowledge
1.1.6 Model a data distribution with a normal distribution.	**1.1.6a** The normal distribution is a model of a data distribution defined by its mean and standard deviation. The normal distribution is bell-shaped and symmetric about the mean. In a normal distribution, the frequency of data values tapers off at one standard deviation above or below the mean. **1.1.6b** When a normal distribution is used to model a data distribution, approximately 68% of the data values fall within one standard deviation of the mean. Approximately 95% of the data values fall within two standard deviations of the mean. Over 99% of the data values fall within three standard deviations of the mean. **1.1.6c** For normally distributed data, the mean and median are the same number, and they correspond to the mode, which is the value in the distribution with the highest frequency.

SUPPORTING STUDENTS

BEFORE THE TASK

In this practice performance task, students are expected to use their understanding of measures of center and spread to analyze a novel situation. Before beginning the task, students might benefit from some warm-up questions related to the possible shapes of graphs of data distributions and summary statistics. You could begin the class by posing some questions to students, such as:

- What is a boxplot? What do we use boxplots for?

  Boxplots are a type of data display that shows the spread of a data distribution. Boxplots split the data into four groups with an equal number of data values in each group.

- What is a histogram? What do we use histograms for?

  Histograms are a type of data display that splits a data distribution into intervals of equal width. Histograms are useful for showing the shape of a distribution.

- What are the mean and the median? What do we use the mean and median for?

  The mean and median are measures of the center of a data distribution. We use them to identify a typical value of the data distribution. The mean is a value that balances the deviations of all other data values. The median is a value in the middle of an ordered data set.

- What is a standard deviation? What do we use the standard deviation for?

  The standard deviation is a measure of the spread of a data set. The standard deviation can be thought of as an average distance that a data value is from the mean. The standard deviation is useful for determining if the values in a distribution are relatively close together or far apart.

- What is the normal distribution? What does a normally distributed data set look like?

  The normal distribution is a model of a data distribution that is symmetric about its mean. In a normally distributed data set, 64% of the data will be located within one standard deviation of the mean, 95% of the data will be located within two standard deviations of the mean, and over 99% of the data will be located within three standard deviations of the mean. The graph of the normal distribution is bell shaped.

DURING THE TASK

Because this is a practice performance task, you could choose to have students engage in the task differently than they would in a conventional assessment. Here are some possible implementation strategies:

- Students could work in pairs to complete the task. It is not recommended that students work in small groups. There is ample work and enough potential discussion areas for two students, but in groups of more than two, some students may not have an opportunity to meaningfully engage in the task.
- You could chunk the task into its four parts and have students complete one part at a time. Students could check their solutions to each part with you or the scoring guidelines before moving on to the next part. During the check, spend a few moments discussing the solution with students. Focus on what changes, if any, they could make to their solution to craft a more complete response the next time they engage in a performance task.
- Have students complete the task individually. Then distribute the scoring guidelines to students to have them score their own responses or those of a classmate. Finally, have students reflect on their solutions and the scoring guidelines and make recommendations to themselves about what they could do to craft a more complete response the next time they engage in a performance task.

AFTER THE TASK

Whether you decide to have students score their own solutions, have students score their classmates' solutions, or score the solutions yourself, the results of the practice performance task should be used to inform instruction.

Students should understand that converting their score into a percentage does not provide a useful measure of how they performed on the task. You can use the suggested scoring conversion guide that follows the scoring guidelines to discuss their performance.

UNIT 1

SCORING GUIDELINES

There are 12 possible points for this performance task.

Student Stimulus and Part (a)

(a) Describe the distribution of the number of items purchased per customer. Be sure to indicate the center, shape, and variability of the distribution.

Sample Solutions	Points Possible
Description of center: The median number of items purchased is 32. Since the distribution appears to be skewed right, we would expect the mean to be greater than the median. Description of shape: The distribution of the number of items purchased per customer appears to be skewed right because the right side of the boxplot is longer than the left side of the boxplot. Description of variability: The interquartile range is the difference between the upper and lower quartile values (also called quartile 3 and quartile 1), which is $35 - 22 = 13$, and the range is the difference between the maximum and minimum values, which is $62 - 17 = 45$.	**3 points maximum** 1 point for identifying the median OR for explaining that the shape suggests the mean is greater than the median 1 point for describing the shape of the distribution 1 point for identifying the interquartile range OR for identifying the range as a measure of spread *Scoring note*: A student may simply state the median to receive the point for the description of center. It is not necessary to state the relationship between the mean and the median. If students use an incorrect definition of the range, such as "the range is from 17 to 62," the student should not receive the point for describing the variability.

Targeted Feedback for Student Responses
If several students do not describe all three components of the distribution in part (a), it could mean students need help describing a distribution in terms of its center, shape, and/or variability of a distribution presented as a boxplot.

TEACHER NOTES AND REFLECTIONS

Student Stimulus and Part (b)

(b) The manager suspected that the distribution of the number of items purchased per customer was approximately normally distributed. Based on your description of the distribution, do you agree with the manager? Explain your reasoning.

Sample Solutions	Points Possible
I do not agree with the manager that the distribution is approximately normal. A normal distribution is symmetric with a mean that is equal to the median. Since this distribution appears to be skewed right, I would expect the mean to be greater than the median.	**3 points maximum** 1 point for describing a normal distribution as symmetric 1 point for correctly describing the relationship between the mean and median in a skewed or symmetric distribution 1 point for concluding that the distribution does not appear to be approximately normal based on the skewness and the fact that the mean appears to be greater than the median *Scoring note*: Students should receive full credit if they have already stated the relationship between the mean and median in part (a).

Targeted Feedback for Student Responses
If several students make mistakes in part (b), it could mean students need help connecting the symmetry or skew of a distribution to an understanding of the normal distribution.

TEACHER NOTES AND REFLECTIONS

Student Stimulus and Part (c)

(c) The manager lets you review the receipts used to make the boxplot. The actual numbers of items purchased by the customers in the first 30 minutes the store was open were: 17, 18, 20, 21, 21, 22, 24, 25, 26, 27, 31, 33, 33, 34, 34, 35, 35, 37, 42, 42, 46, and 62. Calculate the following statistics:

Mean:

Median:

Standard deviation:

Proportion of customers that purchased items within 1 standard deviation of the mean:

Proportion of customers that purchased items within 2 standard deviations of the mean:

Proportion of customers that purchased items within 3 standard deviations of the mean:

Explain how the statistics you calculated support or contradict your answer from part (b).

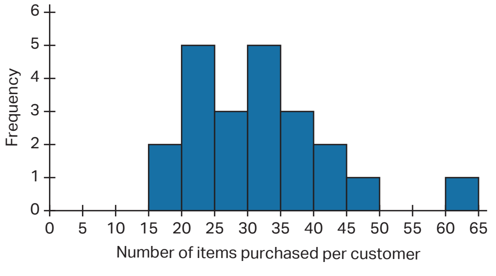
Sample Solutions	Points Possible
The distribution of the numbers of items purchased by customers has these statistics: ▪ mean ≈ 31.136 items. ▪ median = 32 items ▪ standard deviation ≈ 10.476 items The intervals around the standard deviation are: ▪ $\mu \pm \sigma$: from 20.660 to 41.612 items ▪ $\mu \pm 2\sigma$: from 10.184 to 52.088 items ▪ $\mu \pm 3\sigma$: from −0.292 to 62.564 items	**3 points maximum** 1 point for determining all three statistics: the mean, median, and standard deviation 1 point for determining all three of the intervals and calculating the proportions within those intervals 1 point for concluding that the mean and median are actually close to each other and that the proportions are what we would expect from a distribution that was approximately normal
There are 15 data points within one standard deviation of the mean, 21 data points within two standard deviations of the mean, and 22 data points with three standard deviations of the mean. The corresponding proportions are approximately 0.6818, 0.9545, and 1.000. These statistics contradict my conclusion from part (b). The mean and the median are approximately equal and the proportions of data within 1, 2, and 3 standard deviations are approximately 68%, 95%, and 99.7%, respectively, which is what we would predict in a normal distribution.	*Scoring note*: If students miscalculate the mean or standard deviation, but use their incorrect mean or standard deviation in the correct manner to determine intervals and the proportion of data in those intervals, the maximum score a student can receive in part (c) is 2.

Targeted Feedback for Student Responses
If several students make mistakes determining the summary statistics, it could mean students need practice calculating the mean, median, or standard deviation. If several students cannot conclude that the proportions of data in the interval are consistent with the empirical rule, then they may need additional practice with the normal distribution.

TEACHER NOTES AND REFLECTIONS

Student Stimulus and Part (d)

(d) On the grid provided, construct a histogram for the number of items purchased per customer. Use bin widths of 5, starting at 15. Based on your histogram and your answers to parts (a) through (c), what can be said about a distribution that follows the empirical rule? Explain your conclusion.

Sample Solutions	Points Possible
A histogram for the number of items purchased per customer is shown below.  Although the distribution is consistent with the empirical rule, the distribution is not approximately normal. It appears to be skewed right, with an extreme value on the right side of the histogram. We can conclude that some distributions have intervals around the mean that correspond to one, two, and three standard deviations that include 68%, 95%, and 99.7% of the data, but are not normally distributed.	**3 points maximum** 1 point for constructing the histogram 1 point for stating the histogram is skewed right 1 point for concluding that a distribution need not be approximately normal even if approximately 68% of the data falls within one standard deviation of the mean, approximately 95% of the data falls within two standard deviations of the mean, and approximately 99.7% of the data falls within 3 standard deviations of the mean *Scoring note*: If students do not mention an extreme value but do describe the distribution as skewed right, the student can still score 3 points in part (d).

Continues on next page.

Targeted Feedback for Student Responses

If several students make mistakes in part (d), they may not understand that having proportions that match the empirical rule proportions is not sufficient for determining whether or not the data is normally distributed. Students could benefit from closer comparisons of normal distributions and other distributions.

TEACHER NOTES AND REFLECTIONS

Points Received	Appropriate Letter Grade (If Graded)	How Students Should Interpret Their Score
11 or 12 points	A	"I know all of this data analysis really well."
8 to 10 points	B	"I know all of this data analysis well, but I made a few mistakes."
5 to 7 points	C	"I know some of this data analysis well, but not all of it."
2 to 4 points	D	"I only know a little bit of this data analysis."
0 or 1 points	F	"I don't know much of this data analysis at all."

UNIT 1

LESSON 1.7

Introduction to Probability

OVERVIEW

LESSON DESCRIPTION

Part 1: Exploring Carnival Games

In this part of the lesson, students are introduced to probability by watching a short video and then discussing its content as a class. The video shows a number of carnival games and uses probabilistic reasoning to explain how the designs of the games favor the carnival. Concepts discussed include probability as a fraction, random events, bias, long-term behavior, and expected value. The carnival scenario provides a good introduction to concrete applications of probability and how to use probability to make decisions.

Part 2: Analyzing a Dart Game

In this part of the lesson, students consider a dart game and use mathematical reasoning to determine if the game is favorable or unfavorable to the player by analyzing the probability of winning.

Part 3: Modeling an Experiment with a Simulation

Theoretical experiments like the one in Part 2 are important in developing students' understanding of probability. However, students should also gain experience in using simulations to answer probability questions. This part of the lesson introduces students to number cube rolling simulations.

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~135 minutes

LESSON SEQUENCE

This lesson is part of a lesson sequence (~405 minutes total) that includes Lessons 1.7 through 1.11.

HANDOUTS

Lesson

- 1.7.A: The Square Dart Board
- 1.7.B: The Board Game Problem

MATERIALS

- number cubes or access to a number cube rolling simulation
- internet access to the video "Carnival Scam Science" (11:36), available at **youtube.com/watch?v=tk_ZlWJ3qJI**

CONTENT FOCUS

This lesson begins students' exploration of probability in this course. For students with some experience with probability from their prior mathematics courses, this lesson provides a review of key terms and basic probability calculations. For students with limited prior exposure to probability, this lesson serves as an introduction to these important concepts and basic calculations. Through this lesson, students develop the understanding that probability does not predict the future, but allows us to make informed decisions about particular outcomes based on the long-term behavior of an event.

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ Probabilistic reasoning allows us to anticipate patterns in data.	
Learning Objectives	**Essential Knowledge**
1.2.2 Determine the probability of an event.	**1.2.2a** The sample space is the set of all outcomes of an experiment or random trial. An event is a subset of the sample space. **1.2.2b** Probabilities are numbers between 0 and 1 where 0 means there is no possibility that an event can occur, and 1 means the event is certain to occur. The probability of an event occurring can be described numerically as a ratio of the number of favorable outcomes to the number of total outcomes in a sample space. **1.2.2c** A probability distribution is a function that associates a probability with each possible value or interval of values for a random variable. The sum of the probabilities over all possible values of the independent variable must be 1.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

Carla designs a new two-player game. In this game, each player rolls a different six-sided number cube. Cube A has four faces labeled with a 5 and two faces labeled with a 1. Cube B has four faces labeled with a 3 and two faces labeled with a 7. For each number cube, each face is equally likely to land face up. The first player selects one number cube and rolls it, while the second player rolls the other number cube. The winner of the game is the player whose cube shows the higher number face up.

(a) Assume you are the first player and select Cube B. Describe the set of outcomes in which you win the game.

(b) Suppose that you play the game many times. If you always go first, and always select Cube B, determine your probability of winning the game.

PART 1: EXPLORING CARNIVAL GAMES

In this part of the lesson, students are introduced to probability by watching a short video and then discussing its content as a class. The video shows a number of carnival games, and uses probabilistic reasoning to explain how the designs of the games favor the carnival. Concepts discussed include probability as a fraction, random events, bias, long-term behavior, and expected value. The carnival scenario provides a good introduction to concrete applications of probability, and how to use probability to make decisions.

Instructional Rationale

This course moves away from the traditional "marble from a bag" probability scenarios that students likely experienced in middle school. Throughout this key concept, students are expected to use probabilistic reasoning to make decisions by calculating the likelihood of an event occurring.

- Begin the lesson by showing students an 11.5-minute video, narrated by former NASA engineer Mark Rober, as a way to introduce new probability concepts: **youtube.com/watch?v=tk_ZlWJ3qJI**.
- After watching the video, take a few moments to check students' understanding of probability using questions similar to these:
    - What were the three categories of games that were discussed?

      The three categories of games were random chance games, skill-based games, and near-impossible games.
    - What is an example of a random-chance game, and how do you calculate your probability of winning? What makes random-chance games harder to win than the calculated probability of winning?

      The random chance games shown in the video were various ball-toss games. The probability of winning is calculated by dividing the number of possible winning outcomes by the number of total outcomes. Carnivals often use balls that bounce a lot to make these games more difficult to win.

Classroom Ideas
You could have students watch this video at home or a day before the start of the unit to save classroom time, depending on availability of technology.

Meeting Learners' Needs
Some students will benefit from using the note-taking guide provided on **Handout 1.6.A: A Note-Taking Guide for the Video**.

- What are some examples from the video of carnival games with a lower probability of winning than a player might assume? Explain how the carnival has created this inflated confidence for players.

  The games shown in the video that have a lower probability of winning than a player might assume are the basketball toss, the table tennis ball toss, the pitching game, and the milk bottles game. The height of the basketball hoop is slightly different than the standard height players are familiar with, which makes it harder to win. Participants might throw too hard, and therefore less accurately. The table for the table tennis ball toss is slightly slanted, which affects the horizontal speed of the ball. The pitching game does not accurately measure the speed of the ball. The milk bottles are metal and therefore heavier than similarly sized bottles.

- In the video, Mark Rober says that the "odds are heavily stacked against you." In your own words, explain what this means to you.

  Carnival games are designed so that participants overestimate their chances of winning. It is much more likely for a person to lose than to win.

- Now that students have been casually introduced to probability, ask students what words they think of when they hear the term *probability* and collect the answers in a central location accessible by everyone. You will not formally define probability right now, but it is valuable to brainstorm at this point so students have some language to use during their academic conversations in Part 2 of the lesson.

PART 2: ANALYZING A DART GAME

In this part of the lesson, students consider a dart game and use mathematical reasoning to determine if the game is favorable or unfavorable to the player by analyzing the probability of winning.

- Provide students with **Handout 1.7.A: The Square Dartboard**. Have students read the handout to themselves, or select a student volunteer to read it aloud. Students can work in small groups of two or three to answer the question.
- Before letting students work on the task, check to make sure they understand the game components. For example, some students may wonder why the dart thrower is blindfolded. Being blindfolded means that there is no skill advantage for the thrower, because they cannot see the dartboard and aim for a particular region.

Classroom Ideas
To get students to understand the dart game and its mechanics you could project the square dartboard and have students try to throw objects like wadded-up paper or bean bags at it.

- As you circulate around the room, listen for the words that students use while working on the task, such as *chance* and *random*. You should ask students to explain what they mean as a way to check their understanding of the words. If a student's definition is not correct, take a moment to address it. If a student's definition is correct, you can ask that student to define the term later in the class. Several terms that students are likely to use are given in the table below.

Term	Definition and Notes
Probability	Probability is the chance, or likelihood, of an event occurring. Probability is calculated as a ratio of the number of favorable outcomes and the number of total outcomes, where all outcomes are equally likely. Therefore, probability is always a number between 0 and 1. Probability is sometimes reported as a percent between 0% and 100%. For example, the probability of drawing a heart card from a standard deck of cards is $\frac{13}{52}$, which can be reduced to $\frac{1}{4}$.
Chance, likelihood	These two words are often used synonymously and refer to the probability of an event. Sometimes, the word *chance* is used to refer to the odds of an event, not the probability of an event.
Odds	Odds are a ratio of the number of ways an event can occur to the number of ways an event cannot occur. Odds are usually written with a colon, instead of a fraction bar. For example, the odds of drawing a heart card from a standard deck of cards is 13 : 39, which can be reduced to 1 : 3.
Trial	A trial refers to a procedure that can be repeated and is usually reserved for experimentation or simulation, in which the experimenter adjusts a factor in anticipation of some result.
Outcome	The result of any trial is an outcome. For example, when flipping a coin, the trial is flipping the coin and the outcome is getting heads (or tails).
Event	An event is a collection of outcomes. For example, the event of rolling an odd number with a standard number cube has the outcomes {1, 3, 5}.
Sample space	The sample space is the set of all possible outcomes. For example, the sample space of rolling a standard number cube is {1, 2, 3, 4, 5, 6}.
Random	Random means that there is no bias. The next outcome of a random event is not known, but over repeated trials, the proportions of specific outcomes could be predictable.
Fair	A fair game is one in which you expect to win as often as you expect to lose.

UNIT 1

- Once students have been working on the problem for about four minutes, you may want to do a quick check for understanding of the problem's two main ideas using questions like the ones below:
  - Let's assume that every dart throw (a trial) is random. Is there an equal chance of hitting any point on the target? Why or why not?

    Hitting any point on the board is equally likely. There is an equal chance of hitting every point on the board, if we ignore the color of the region.
  - Is the probability of hitting a yellow region equal to the probability of hitting a blue region? Why or why not?

    The probability of a dart hitting a yellow region is not equal to the probability of a dart hitting a blue region, because the yellow regions only account for $\frac{1}{4}$ of the area of the dart board while the blue regions account for $\frac{3}{4}$ of the area of the dart board.

Meeting Learners' Needs

One way to help students succeed in academic conversations is to provide sentence stems they can complete, such as "I think that ... because ..."; "In addition ..."; and "I don't understand why ..."

Guiding Student Thinking

You may have to clarify the difference between theoretical and experimental probability. Theoretical probability is calculated based on reasoning about an event without performing any trials. Experimental probability is calculated from the results of an experiment. For example, the theoretical probability of getting heads in a coin toss is $\frac{1}{2}$, but experimental probability after a series of 10 trials could be $\frac{7}{10}$.

- After students have had about 10 more minutes to discuss the problem, bring the class back together and allow student groups to present their solutions.
- Highlight strategies for determining if the game is favorable to the player or unfavorable to the player. One thing that students should state in their responses is that by using the proportions of the yellow and blue areas, we can estimate the probability of losing or winning over many, many repeated trials. Such estimates are useful for making decisions when the outcomes are uncertain. Some sample solutions for the task are given in the Assess and Reflect section of the lesson.
- At this point in the lesson, you can introduce the notation "*P*(event)" to students. Some students may already be using this notation, depending on their previous experience with probability. Explain that the notation "*P*(event)" means "the probability of a specified event occurring." In the dart board problem, we might use

the notation "*P*(blue)" to mean "the probability that a dart thrown at the board hits a blue region."

PART 3: MODELING AN EXPERIMENT WITH A SIMULATION

Theoretical experiments like the one in Part 2 are important in developing students' understanding of probability. However, students should also gain experience in using simulations to answer probability questions. This part of the lesson introduces students to a number cube rolling simulation. If you used a coin flip simulation in Lesson 1.6, students may see similarities between the two scenarios.

- If you have not already clarified the difference between theoretical and experimental probability, this is an appropriate time to address it. Let students know that the dart board task involved theoretical probabilities because they did not perform any experiments to determine whether or not the game was favorable to the player. One hallmark of a theoretical probability problem is that the problem is often solved by applying reasoning to information given in the problem. By contrast, an experimental probability problem is often solved by performing an experiment or simulation and applying reasoning to the results to find a solution.
- Provide students with **Handout 1.7.B: The Board Game Problem**. Students could work in pairs or small groups of three to complete the task.
- As you circulate around the room, listen to students' ideas for an experiment. They will likely recognize that they should perform several rolls of the number cubes to get some insight into how often the sum of the numbers shown on the tops of the cubes is 5.
- Once students have an idea for how to perform the experiment, you can distribute number cubes to the small groups. Each person in the group could roll their own pair of number cubes multiple times so that the group can relatively quickly accumulate data for a large number of trials.

Guiding Student Thinking

You might have to remind students that probability is about the behavior of an event over a large number of trials. This is important because they may not see patterns emerge in the sum of the number cube rolls until they have data from several hundred trials.

- Allow students a few minutes to roll their number cubes and record the results as many times as they can. Then, as students start to experience some frustration about the number of rolls they would need to perform, you can introduce a roll simulator as an alternative approach.

- There are several good number-cube roll simulators available online. You could allow students to try different ones to compare their advantages and disadvantages.
- Ask students to use their simulator to perform 10 trials, 100 trials, 1,000 trials, and 10,000 trials. For each set of trials, have students determine the experimental probability of rolling two numbers whose sum is 5.

Number of Trials	Experimental Probability of Rolling a Sum of 5
10	
100	
1,000	
10,000	

- Finally, have students return to the board game task to answer the problem. Students must use the information from their experiments to conclude whether or not they are likely to lose the game on the next turn. Give students two or three minutes to form a conclusion about the task.
- Provide time for several groups to share their conclusions with the class. Several groups will likely conclude that they are not likely to lose the game on the next turn. This debrief is an opportunity to engage students in an academic conversation, especially if some groups can present evidence and argue for different conclusions. The conclusion itself is not as important as using the available probability evidence to reach a reasoned conclusion.
- As a follow-up, display a graph of 10,000 trials and the totals of each possible sum, which is shown on the next page. Then ask students to analyze the graph:
    - How is the shape of the graph familiar?

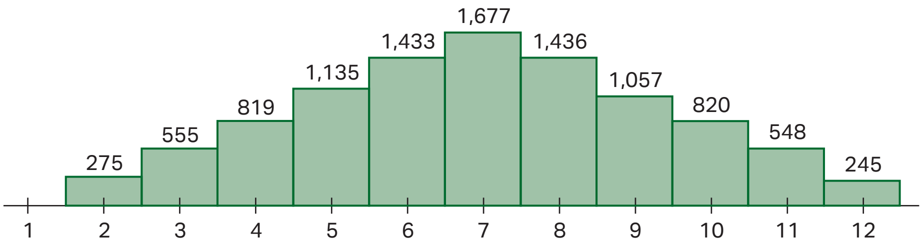
      The graph looks somewhat like a normal distribution.
    - Does it make sense to you that the graph has this shape?

      Because there are more ways to roll a sum of 7 than any of the other sums, it makes sense that there is a peak in the center of the graph.

- What do you wonder about the graph?

  I wonder if the data can be modeled by a normal distribution. The data towards the tails of the graph do not appear to taper off as quickly as in a normal curve.

- Ask students if the shape of the frequency table looks familiar as the number of simulated rolls increases. By examining the following distribution, which represents the sums of two standard number cubes over 10,000 rolls, students might notice that the bar graph has a shape similar to a normal distribution.



The sums of two number cubes over 10,000 trials

- To conclude this lesson, you can have students create entries of important probability terms for a word wall (if you have one) using a graphic organizer such as **Handout 3.1.C: Vocabulary Graphic Organizer**. Alternatively, you could have students offer definitions for important probability terms and have the class reach a consensus about the best definition for each term. You may have to correct some misconceptions about the definitions of the terms.

Meeting Learners' Needs

If you have students who would benefit from a challenge, you can ask them to determine if a normal curve would be a good model for the data and to present a case for or against using a normal curve for this scenario.

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

Carla designs a new two-player game. In this game, each player rolls a different six-sided number cube. Cube A has four faces labeled with a 5 and two faces labeled with a 1. Cube B has four faces labeled with a 3 and two faces labeled with a 7. For each number cube, each face is equally likely to land face up. The first player selects one number cube and rolls it, while the second player rolls the other number cube. The winner of the game is the player whose cube shows the higher number face up.

(a) Assume you are the first player and select Cube B. Describe the set of outcomes in which you win the game.

The player who rolls Cube B can win two different ways. The first way is if they roll a 7, since the other player can only roll a 5 or 1, which are both lower than 7. They can also win if they roll a 3 and the second player rolls a 1. Stated another way, a player who rolls Cube B will only lose if they roll a 3 and the other player rolls a 5.

Outcomes	Cube A: 1	Cube A: 5
Cube B: 3	(1, 3); B wins	(5, 3); A wins
Cube B: 7	(1, 7); B wins	(5, 7); B wins

(b) Suppose that you play the game many times. If you always go first, and always select Cube B, determine your probability of winning the game.

The player rolling Cube A wins when Cube A shows a 5 and Cube B shows a 3. The probability of rolling a 5 on Cube A is $\frac{4}{6}$ and the probability of rolling a 3 on Cube B is $\frac{4}{6}$. That means there are 16 possible winning outcomes out of 36 total outcomes for the player rolling Cube A. The other 20 possible outcomes result in the player rolling Cube B winning. A table of the sample space is shown on the next page. That means that there is a probability of $\frac{20}{36}$, or $\frac{5}{9}$, that the player rolling Cube B will win.

UNIT 1

Sample Space	A: 1	A: 1	A: 5	A: 5	A: 5	A: 5
B: 3	(1, 3) B wins	(1, 3) B wins	(5, 3) A wins	(5, 3) A wins	(5, 3) A wins	(5, 3) A wins
B: 3	(1, 3) B wins	(1, 3) B wins	(5, 3) A wins	(5, 3) A wins	(5, 3) A wins	(5, 3) A wins
B: 3	(1, 3) B wins	(1, 3) B wins	(5, 3) A wins	(5, 3) A wins	(5, 3) A wins	(5, 3) A wins
B: 3	(1, 3) B wins	(1, 3) B wins	(5, 3) A wins	(5, 3) A wins	(5, 3) A wins	(5, 3) A wins
B: 7	(1, 7) B wins	(1, 7) B wins	(5, 7) B wins	(5, 7) B wins	(5, 7) B wins	(5, 7) B wins
B: 7	(1, 7) B wins	(1, 7) B wins	(5, 7) B wins	(5, 7) B wins	(5, 7) B wins	(5, 7) B wins

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 1.7.A: The Square Dart Board

Sample response 1: If you assume your darts are going to hit the target at random, the probability of wining is $\frac{1}{4}$, and the probability of losing is $\frac{3}{4}$. So, if you threw 100 darts, you would win, on average, 25 times, receiving \$100. Because you have spent \$50 to play, you have a net gain of \$50 from those wins. However, you would lose on average 75 times, for a net loss of \$150. That means if you threw 100 darts at a cost of \$200, you should expect to win about \$50 and lose about \$150 for a total loss of \$100. Stated another way: you should expect to lose \$1 for every \$2 you spend. Therefore, the game is not favorable to the player.

Sample response 2: Make a table of values for different numbers of darts thrown. In this approach, the number of darts thrown is a multiple of 4 to make the calculations easier.

Number of Darts Thrown	Average Number of Wins	Total Amount Won (# of wins) ($2)	Average Number of Losses	Total Amount Lost (# of losses) (–$2)	Net Gain
4	1	+2	3	–6	–4
12	3	+6	9	–18	–12
40	10	+20	30	–60	–40
100	25	+50	75	–150	–100
400	100	+200	300	–600	–400
1,000	250	+500	750	–1,500	–1,000
4,000	1,000	+2,000	3,000	–6,000	–4,000
n	$\frac{1}{4}n$	$\left(\frac{1}{4}n\right)(2)$ $\frac{1}{2}n$	$\frac{3}{4}n$	$\left(\frac{3}{4}n\right)(-2)$ $-\frac{3}{2}n$	$-n$

In general, you should expect to lose $1 per dart throw. The game is not favorable to the player.

Guiding Student Thinking

Some students may be confused about why the amount won is $2 and not $4. This is because the amount you receive, $4, does not account for the amount you spent to play the game, $2. Therefore, the amount won is $4 – $2 = $2.

Handout 1.7.B: The Board Game Problem

As students perform more trials in their simulation, the experimental probability of rolling a 5 should approach the theoretical probability of rolling a 5.

Sample solution: Because we rolled a sum of 5 exactly 3 times in 30 trials, the probability would be about 10%. Because our trials show that a sum of 5 would be expected about only once every nine rolls of the number cubes, it is possible, but not likely, to lose on the next turn.

Note that the theoretical probability of rolling a 5 is $\frac{4}{36}$, or approximately 0.111. A table of the sample space is shown here:

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

UNIT 1

LESSON 1.8

Venn Diagrams

OVERVIEW

LESSON DESCRIPTION

Part 1: Organizing Categorical Data with Venn Diagrams

In this part of the lesson, students sort through categorical data and answer questions about the quantities of data in several categories. The primary goal of this activity is for students to recognize the utility of Venn diagrams for organizing data values that fit into one or more different categories.

Part 2: Using Venn Diagrams to Solve Probability Problems

In this part of the lesson, students explore more complex Venn diagrams and use them to answer probability questions. The main goals of this part of the lesson are to introduce the concept of conditional probability and to formally define the intersection and union of two sets.

CONTENT FOCUS

This lesson has several major goals. The first is to have students explore Venn diagrams as useful displays for organizing categorical data that also allow probabilities associated with the data to be determined. Some students may have some familiarity with Venn diagrams from previous courses, but others may not have used them before, either for organizing information or for calculating probabilities.

The second goal is to formalize the concepts and notation for union and intersection. This lesson leverages what students already know about "and" and "or" statements to help them make sense of union and

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~45 minutes

LESSON SEQUENCE

This lesson is part of a lesson sequence (~405 minutes total) that includes Lessons 1.7 through 1.11.

HANDOUTS

Lesson

- 1.8.A: Organizing Categorical Data
- 1.8.B: Calculating Probabilities with Venn Diagrams

Practice

- 1.8.C: Practice Using Venn Diagrams to Determine Probabilities

MATERIALS

- calculator

intersection concepts. It is not necessary for students to learn formulas for determining probabilities involving unions and intersections. It is more valuable for students to develop reasoning skills related to probabilities with the help of a Venn diagram.

The final goal of the lesson is to introduce students to conditional probability. This essential concept will recur throughout other lessons in Key Concept 1.2, so students are not expected to master it through this lesson alone. However, by the end of the lesson, students should understand that in problems involving conditional probability, the sample space of an event is constrained in some way. As with unions and intersections, it is more valuable for students to develop and use reasoning skills to answer a conditional probability problem than to memorize formulas associated with probability.

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ Probabilistic reasoning allows us to anticipate patterns in data.	
Learning Objectives	**Essential Knowledge**
1.2.1 Create or analyze a data display for a categorical data set.	**1.2.1a** Venn diagrams and contingency tables are common displays of categorical data and are useful for answering questions about probability. **1.2.1b** The intersection of two categories is the set of elements common to both categories. **1.2.1c** The union of two categories is the set of elements found by combining all elements of both categories.
1.2.3 Calculate relative frequencies, joint frequencies, marginal frequencies, or conditional probabilities for a categorical data set.	**1.2.3b** Joint frequencies are events that co-occur for two or more variables. They are the frequencies displayed in cells in a two-way contingency table. **1.2.3d** The conditional probability of *B*, given *A* has already occurred, is the proportion of times *B* occurs when restricted to events only in *A*.

UNIT 1

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

A survey of a random sample of students was conducted by a school district to determine if new menu items should be offered in the cafeteria to accommodate students' dietary restrictions. One survey question asked if a student is a vegetarian, while another question asked if a student has a tree nut allergy. Of the 50 students surveyed, 8 stated they are vegetarians, 3 said they have tree nut allergies, and 1 said they are both vegetarians and have tree nut allergies.

(a) Create a Venn diagram to display the results of the survey.

(b) From the students surveyed, what is the probability that a randomly selected student is a vegetarian and does not have a tree nut allergy?

(c) From the students surveyed, what is the probability that a randomly selected student is a vegetarian or does not have a tree nut allergy?

(d) From the students surveyed, what is the probability that a randomly selected student is a vegetarian given that the student does not have a tree nut allergy?

PART 1: ORGANIZING CATEGORICAL DATA WITH VENN DIAGRAMS

In this part of the lesson, students sort through categorical data and answer questions about the quantities of data in several categories. The primary goal of this activity is for students to recognize the utility of Venn diagrams for organizing data values that fit into one or more different categories.

- Begin the lesson by presenting the scenario below to students, which is also on **Handout 1.8.A: Organizing Categorical Data**. Arrange students into pairs and give them about 10 minutes to work toward solutions for the questions.

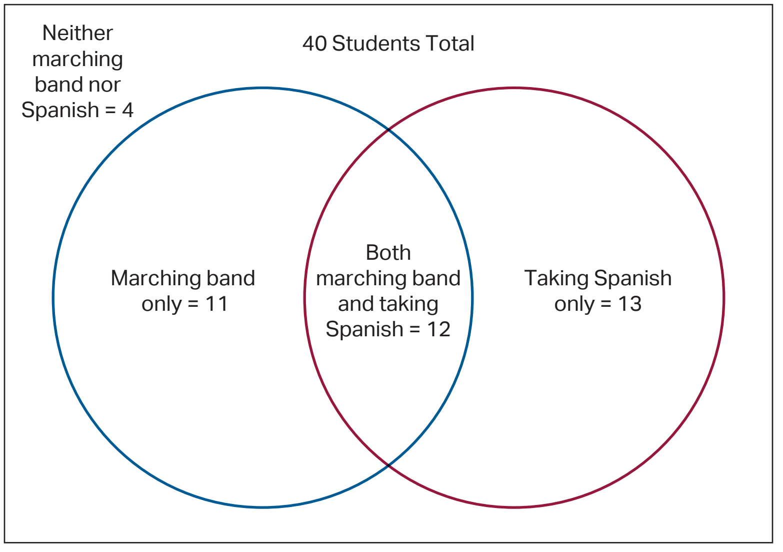
> Ms. Lucas conducted a survey of 40 randomly chosen students she teaches. The survey results showed that 23 are in the marching band and 25 are taking Spanish class. There are 12 students both in the marching band and taking Spanish class.

Instructional Rationale

It is important for students to recognize the need for a data organizer, like a Venn diagram, before suggesting one to them. That is, students will only appreciate its usefulness if they experience difficulty in solving a problem without an organizer. Therefore, refrain from providing a Venn diagram template to students at the beginning of the problem.

- As you circulate around the room, encourage students to think about different ways to organize the information in the problem. Some students may use a Venn diagram if they have had previous experience with it. Other students may organize their data into a table. Make note of the different organizing structures that students use so you can ask them to share their strategies with the class.
- After about 10 minutes, bring the class back together. Select several student pairs to share their answers and visual data displays. If any group used a Venn diagram to organize the information, have them share that strategy. Solutions for the handout questions are given in the Assess and Reflect section of this lesson.
- Two important terms to identify and formally define in a mathematical sense are *and* and *or*. Mathematically, when the conjunction *and* joins two categories, a new category is formed, composed of elements common to both categories. When the conjunction *or* joins two categories, a new category is formed that combines all elements of both categories.

- If no students use a Venn diagram to organize their data, this is a good opportunity to show them a basic two-set Venn diagram. Be sure to discuss with students how to use the information in the prompt to determine how many students fall into each category, including students that fall into neither category.



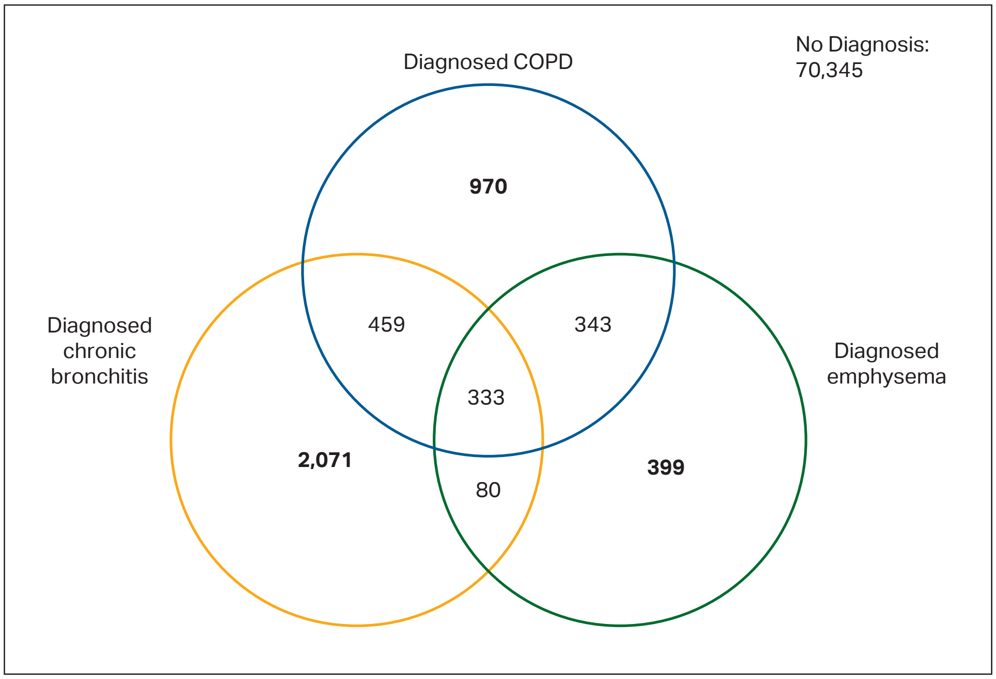
Guiding Student Thinking

At this point, students only need to be able to reason through finding the number of students in each category. It is much more important that they know what the categories represent and have some techniques for determining the values in each category. It is not necessary to introduce formulas at this point, as they may be unnecessarily confusing for many students.

PART 2: USING VENN DIAGRAMS TO SOLVE PROBABILITY PROBLEMS

In this part of the lesson, students explore more complex Venn diagrams and use them to answer probability questions. The main goals of this part of the lesson are to introduce the concept of conditional probability and to formally define the intersection and union of two sets.

- Now that students have seen a two-set Venn diagram, they are ready to explore a three-set Venn diagram. Display the Venn diagram from **Handout 1.8.B: Calculating Probabilities with Venn Diagrams** for students. The data in the diagram is taken from a recent study on the characteristics of chronic obstructive pulmonary disease (COPD). The researchers were attempting to differentiate among the treatments and outcomes of COPD, chronic bronchitis, and emphysema. The data was gathered from a 2010 US National Health and Wellness Survey of 75,000 Americans.



- Allow students a minute to make some general observations and ask questions about the Venn diagram. They may notice that there are three sets instead of two, or that the number of people with no diagnosis of any of the three lung diseases is very high.
- All the questions that students should answer about this situation appear on Handout 1.8.B. An appropriate technique for this activity would be to chunk the questions into groups of two. Every two questions on the handout are designed to be answered together, as the answer for each odd-numbered question informs the following even-numbered question. Having students work on two questions at a time allows you to make sure that students understand the important concepts of the lesson as they are revealed through the questions.

Meeting Learners' Needs

If students are having trouble calculating probability on the handout problems, you may have to remind them that the probability of an event is the ratio of the number of favorable outcomes to the number of total outcomes in the sample space. For example, when flipping two coins, the probability of getting exactly one tail would be $\frac{2}{4}$, because in the sample space, {HH, HT, TT, TH}, there are two favorable outcomes, {HT, TH}, out of the four total outcomes.

- The goals of problems 1 and 2 are to orient students to the intersection of all three sets and to have students begin to think about using Venn diagrams to calculate probability. While these problems use the concept of intersection and joint probability, you can introduce the mathematical notation and terminology after debriefing problems 3 and 4. You may have to remind students that probability is a number between 0 and 1, inclusive. The preferred ways to report probability for this course are as fractions and decimals.
  - Where in the diagram should we look to determine the number of people diagnosed with all three lung diseases?

    The overlap (or intersection) of the three sets is where to find the number of people diagnosed with all three lung diseases. This is the region bounded by three arcs in the center of the Venn diagram.
  - How many total people were there in the entire survey (not just those people with lung disease)?

    There are 75,000 people represented in the survey.
  - What is the probability that a randomly selected person from the survey was diagnosed with all three lung diseases?

    There are 333 people who were diagnosed with all three lung diseases. There were 75,000 people represented in the survey. The probability that a randomly selected person from the survey was diagnosed with all three lung diseases is $\frac{333}{75,000}$, or approximately 0.004.
- The goal of problems 3 and 4 is to have students consider the intersection of two sets in a three-set Venn diagram and to use that information to calculate probability.
- When debriefing these problems, introduce the symbol "$\cap$" to indicate the intersection of sets and the term *joint probability*. You should connect "$\cap$" notation to the mathematical definition of the word *and* that students learned in the first part of the lesson. The intersection of two sets is the collection of elements common to both sets. Joint probability refers to the probability that two or more events occur simultaneously.
  - Where in the diagram should we look to determine the intersection of people diagnosed with both COPD and emphysema?

    The intersection of the two sets is the pointed oval shaped region on the right side of the Venn diagram formed by the overlap of the blue and green circles.

UNIT 1

- What is the joint probability that a person selected randomly from the people in the survey was diagnosed with both COPD and emphysema?

  There are 676 people who were diagnosed with both COPD and emphysema. There were 75,000 people represented in the survey. The joint probability that a randomly selected person from the survey was diagnosed with both COPD and emphysema is $\frac{676}{75{,}000}$, or approximately 0.009.

- How could we express the answer to problem 4 using the mathematical notation for intersection?

  We could express the answer to problem 4 as $P(\text{COPD} \cap \text{emphysema}) = \frac{676}{75{,}000} \approx 0.009$.

- How could we express the answer to problem 2 using the notation for intersection?

  We could express the answer to problem 2 as $P(\text{COPD} \cap \text{emphysema} \cap \text{chronic bronchitis}) = \frac{333}{75{,}000} \approx 0.004$.

▪ The goal of problems 5 and 6 is to have students consider the union of two sets in a three-set Venn diagrams and to use that information to calculate probability.

▪ When debriefing these problems, introduce the symbol "$\cup$" to indicate the union of sets. You should connect "$\cup$" notation to the mathematical definition of the word *or* that students learned in the first part of the lesson. The union of two sets is formed by combining elements of both sets.

- Where in the diagram should we look to determine the union of people diagnosed with chronic bronchitis or emphysema?

  The union of the two sets is all people contained in the yellow circle and in the green circle, including their overlapping region, at the bottom of the diagram.

- What is the probability that a person selected randomly from the people surveyed was diagnosed with either chronic bronchitis or emphysema?

  There are 3,685 people who were diagnosed with either chronic bronchitis or emphysema. There were 75,000 people represented in the survey. The probability that a randomly selected person from the survey was diagnosed with either chronic bronchitis or emphysema is $\frac{3{,}685}{75{,}000}$, or approximately 0.049.

- How could we express the answer to problem 6 using mathematical notation?

  We could express the answer to problem 6 as $P(\text{chronic bronchitis} \cup \text{emphysema}) = \frac{3{,}685}{75{,}000} \approx 0.049$.

- The goal of problems 7 and 8 is to introduce students to conditional probability.
- When debriefing these problems, introduce the term *conditional probability* to refer to the probability of an event occurring given that another event has occurred. This has the effect of changing the sample space and therefore the total number of outcomes. Introduce the notation "$P(B \mid A)$" to mean "the probability of event B occurring given that A has occurred."
  - Where in the diagram should we look to determine the number people diagnosed with COPD?

    The number of people diagnosed with COPD corresponds to all regions enclosed by the blue circle.

  - What is the probability that a randomly selected person from the survey was diagnosed with chronic bronchitis, given that the person was already diagnosed with COPD?

    There are 2,105 people who were diagnosed with COPD. Of that number, 792 people were also diagnosed with chronic bronchitis. The probability that a randomly selected person from the survey was diagnosed with chronic bronchitis given that they were diagnosed with COPD is $\frac{792}{2{,}105}$, or approximately 0.376.
  - How could we express the answer to problem 8 using mathematical notation?

    We could express the answer to problem 8 as $P(\text{chronic bronchitis} \mid \text{COPD}) = \frac{792}{2{,}105} \approx 0.376$.
- The goal of problems 9 and 10 is for students to analyze whether or not the order of the events in a conditional probability problem matters. When debriefing these problems, reinforce that the notation "$P(B \mid A)$" means "the probability of event B occurring given that A has occurred."
  - Where in the diagram should we look to determine the number people diagnosed with chronic bronchitis?

    The number of people diagnosed with chronic bronchitis corresponds to all regions enclosed by the yellow circle.

  - What is the probability that a randomly selected person from the survey was diagnosed with COPD, given that the person was already diagnosed with chronic bronchitis?

    There are 2,943 people who were diagnosed with chronic bronchitis. Of that number, 792 people were also diagnosed with COPD. The probability that a randomly selected person from the survey was diagnosed with chronic bronchitis given that they were diagnosed with COPD is $\frac{792}{2{,}943}$, or approximately 0.269.

- How could we express the answer to problem 10 using mathematical notation?

  We could express the answer to problem 10 as $P(\text{COPD} \mid \text{chronic bronchitis}) = \frac{792}{2,943} \approx 0.269$.

- As a quick formative assessment at the conclusion of the handout discussion, you can pose these two questions to students, which require them to use many of the concepts they explored through the handout:
  - Given that a person was diagnosed with at least one of these three lung diseases, what is the probability that a randomly selected person from the survey was diagnosed with COPD? How would you express the problem using mathematical notation?

    There are 4,655 people diagnosed with at least one lung disease and 2,105 people diagnosed with COPD. The probability that a randomly selected person was diagnosed with COPD given that they were diagnosed with at least one lung disease is $\frac{2,105}{4,655}$, or approximately 0.452. This can be expressed in mathematical notation as $P(\text{COPD} \mid \text{at least one lung disease}) = \frac{2,105}{4,655} \approx 0.452$.
  - Given that a person was diagnosed with at least one of these three lung diseases, what is the probability that they were diagnosed with all three lung diseases?

    There are 4,655 people diagnosed with at least one lung disease and 333 people diagnosed with all three lung diseases. The probability that a randomly selected person was diagnosed with all three lung diseases given that they were diagnosed with at least one lung disease is $\frac{333}{4,655}$, or 0.072. This can be expressed in mathematical notation as $P(\text{all lung diseases} \mid \text{at least one lung disease}) = \frac{333}{4,655} \approx 0.072$.
- At this point in the lesson, it is appropriate to provide students with some time to practice using Venn diagrams to answer probability questions. **Handout 1.8.C: Practice Using Venn Diagrams to Determine Probabilities** has two new scenarios for students to explore. For problem 1, students need to create a Venn diagram and then answer questions about the scenario. For problem 2, students are given a Venn diagram representing the popularity of smoking cessation programs and asked to use it to determine probabilities and make decisions about them. You can point out to students that the Venn diagram in problem 2 is a display called a proportional Venn diagram, which means that the sizes of the circles in the diagram are proportional to the population represented in the diagram.

UNIT 1

ASSESS AND REFLECT ON THE LESSON

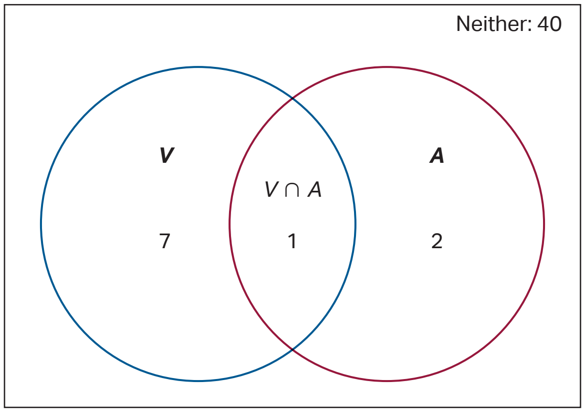
FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

A survey of a random sample of students was conducted by a school district to determine if new menu items should be offered in the cafeteria to accommodate students' dietary restrictions. One survey question asked if a student is a vegetarian, while another question asked if a student has a tree nut allergy. Of the 50 students surveyed, 8 stated they are vegetarians, 3 said they have tree nut allergies, and 1 said they are both vegetarians and have tree nut allergies.

(a) Create a Venn diagram to display the results of the survey.

Let V be the event that a student responded that they are a vegetarian and A be the event that a student responded that they have a tree nut allergy. The Venn diagram that represents the results of the survey is shown below.



(b) From the students surveyed, what is the probability that a randomly selected student is a vegetarian and does not have a tree nut allergy?

The randomly selected students would have to be in the V circle but not in the A circle. There are 7 such students, so the probability is $\frac{7}{50} = 0.14$.

(c) From the students surveyed, what is the probability that a randomly selected student is a vegetarian or does not have a tree nut allergy?

The randomly selected student would have to be in the V circle or not in the A circle. There are $7 + 40 = 47$ such students, so the probability is $\frac{47}{50} = 0.94$.

(d) From the students surveyed, what is the probability that a randomly selected student is a vegetarian given that the student does not have a tree nut allergy?

There are 50 – 3 = 47 students who do not have a tree nut allergy. Of the students who do not have tree nut allergies, there are 7 students who are vegetarians. Therefore, the probability is $\frac{7}{47} \approx 0.149$.

Guiding Student Thinking

Some students may struggle with the formative assessment item because of its use of the negation of the tree nut allergy category. These students can benefit from talking through the problem to identify the categories that would be included for students who do not have a tree nut allergy. They may find it helpful to shade the regions that correspond to the categories identified in the problem.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 1.8.A: Organizing Categorical Data

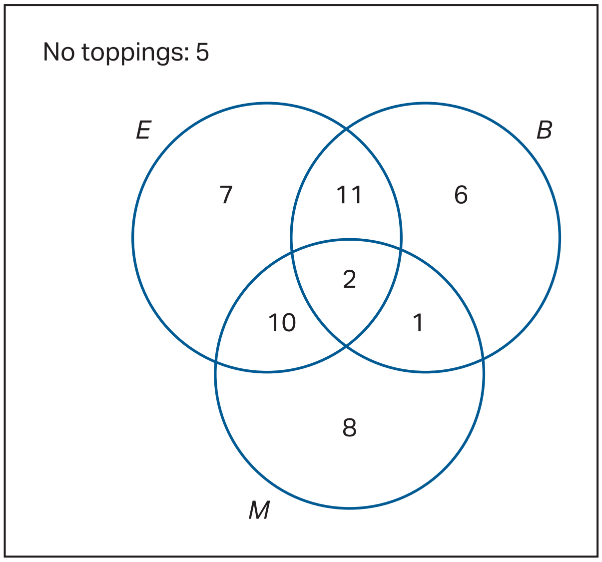
1. There are 23 – 12 = 11 students in the marching band and not taking Spanish.
2. There are 25 – 12 = 13 students taking Spanish and not in the marching band.
3. There are 40 – 11 – 12 – 13 = 4 students neither in the marching band nor taking Spanish class.
4. There are 11 + 12 + 13 = 36 students either in the marching band or taking Spanish class.

Handout 1.8.B: Calculating Probabilities with Venn Diagrams

See lesson for answers.

Handout 1.8.C: Practice Using Venn Diagrams to Determine Probabilities

1. (a) Let E stand for students who would choose eggplant, let B stand for students who would choose broccoli, and let M stand for students who would choose mushrooms. A Venn diagram to represent the results of the survey is:



Guiding Student Thinking

Some students may need help when considering values for overlapping categories. For example, students may need to be reminded that the category "eggplant and broccoli" does not exclude those students who would also order mushrooms.

(b) $P(\text{no topping}) = \dfrac{5}{50} = 0.1$

(c) $P(M \cup E) = \dfrac{39}{50} = 0.78$

(d) $P(B \cap M) = \dfrac{3}{50} = 0.06$

(e) $P(M \mid B) = \dfrac{3}{20} = 0.15$

2. (a) There are 1,466 people in the study.
   (b) $P(\text{patch} \cup \text{gum}) = \dfrac{1{,}208}{1{,}466} \approx 0.824$
   (c) $P(\text{no patch} \cap \text{no gum}) = \dfrac{258}{1{,}466} \approx 0.176$
   (d) $P(\text{patch} \cap \text{gum}) = \dfrac{146}{1{,}466} \approx 0.0996$
   (e) $P(\text{lozenge} \mid \text{gum}) = \dfrac{142}{487} \approx 0.292$
   (f) $P(\text{lozenge} \mid \text{no gum}) = \dfrac{232}{979} \approx 0.237$
   (g) $P(\text{no patch} \mid \text{no lozenge}) = \dfrac{325}{1{,}092} \approx 0.298$
   (h) I would choose to stock the patch, because it seems to be the most popular nicotine replacement therapy.

UNIT 1

LESSON 1.9

Contingency Tables

OVERVIEW

LESSON DESCRIPTION

Part 1: Connecting Venn Diagrams and Contingency Tables
In Lesson 1.8, students learned that Venn diagrams provide a powerful way to organize categorical data. Another extremely useful display for categorical data is a contingency table, which is also called a two-way table. In this part of the lesson, students learn how to create contingency tables and use them to calculate probabilities related to the data sets they display.

Part 2: Using Contingency Tables to Calculate Conditional Probabilities
This part of the lesson affords students an opportunity to explore how contingency tables can be used to answer questions about categorical data sets by finding relative frequencies, joint frequencies, marginal frequencies, and conditional probabilities. The categorical variables in this part of the lesson will often have more than two properties to consider.

CONTENT FOCUS

Contingency tables, like Venn diagrams, provide a way to display categorical data. Because they are used to show the relationship between two categorical variables, they are also called two-way tables. Contingency tables are particularly helpful displays when both of the categories have multiple properties. For example, while a Venn diagram can show the relationship between the categories "is in an extracurricular activity" and "takes a world language class," a contingency table can be used

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Connections Among Multiple Representations

SUGGESTED TIMING

~90 minutes

LESSON SEQUENCE

This lesson is part of a lesson sequence (~405 minutes total) that includes Lessons 1.7 through 1.11.

HANDOUTS

Lesson

- 1.9.A: Connecting Venn Diagrams and Contingency Tables

Practice

- 1.9.B: Using Contingency Tables to Calculate Probabilities

MATERIALS

- calculator

to further sort the extracurricular activities into marching band, drama club, and soccer, and the world language classes into Spanish, Arabic, and Chinese. As with Venn diagrams, contingency tables can be useful for answering probability questions about categorical data sets.

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ Probabilistic reasoning allows us to anticipate patterns in data.	
Learning Objectives	**Essential Knowledge**
1.2.1 Create or analyze a data display for a categorical data set.	**1.2.1a** Venn diagrams and contingency tables are common displays of categorical data and are useful for answering questions about probability. **1.2.1b** The intersection of two categories is the set of elements common to both categories. **1.2.1c** The union of two categories is the set of elements found by combining all elements of both categories. **1.2.1d** For categorical data, variability is determined by comparing relative frequencies of categories.
1.2.3 Calculate relative frequencies, joint frequencies, marginal frequencies, or conditional probabilities for a categorical data set.	**1.2.3a** Relative frequencies are the number of times an event occurs divided by the total number of observations. They can be used to estimate probabilities of future events occurring. **1.2.3b** Joint frequencies are events that co-occur for two or more variables. They are the frequencies displayed in cells in a two-way contingency table. **1.2.3c** Marginal frequencies are events that summarize the frequencies across all levels of one variable while holding the second variable constant. They are the row totals and column totals in a two-way contingency table. **1.2.3d** The conditional probability of B, given A has already occurred, is the proportion of times B occurs when restricted to events only in A.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

A recent national poll was conducted with registered voters to gather information about support for increasing the federal minimum wage. The results of the poll are displayed below.

	In Favor of Increasing	Against Increasing	Total
18 to 34 Years Old	207	43	250
35 to 49 Years Old	221	39	260
50 to 64 Years Old	223	57	280
Over 65 Years Old	164	46	210
Total	815	185	1,000

Data from Hill-HarrisX poll, **thehill.com/hilltv/what-americas-thinking/426780-poll-a-majority-of-voters-want-a-15-minimum-wage**

(a) What is the probability that a randomly selected person from those surveyed is 18 to 34 years old or in favor of increasing the minimum wage? Explain your answer.

(b) What is the probability that a randomly selected person is 18 to 34 years old and in favor of increasing the minimum wage? Explain your answer.

(c) What is the probability that a randomly selected person is 18 to 34 years old given that they are in favor of increasing the minimum wage? Explain your answer.

(d) Which is more likely: randomly selecting a person from the survey who is against increasing the minimum wage, given that the person selected is 50 to 64 years old, or randomly selecting a person from the survey who is against increasing the minimum wage, given that the person selected is over 65 years old? Explain your answer.

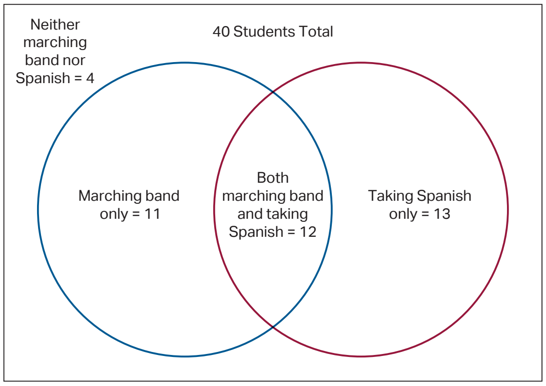
PART 1: CONNECTING VENN DIAGRAMS AND CONTINGENCY TABLES

In Lesson 1.8, students learned that Venn diagrams provide a powerful way to organize categorical data. Another extremely useful display for categorical data is a contingency table, which is also called a two-way table. In this part of the lesson students learn how to create contingency tables and use them to calculate probabilities related to the data sets they display.

Instructional Rationale

It is important that students see the advantages of both Venn diagrams and contingency tables and how they relate to each other. They are both common displays of categorical data. Students should be familiar with both displays so they can make decisions about which display would be the more appropriate way to organize a particular data set and better help them answer probability questions about the data.

- Begin this lesson by having students closely observe the Venn diagram they explored in Lesson 1.8, which involved students taking Spanish class and being in the marching band. This is also available on **Handout 1.9.A: Connecting Venn Diagrams and Contingency Tables**. The handout has a blank contingency table for students to complete.



	In the Marching Band	Not in the Marching Band	Total
Taking Spanish Class			
Not Taking Spanish Class			
Total			40

Handout 1.9.A

- Some students may be familiar with contingency tables, while others may not. You may have to identify some features of the table for students. The individual cells in the table show the intersection of two categories, which is why these organizers are sometimes called two-way tables. Often each category will have multiple properties. In the case of the Spanish and marching band diagram, the category of Spanish class has the properties of taking Spanish class or not taking Spanish class, and the category of marching band has the properties of being in marching band or not being in marching band.
- The categories are already populated in Handout 1.9.A, so students can work with a partner to determine the quantities for each cell. As students work, circulate around the room to support them as necessary. It is helpful to ask students questions to encourage them to reason through the number of students who fall into each category. The solutions are shown here:

Meeting Learners' Needs

If students are having trouble getting started, you can help them by posing questions about what each cell represents. For example, you could ask "Look at the first row, first column. How many students are in the marching band and are taking Spanish?" and then show them that that number (12) should be written in the cell. You can also ask how many students there are total (40) and show them where that number would go to help them make sense of the rest of the table.

	In the Marching Band	Not in the Marching Band	Total
Taking Spanish Class	12	13	25
Not Taking Spanish Class	11	4	15
Total	23	17	40

- After a few minutes, or when most students have completed the table, ask them as a whole class what they notice about the table. They may offer many observations, but be sure that they identify that the right column and the bottom row each sum to the total number of students.
- To provide students with some practice interpreting the contingency table, ask questions such as those on the next page. Ensure that students provide evidence from specific cells in the table rather than from the Venn diagram.

- How many students are in the marching band and not taking Spanish?

  There are 11 students in the marching band and not taking Spanish. This is in the first column, second row.

- How many students total are not in the marching band?

  There are 17 students not in the marching band. This is in the second column, last row.

- What is the probability that a randomly selected student from the survey is not in the marching band?

  The probability is $\frac{17}{40}$, because there are 17 students not in the marching band, which is the total in the marching band column, and there are 40 students total.

- What is the probability that a randomly selected student is taking Spanish given that they are in the marching band?

  The probability is $\frac{12}{23}$. There are 23 students in the marching band, and 12 of those students are taking Spanish.

- What is the probability that a randomly selected student is in the marching band given that they take Spanish?

  The probability is $\frac{12}{25}$. There are 25 students taking Spanish, and 12 of those students are in the marching band.

- What are some advantages and disadvantages of Venn diagrams and contingency tables?

  Venn diagrams can be helpful for visualizing categorical data, especially the concepts of intersection and union. Venn diagrams are also useful when there are more than two categories. Contingency tables are helpful for conditional probability questions. Contingency tables are also easier to read if the categorical variables in a data set each have multiple properties.

- Finally, use the contingency table to introduce the terms *joint frequency* and *marginal frequency*. You can ask students to recall joint probabilities (the probability of two events occurring at the same time) and to use that concept to identify which cells in the table would be considered joint frequencies.

Classroom Ideas

One effective way to engage students in these questions would be to have them write their answers on a personal whiteboard or by using a form of interactive response technology. This would allow you to quickly determine which students need support and guidance with contingency tables and/or probability.

	In the Marching Band	Not in the Marching Band	Total
Taking Spanish Class	12 joint frequency	13 joint frequency	25 marginal frequency
Not Taking Spanish Class	11 joint frequency	4 joint frequency	15 marginal frequency
Total	23 marginal frequency	17 marginal frequency	40 marginal frequency

PART 2: USING CONTINGENCY TABLES TO CALCULATE CONDITIONAL PROBABILITIES

This part of the lesson affords students an opportunity to explore how contingency tables can be used to answer many questions about categorical data sets by finding relative frequencies, joint frequencies, marginal frequencies, and conditional frequencies. The categorical variables in this part of the lesson will often have more than two properties to consider.

- This part of the lesson calls back to some of the categorical data collected in Lesson 1.2. If you did not collect data at that time, you can ask students these two questions and gather the information now:
  - What superpower would you choose: flying, invisibility, super strength, or telepathy?
  - How many pets do you have?
- Once you have the data, have students work in groups of three to create a contingency table from the information.

Guiding Student Thinking

While the superpower question makes sense as a categorical variable, students may not be sure how to handle the number of pets question. This is a good opportunity to explore using a quantitative variable categorically. Some groups may want to create a row for every value given as a response to the number of pets question. This is acceptable, but encourage them to consider chunking the data instead. One possible solution is to sort the pet data into "has pets" and "does not have pets." Another possibility is "has no pets," "has one pet," and "has more than one pet."

- A sample contingency table is shown below.

	Flying	Invisibility	Super Strength	Telepathy	Total
Has More than One Pet	4	3	1	3	11
Has One Pet	3	2	4	5	14
Has No Pets	7	5	0	3	15
Total	14	10	5	11	40

- To connect contingency tables to the statistical concepts from Key Concept 1.1, you can ask students to think about whether or not the contingency table they made could be thought of as a frequency distribution. A contingency table is a lot like a histogram in terms of the input and output. That is, the input is a data value of a categorical variable and the output is a frequency. Since each data value will be associated with only one frequency, the frequency distribution shown in the contingency table is a function.
- Ask students some questions like the ones below to see if they can make sense of the contingency table as a function. The final two questions are intended to challenge students a bit, and may take some time to work through.
  - Let's suppose that the function f maps a category to its frequency. What would be the value of f(has no pets)? How do you know?

    f(no pets) = 15. This is because the sum of the cells in the "has no pets" row is 15.
  - What would be the value of f(wants a superpower of flying)? How do you know?

    f(flying) = 14. This is because the sum of the cells in the "flying" column is 14.
  - What would be the value of f(has one pet $\cap$ wants telepathy)? How do you know?

    f(one pet $\cap$ telepathy) = 5. This is because the value of the cell in the "one pet" row and the "telepathy" column is 5.
  - What would be the value of f(has more than one pet $\cup$ wants invisibility)? How do you know?

    f(more than one pet $\cup$ invisibility) = 18. This is determined by the sum of all the cells in the "more than one pet" row and the "invisibility" column. The sum only includes the cell representing the intersection (more than one pet $\cap$ invisibility) once.

- What would be the value of f(has no pets $\cap$ does not want to fly)? How do you know?

  f(no pets $\cap$ not flying) = 8. This is determined by the sum of all the cells in the "no pets" row that are not in the "flying" column.

- What would be the value of f(has more than one pet $\cup$ does not want super strength)? How do you know?

  f(more than one pet $\cup$ not super strength) = 36. This is determined by the sum of all the cells in the "more than one pet" row and all columns of the other rows that are not "super strength". Alternatively, the cells not to include in the sum are (one pet $\cap$ super strength) and (no pets $\cap$ super strength) because they do not fall into either of the categories of "more than one pet" or "does not want super strength."

- Next, have students modify their table so that it shows the relative frequencies of the categories. A relative frequency is the ratio of a frequency to the total number of data points. In the case of the pets and superpowers contingency table, it means creating a ratio with each of the cell entries and 40, the total number of student responses represented in the table.

Meeting Learners' Needs

You can help students who are struggling with the function notation make sense of it by reframing the question as "How many people fall into the category ...?"

	Flying	Invisibility	Super Strength	Telepathy	Total
Has More than One Pet	$\frac{4}{40}=0.10$	$\frac{3}{40}=0.075$	$\frac{1}{40}=0.025$	$\frac{3}{40}=0.075$	$\frac{11}{40}=0.275$
Has One Pet	$\frac{3}{40}=0.075$	$\frac{2}{40}=0.05$	$\frac{4}{40}=0.10$	$\frac{5}{40}=0.125$	$\frac{14}{40}=0.35$
Has No Pets	$\frac{7}{40}=0.175$	$\frac{5}{40}=0.125$	$\frac{0}{40}=0$	$\frac{3}{40}=0.075$	$\frac{15}{40}=0.375$
Total	$\frac{14}{40}=0.35$	$\frac{10}{40}=0.25$	$\frac{5}{40}=0.125$	$\frac{11}{40}=0.275$	$\frac{40}{40}=1.0$

UNIT 1

- Ask students what they notice and what they wonder about the relative frequency table. Students may observe that all of the cell entries are fractions or decimals, or they may observe that the patterns of row and column totals are still present. They may wonder if the relative frequency table could be used find probabilities.
- Now, to help students connect the relative frequency table to probability, ask them some questions like the ones below. Students may notice that some of the questions are similar to the ones you asked about the frequency distribution earlier in the lesson. However, this time you are asking about probability. The final two questions are conditional probability questions and may take some time to work through.
  - What is the probability that a randomly selected student has no pets? How do you know?
    $P(\text{no pets}) = \frac{15}{40} = 0.375$. This is because the sum of the cells in the "has no pets" row is 15 and there are 40 students total.
  - What is the probability that a randomly selected student wants a superpower of flying? How do you know?
    $P(\text{flying}) = \frac{14}{40} = 0.35$. This is because the sum of the cells in the "flying" column is 14 and there are 40 students total.
  - What is the probability that a randomly selected student has one pet and wants telepathy? How do you know?
    $P(\text{one pet} \cap \text{telepathy}) = \frac{5}{40} = 0.125$. This is because the value of the cell in the "one pet" row and the "telepathy" column is 5, and there are 40 students total.
  - What is the probability that a randomly selected student has more than one pet or wants invisibility? How do you know?
    $P(\text{more than one pet} \cup \text{invisibility}) = \frac{18}{40} = 0.45$. This is because the sum of all the cells in the "more than one pet" row and the "invisibility" column, while counting the cell representing the intersection only once, is 18 and there are 40 students total.
  - What is the probability that a randomly selected student has no pets given that they want to fly? How do you know?
    $P(\text{no pets} \mid \text{flying}) = \frac{7}{14} = 0.5$. This is because the sample space is restricted to the "flying" column so there are only 14 students. Of those 14 students, 7 of them have no pets.

- What is the probability that a randomly selected student wants super strength given that they have more than one pet? How do you know?

  $P(\text{super strength} \mid \text{more than one pet}) = \frac{1}{11} \approx 0.09$. This is because the sample space is restricted to the "more than one pet" row, so there are only 11 students. Of those 11 students, 1 of them wants super strength.

Instructional Rationale

It is important for students to understand conditional probabilities conceptually rather than rely on formulas. As they become more familiar with the power of two-way tables, they will see how column totals will become the denominator of a conditional probability problem when the column category is the given condition, and that the row totals are the denominators of a conditional probability problem when the row category is the given condition.

- At this point, you can define a probability distribution for students.

  A probability distribution is a function that associates a probability with each possible value or interval of values for a random variable. The sum of the probabilities of all possible values of the independent variable is 1.

- The relative frequency version of the contingency table is a probability distribution, just as the original contingency table is a frequency distribution. We know this is a function because the input is a data value of a categorical variable and the output is a probability, and each data value will be associated with only one probability.
- To conclude the lesson, have students complete **Handout 1.9.B: Using Contingency Tables to Calculate Probabilities**. An important component of this practice opportunity is debriefing the problems as a class. Solutions for the handout are given in the Assess and Reflect section of the lesson.

Meeting Learners' Needs

Some students may need additional instruction with respect to contingency tables. The 12-minute video from MIT OpenCourseware linked below describes how to evaluate conditional probabilities using both contingency tables and tree diagrams. Tree diagrams are outside the scope of this course, but you could have students who would benefit from a challenge try to use them in addition to contingency tables: **youtube.com/watch?v=JGeTcRfKgBo**.

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

A recent national poll was conducted with registered voters to gather information about support for increasing the federal minimum wage. The results of the poll are displayed below.

	In Favor of Increasing	Against Increasing	Total
18 to 34 Years Old	207	43	250
35 to 49 Years Old	221	39	260
50 to 64 Years Old	223	57	280
Over 65 Years Old	164	46	210
Total	815	185	1,000

(a) What is the probability that a randomly selected person from those surveyed is 18 to 34 years old or in favor of increasing the minimum wage? Explain your answer.

The word "or" indicates a union of the categories "18 to 34 years old" and "in favor of increasing." There are 858 people in these two categories (the sum of the entries in the "18 to 34 years old" row and the "in favor" column). The probability is $\frac{858}{1{,}000} = 0.858$.

(b) What is the probability that a randomly selected person is 18 to 34 years old and in favor of increasing the minimum wage? Explain your answer.

The word "and" indicates an intersection of the categories "18 to 34 years old" and "in favor of increasing." There are 207 people in these two categories (the cell in both the "18 to 34 years old" row and the "in favor" column). The probability is $\frac{207}{1{,}000} = 0.207$.

(c) What is the probability that a randomly selected person is 18 to 34 years old given that they are in favor of increasing the minimum wage? Explain your answer.

The word "given" indicates that this is a conditional probability problem, which means that the sample space is a subset of the total number of individuals surveyed. There are 815 people in favor of increasing the minimum wage, of which 207 are 18 to 34 years old. So, the probability is $\frac{207}{815} \approx 0.254$.

(d) Which is more likely: randomly selecting a person from the survey who is against increasing the minimum wage, given that the person selected is 50 to 64 years old, or randomly selecting a person from the survey who is against increasing the minimum wage, given that the person selected is over 65 years old? Explain your answer.

The word "given" indicates that this is a conditional probability problem, which means the sample space is a subset of the total number of individuals surveyed. Of the 280 people aged 50 to 64 years old in the survey, 57 are against increasing the minimum wage. So, the probability is $\frac{57}{280} \approx 0.203$. Of the 210 people over 65 years old in the survey, 46 are against increasing the minimum wage. So, the probability is $\frac{46}{210} \approx 0.219$. This means it is more likely that a person randomly selected from the survey will be against increasing the minimum wage given that the person is over 65 years old than a person randomly selected from the survey will be against increasing the minimum wage given that the person is 50 to 64 years old.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 1.9.A: Connecting Venn Diagrams and Contingency Tables

See lesson for answers.

Handout 1.9.B: Using Contingency Tables to Calculate Probabilities

1. (a) $P(\text{middle school}) = \frac{1{,}000}{3{,}000} \approx 0.333$

   (b) $P(\text{high school} \cap \text{at least once a week}) = \frac{259}{3{,}000} \approx 0.086$

   (c) $P(\text{primary school} \cup \text{less than once a week}) = \frac{2{,}485}{3{,}000} \approx 0.828$

(d) $P(\text{middle school} \mid \text{at least once a week}) = \dfrac{256}{557} \approx 0.460$

(e) $P(\text{at least once a week} \mid \text{high school}) = \dfrac{259}{1{,}000} \approx 0.259$

(f) Cyberbullying becomes and remains a more common occurrence for students from middle school onward.

(g) The survey does not address other frequencies of cyberbullying. For example, the category "less than once a week" probably includes students who never experience cyberbullying as well as students who experience cyberbullying less than once a week but at least once a month. The survey does not help us answer how widespread cyberbullying is because the categories are not differentiated enough.

2. (a) and (b)

	Used a Patch	**Did Not Use a Patch**	**Total**
Used a Lozenge or Gum	206 $\dfrac{206}{1{,}466} \approx 0.141$	513 $\dfrac{513}{1{,}466} \approx 0.350$	719 $\dfrac{719}{1{,}466} \approx 0.490$
Used Neither Lozenge nor Gum	661 $\dfrac{661}{1{,}466} \approx 0.451$	86 $\dfrac{86}{1{,}466} \approx 0.059$	747 $\dfrac{747}{1{,}466} \approx 0.510$
Total	867 $\dfrac{867}{1{,}466} \approx 0.591$	599 $\dfrac{599}{1{,}466} \approx 0.409$	1,466 $\dfrac{1{,}466}{1{,}466} = 1$

(c) $P(\text{patch}) = \dfrac{867}{1{,}466} \approx 0.591$

(d) $P(\text{no patch} \cap \text{no lozenge} \cap \text{no gum}) = \dfrac{86}{1{,}466} \approx 0.059$

(e) $P(\text{patch} \mid \text{lozenge or gum}) = \dfrac{206}{719} \approx 0.287$

(f) $P(\text{lozenge or gum} \mid \text{patch}) = \dfrac{206}{867} \approx 0.238$

3. (a) $P(\text{woman} \cap \text{master's degree}) = \dfrac{465}{3{,}892} \approx 0.119$; $P(\text{man} \cap \text{bachelor's degree}) = \dfrac{822}{3{,}892} \approx 0.211$. Because $0.119 < 0.211$, it is more likely to randomly select a man with a bachelor's degree than a woman with a master's degree.

   (b) $P(\text{doctoral degree} \mid \text{woman}) = \dfrac{94}{2{,}274} \approx 0.041$; $P(\text{woman} \mid \text{doctoral degree}) = \dfrac{94}{178} \approx 0.528$. Because $0.041 < 0.528$, it is more likely that a randomly selected person is a woman given that that the person has a doctoral degree.

4. (a)

	Brightest	Bright	Drab	Drabbest	Total
Deep Pool	$\frac{4}{664} \approx 0.006$	$\frac{11}{664} \approx 0.017$	$\frac{25}{664} \approx 0.037$	$\frac{170}{664} \approx 0.256$	$\frac{210}{664} \approx 0.316$
Shallow Pool	$\frac{204}{664} \approx 0.307$	$\frac{18}{664} \approx 0.027$	$\frac{5}{664} \approx 0.008$	$\frac{9}{664} \approx 0.014$	$\frac{236}{664} \approx 0.355$
Pool Behind Natural Dam	$\frac{2}{664} \approx 0.003$	$\frac{98}{664} \approx 0.148$	$\frac{111}{664} \approx 0.167$	$\frac{7}{664} \approx 0.011$	$\frac{218}{664} \approx 0.328$
Total	$\frac{210}{664} \approx 0.316$	$\frac{127}{664} \approx 0.191$	$\frac{141}{664} \approx 0.212$	$\frac{186}{664} \approx 0.280$	$\frac{664}{664} = 1$

   (b) $P(\text{brightest color variant} \mid \text{shallow pool}) = \dfrac{204}{236} \approx 0.864$

   (c) The events are equally likely because $P(\text{deep pool}) = P(\text{brightest color variant}) = \dfrac{210}{664} \approx 0.316$.

   (d) There are $141 + 186 = 327$ drab or drabbest guppies. $P(\text{deep pool} \mid \text{drab} \cup \text{drabbest}) = \dfrac{195}{327} \approx 0.596$ and $P(\text{not a deep pool} \mid \text{drab} \cup \text{drabbest}) = \dfrac{132}{327} \approx 0.404$. Therefore it is possible to conclude that a randomly selected drab or drabbest guppy is most likely to have come from a deep pool.

LESSON 1.10

Independent Events

OVERVIEW

LESSON DESCRIPTION

Part 1: Identifying Independent Events Using Multiplication
In the first part of the lesson, students determine if two events are independent using multiplication. Multiplying the probabilities of two independent events to determine the probability of the compound event connects this part of the lesson to the types of probability problems that students are likely to be familiar with from previous courses. Through this activity, students learn that two events are independent if the product of their individual probabilities is equal to their joint probability.

Part 2: Identifying Independent Events Using Conditional Probability
In the second part of the lesson, students determine if two events are independent using conditional probability. Students learn that two events are independent if the probability of one event occurring given that the other event has occurred is equal to the probability of the event by itself. If the occurrence of one event has an effect on the probability of the second event, then the events are not independent.

CONTENT FOCUS

Two events are described as independent if the outcome of one of the events does not affect the probability of the other. Mathematically, there are two methods for determining whether events

AREA OF FOCUS

- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~90 minutes

LESSON SEQUENCE

This lesson is part of a lesson sequence (~405 minutes total) that includes Lessons 1.7 through 1.11.

HANDOUTS

Lesson

- 1.10.A: Identifying Independent Events Using Multiplication

Practice

- 1.10.B: Identifying Independent Events Using Conditional Probability

MATERIALS

- calculator

are independent. The first method involves multiplication: if the product of the probabilities of the individual events is equal to their joint probability, then they are independent. Because one event's outcome does not affect the sample space for the second event, their individual probabilities can simply be multiplied to determine their joint probability. The second method involves conditional probability: if two events are independent, then assuming one event has occurred does not affect the probability of the other event occurring. If the occurrence of one event affects the sample space of the second event, then the events are not independent.

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ Probabilistic reasoning allows us to anticipate patterns in data.	
Learning Objectives	**Essential Knowledge**
1.2.4 Determine if two events are independent.	**1.2.4a** Two events, *A* and *B*, are independent if the occurrence of *A* does not affect the probability of *B*. **1.2.4b** Two events, *A* and *B*, are independent if the probability of *A* and *B* occurring together is the product of their probabilities.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

> One card will be selected from a well-shuffled deck of 52 standard playing cards. Let *H* be the event that the card drawn is a heart, *Q* be the event that the card drawn is a queen, and *R* be the event that the card drawn is a red card.
>
> (a) Are the events *H* and *Q* independent? Justify your reasoning using multiplication.
>
> (b) Are the events *H* and *R* independent? Justify your reasoning using conditional probability.

PART 1: IDENTIFYING INDEPENDENT EVENTS USING MULTIPLICATION

In the first part of the lesson, students determine if two events are independent using multiplication. Multiplying the probabilities of two independent events to determine the probability of the compound event connects this part of the lesson to the types of probability problems that students are likely to be familiar with from previous courses. If the occurrence of one event has an effect on the probability of the second event, then the events are not independent. Through this activity, students learn that two events are independent if the product of their individual probabilities is equal to their joint probability.

Instructional Rationale

This lesson begins with a relatively familiar context so students can connect prior knowledge to this course. It is important to allow them the necessary space and time to make sense of new information about independence, including how it fits with or challenges their understanding of probability.

- Begin by asking students to determine the probabilities of three events. You could post them in a central location in the room or provide each student with their own copy of the problems.

  1. If you flip a fair coin, what is the probability that it lands heads up?
  2. If you roll a standard six-sided number cube, what is the probability that it shows a 5?
  3. If you flip a coin and roll a standard six-sided number cube, what is the probability that the coin lands heads up and the number cube shows a 5?

- Most students should reason that for event 1, $P(\text{head}) = \frac{1}{2}$, and that for event 2, $P(5) = \frac{1}{6}$. Some students may struggle with identifying the probability for event 3 even if they have experience using multiplication for compound events. It is valuable for them to write out the sample space for flipping a coin and rolling a number cube.

  - What is the sample space for flipping a coin and rolling a number cube? How many outcomes are there?

    The sample space has 12 outcomes: {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}.

  - Use the sample space you wrote to determine the probability that a flipped coin lands heads up.

    $P(\text{H}) = \frac{6}{12} = \frac{1}{2}$ because there are six outcomes with heads.

- Use the sample space you wrote to determine the probability that a rolled number cube will show a 5.

  $P(5) = \frac{2}{12} = \frac{1}{6}$ because there are two outcomes with a 5.

- Use the sample space you wrote to determine the probability that a flipped coin will show heads and a rolled number cube will show a 5.

  $P(\text{H5}) = \frac{1}{12}$ because there is only one outcome in the sample space that has heads and a 5.

- The next set of questions asks students to consider the relationship between the individual probabilities and the compound probability. The main takeaway for students is that the compound probability is the product of the individual probabilities, but that this relationship is true in this case because the events are independent. That is, the individual outcomes do not affect each other.

  - Look at the individual probabilities of having heads showing on the coin, $P(\text{H})$, and having a 5 showing on the number cube, $P(5)$. How could you use $P(\text{H})$ and $P(5)$ to determine the probability of the coin showing heads and the number cube showing a 5?

    $P(\text{H5})$ is equal to the product of $P(\text{H})$ and $P(5)$.

  - Think about the two events: flipping a coin and rolling a number cube. Does the outcome of one event affect the outcome of the other event? That is, does having heads (or tails) showing on the coin have any affect on the roll of the number cube?

    These events do not affect each other. The coin flip does not influence the number cube roll and the number cube roll does not influence the coin flip.

- At this point, you can define the term *independence* for students: *Two events A and B are called independent if* $P(A) \bullet P(B) = P(A \cap B)$. *Also, if* $P(A) \bullet P(B) = P(A \cap B)$, *then the events A and B are independent.*
- To make this definition clearer for students, you can explain that with independent events, the outcome of one event has no effect on the outcome of other events. Meanwhile, with dependent events, the outcome of one event does have an effect on the outcomes of other events. You can have students generate some examples of events that are dependent, such as dealing cards for a game or picking marbles from a bag without replacing them.
- To contrast the multiplication property for independence with dependence, have students consider a familiar scenario: the superpowers relative frequency contingency table from Lesson 1.9.

	Flying	Invisibility	Super Strength	Telepathy	Total
Has More than One Pet	$\frac{4}{40}=0.10$	$\frac{3}{40}=0.075$	$\frac{1}{40}=0.025$	$\frac{3}{40}=0.075$	$\frac{11}{40}=0.275$
Has One Pet	$\frac{3}{40}=0.075$	$\frac{2}{40}=0.05$	$\frac{4}{40}=0.10$	$\frac{5}{40}=0.125$	$\frac{14}{40}=0.35$
Has No Pets	$\frac{7}{40}=0.175$	$\frac{5}{40}=0.125$	$\frac{0}{40}=0$	$\frac{3}{40}=0.075$	$\frac{15}{40}=0.375$
Total	$\frac{14}{40}=0.35$	$\frac{10}{40}=0.25$	$\frac{5}{40}=0.125$	$\frac{11}{40}=0.275$	$\frac{40}{40}=1.0$

- Consider the events “has one pet” and “wants the superpower of flying.” Do you think these events are independent? Why or why not?

  The outcome of one event might have an effect on the other, but it is hard to tell just by looking at the table.

- Use the multiplication property of independence to determine whether or not the events “has one pet” and “wants the superpower of flying” are independent or dependent.

  $P(\text{has one pet}) = \frac{14}{40}$ and $P(\text{flying}) = \frac{14}{40}$. $P(\text{has one pet} \mid \text{flying}) = \frac{3}{40}$. Using the multiplication property, we see that $\frac{14}{40} \bullet \frac{14}{40} = \frac{196}{1{,}600} \neq \frac{3}{40}$. Therefore, the events are not independent.

- What does it mean in the context of this problem that the events “has one pet” and “wants the superpower of flying” are not independent?

  This means that randomly choosing one person from this group who has one pet does have an effect on the probability of randomly choosing a person from this group who wants to fly.

PART 2: IDENTIFYING INDEPENDENT EVENTS USING CONDITIONAL PROBABILITY

In the second part of the lesson, students determine if two events are independent using conditional probability. Students learn that two events are independent if the probability of one event occurring given that the other event has occurred is equal to the probability of the event by itself. If the occurrence of one event has an effect on the probability of the second event, then the events are not independent.

- Begin this part of the lesson by posing a question to students:
  - We know that two events are independent if the product of their individual probabilities is equal to their joint probability. This means that the outcome of one event does not affect the outcome of the other. Do you think we could use conditional probability to make sense of independence?

Instructional Rationale

This question may be challenging for students to answer. However, it is important to ask because it may seem odd to students to use conditional probability to determine independence without any motivation for a second technique. It is acceptable if students can reason that because the conditional probability of an event changes the sample space, it is reasonable to think that the probabilities of dependent events might be different given different conditions. It is also acceptable if they are unsure what to think.

- To help students see how to use conditional probability to determine dependence, return to the coin flip and number cube roll problem from Part 1. You can lead them through a series of questions to have them consider the conditional probabilities, or you could have them work in pairs to answer the questions.
  - Let's return to the trials involving flipping a coin and rolling a number cube to explore conditional probability. What is the sample space?

    The sample space has 12 outcomes: {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}.

  - We know that the events of a flipped coin showing heads and a rolled number cube showing a 5 are independent. Let's reason with conditional probability. What is the probability that a flipped coin will show heads?

    $P(\text{H}) = \frac{6}{12} = \frac{1}{2}$ because there are six outcomes with heads.

  - What is the probability that a flipped coin will show heads given that the rolled number cube shows a 5?

    $P(\text{H}|5) = \frac{1}{2}$ because the sample space consists only of {H5, T5}. There is one outcome with heads out of the two outcomes.

  - Is the probability of flipping a coin and it showing heads equal to the probability of flipping a coin and it showing heads given that the number cube shows a 5? What can we conclude about the events?

    The probabilities are equal. This means that the outcome of the number cube roll does not have an effect on the probability of flipping the coin and it showing heads. Therefore, the events are independent.

- Now let's look at it the other way. What is the probability the rolled number cube will show a 5?
  $P(5) = \frac{2}{12} = \frac{1}{6}$ because there are two outcomes with a 5.
- What is the probability that a rolled number cube will show a 5 given that a flipped coin shows heads?
  $P(5|\text{H}) = \frac{1}{6}$ because the sample space consists only of {H1, H2, H3, H4, H5, H6}. There is one outcome with a 5 out of the six outcomes.
- Is the probability of the rolled number cube showing a 5 equal to the probability that a rolled number cube will show a 5 given that a flipped coin shows heads? What can we conclude about the events?

  The probabilities are equal. This means that the outcome of the coin flip does not have an effect on the probability of the number cube roll. Therefore, the events are independent.

- At this point, you can provide an alternative definition of the term *independence* for students:
  Two events A and B are called independent if $P(A) = P(A|B)$ *or if* $P(B) = P(B|A)$.
- To further explore the relationship between conditional probability and independence, have students return to the superpowers relative frequency contingency table.

	Flying	Invisibility	Super Strength	Telepathy	Total
Has More than One Pet	$\frac{4}{40} = 0.10$	$\frac{3}{40} = 0.075$	$\frac{1}{40} = 0.025$	$\frac{3}{40} = 0.075$	$\frac{11}{40} = 0.275$
Has One Pet	$\frac{3}{40} = 0.075$	$\frac{2}{40} = 0.05$	$\frac{4}{40} = 0.10$	$\frac{5}{40} = 0.125$	$\frac{14}{40} = 0.35$
Has No Pets	$\frac{7}{40} = 0.175$	$\frac{5}{40} = 0.125$	$\frac{0}{40} = 0$	$\frac{3}{40} = 0.075$	$\frac{15}{40} = 0.375$
Total	$\frac{14}{40} = 0.35$	$\frac{10}{40} = 0.25$	$\frac{5}{40} = 0.125$	$\frac{11}{40} = 0.275$	$\frac{40}{40} = 1.0$

- We know that the events "has one pet" and "wants the superpower of flying" are dependent. Let's explore the conditional probabilities. Compare the probability that a randomly selected student has one pet and the probability that a randomly selected student has one pet given that they want the superpower of flying. How do we know the events are not independent?

  $P(\text{has one pet}) = \frac{14}{40} = 0.35$ and $P(\text{has one pet} \mid \text{flying}) = \frac{3}{14} \approx 0.214$. The outcome of one event affects the probability of the other, so they are not independent.

- Now compare the probability that a randomly selected student wants the superpower of flying and the probability that a randomly selected student wants the superpower of flying given that they have one pet. How do we know the events are not independent?

  $P(\text{flying}) = \frac{14}{40} = 0.35$ and $P(\text{flying} \mid \text{has one pet}) = \frac{3}{14} \approx 0.214$. The outcome of one event affects the probability of the other, so they are not independent.

- At this point, ask students some questions like the ones below to make sure they understand the different ways to test for independence:
  - What does it mean for two events to be independent?

    When two events are independent, the outcome of one event does not affect the probability of the other.

  - How can we determine the independence of events using multiplication?

    If two events are independent, the product of their individual probabilities will be equal to their joint probability. That is, A and B are independent if $P(A) \bullet P(B) = P(A \cap B)$.

  - How can we determine the independence of events using conditional probability?

    If two events are independent, then the probability of one event is equal to the probability of that event given that the other event has occurred. That is, A and B are independent if $P(A) = P(A \mid B)$ or $P(B) = P(B \mid A)$.

- Many students will need some additional practice with determining independence using conditional probability. There are some problems on **Handout 1.10.B: Identifying Independent Events Using Conditional Probability** that you can use with students in class or as an out-of-class assignment. An important part of completing the assignment is allowing students to share their answers with classmates and defend them as appropriate. A debrief of this handout would provide an excellent opportunity for students to engage in mathematical argumentation and academic conversation.

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

One card will be selected from a well-shuffled deck of 52 standard playing cards. Let H be the event that the card drawn is a heart, Q be the event that the card drawn is a queen, and R be the event that the card drawn is a red card.

(a) Are the events H and Q independent? Justify your reasoning using multiplication.

The probability that a randomly selected card is a heart is $\frac{13}{52}$, or $\frac{1}{4}$. The probability that a randomly selected card is a queen is $\frac{4}{52}$, or $\frac{1}{13}$. The probability that a randomly selected card is a queen and a heart is $\frac{1}{52}$. Since the product of the individual probabilities is equal to the probability of the intersection of the two events, the events H and Q are independent. Using mathematical notation, this relationship can be expressed as $P(H) \bullet P(Q) = \left(\frac{1}{4}\right)\left(\frac{1}{13}\right) = \frac{1}{52} = P(H \cap Q)$.

(b) Are the events H and R independent? Justify your reasoning using conditional probability.

The probability that a randomly selected card is a heart is $\frac{13}{52}$, or $\frac{1}{4}$. The probability that a randomly selected card is a heart given that the card is red is $\frac{13}{26} = \frac{1}{2}$. Because $P(H) \neq P(H \mid R)$, we can conclude that the events are dependent. Alternatively, the probability that a randomly selected card is red is $\frac{26}{52}$, or $\frac{1}{2}$. The probability that a randomly selected card is red given that the card is a heart is $\frac{13}{13} = 1$. Because $P(R) \neq P(R \mid H)$, we can conclude that the events are dependent.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 1.10.A: Identifying Independent Events Using Multiplication

1. Each answer is an independent event. The outcome of each question has no effect on the next question, because you are guessing on each question.
2. P(leaving on time) = 0.70 and P(arriving on time) = 0.605; the product is 0.4235. However, P(leaving on time $\cap$ arriving on time) = 0.56. Therefore, the events are not independent.
3. The two events are independent because P(win basketball) • P(whack-a-mole) = 0.15 • 0.32 = 0.048 = P(win Basketball $\cap$ win whack-a-Mole).
4. P(taking Spanish) $= \frac{25}{40}$ and P(not in the marching band) $= \frac{17}{40}$; the product is $\frac{17}{64}$. However, P(taking Spanish $\cap$ not in the marching band) $= \frac{13}{40}$. Therefore, the events are not independent.

Handout 1.10.B: Identifying Independent Events Using Conditional Probability

1. (a) P(brings lunch) $= \frac{90}{360} = \frac{1}{4}$. P(bring lunch | underclass) $= \frac{40}{160} = \frac{1}{4}$. The probabilities are equal, so the events are independent.
   (b) P(buys lunch in the cafeteria) $= \frac{140}{360} = \frac{7}{18}$. P(buys lunch in the cafeteria | upperclass) $= \frac{40}{200} = \frac{1}{5}$. The probabilities are not equal, so the events are not independent.
2. (a) $P(2) = \frac{4}{12} = \frac{1}{3}$ and $P(2 \mid \text{acorn}) = \frac{2}{6} = \frac{1}{3}$. Because the probabilities are equal, we can conclude that the events are independent. Alternatively, P(acorn) $= \frac{6}{12} = \frac{1}{2}$ and $P(\text{acorn} \mid 2) = \frac{2}{4} = \frac{1}{2}$. Because the probabilities are equal, we can conclude that the events are independent.
   (b) P(red) $= \frac{3}{12} = \frac{1}{4}$ and P(red | pencil) $= \frac{3}{6} = \frac{1}{2}$. Because the probabilities are not equal, we can conclude that the events are not independent. Alternatively, P(pencil) $= \frac{6}{12} = \frac{1}{2}$ and P(pencil | red) $= \frac{3}{3} = 1$. Because the probabilities are not equal, we can conclude that the events are not independent.

LESSON 1.11

Modeling Probability with the Normal Distribution

OVERVIEW

LESSON DESCRIPTION

Part 1: Revisiting the Normal Distribution
In this part of the lesson, students revisit how the normal distribution can be used as a model for a probability distribution. This lesson connects the content of Key Concepts 1.1 and 1.2.

Part 2: Modeling Shoe Size with the Normal Distribution
In this part of the lesson, students continue to use the empirical rule (68-95-99.7) and the symmetry of the normal curve to answer questions about normally distributed data sets. Unlike in Part 1, the problems in this section do not provide the total number of data values in data sets, so students rely on probabilities to complete them.

CONTENT FOCUS

This lesson is intended to connect several important ideas from Key Concepts 1.1 and 1.2. In Key Concept 1.1, students learned that quantitative data distributions can be displayed in a histogram and that some of these distributions can be modeled by the normal distribution. In Key Concept 1.2, students learned that categorical data can be displayed in contingency tables. In both displays, the frequencies of the data values are shown. To answer probability questions about categorical data displayed in a contingency table, students transformed the table into a relative frequency contingency table by dividing each frequency by the size of the population. The same transformation can be done with a histogram to create

AREA OF FOCUS

- Greater Authenticity of Applications and Modeling

SUGGESTED TIMING

~45 minutes

LESSON SEQUENCE

This lesson is part of a lesson sequence (~405 minutes total) that includes Lessons 1.7 through 1.11.

HANDOUTS

Lesson

- 1.11.A: Modeling Probability Distributions with the Normal Distribution

Practice

- 1.11.B: Calculating Probabilities with the Normal Distribution

MATERIALS

- calculator

a relative frequency histogram, which then can be used to answer probability questions about quantitative data. Transforming a data distribution by dividing the frequencies by the size of the population (also called a dilation) does not affect the shape of a histogram, so the probability distribution associated with normally distributed data can also be modeled with the normal distribution.

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ Distributions are functions whose displays are used to analyze data sets. ▪ Probabilistic reasoning allows us to anticipate patterns in data.	
Learning Objectives	**Essential Knowledge**
1.2.5 Calculate the probability of a range of values of an independent variable, given a mean, standard deviation, and normal distribution.	**1.2.5a** The normal distribution can be used to model a probability distribution that is bell-shaped and symmetric about the mean. **1.2.5b** When a normal distribution is used as a model of a probability distribution, the probability of a data value occurring above the mean is 0.5 and the probability of a data value occurring below the mean is 0.5. **1.2.5c** When a normal distribution is used as a model of a probability distribution, the probability of data occurring within one standard deviation of the mean is approximately 0.68, the probability of data occurring within two standard deviations of the mean is approximately 0.95, and the probability of data occurring within three standard deviations of the mean is approximately 0.997. These proportions can be used to determine the probability of an event occurring in a population.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

An airline has a flight scheduled for 9 a.m. every morning. The flight departs early on some mornings and late on others. The airline has analyzed past departure data and determined that the distribution of departure times in minutes after 9 a.m. is approximately normally distributed with a mean of 7 minutes and a standard deviation of 4 minutes.

(a) What is the probability that on a randomly selected day, the flight will depart after 9:11 a.m.?

(b) What is the probability that on a randomly selected day, the flight will depart between 8:59 a.m. and 9:11 a.m.?

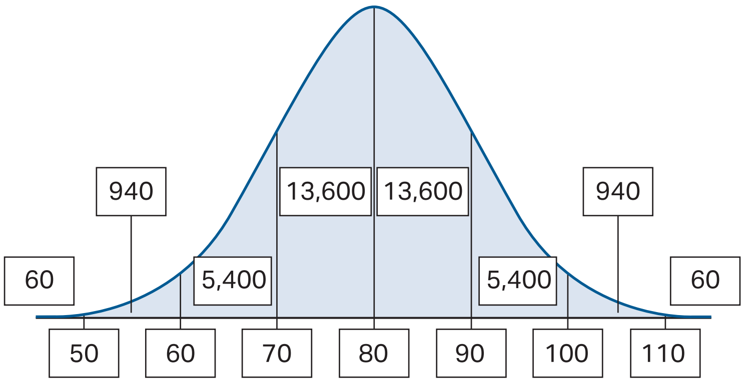
(c) If a random sample of 40 days was selected, how many flights would you expect to depart between 8:59 a.m. and 9:11 a.m.? Justify your response.

UNIT 1

PART 1: REVISITING THE NORMAL DISTRIBUTION

In this part of the lesson, students revisit how the normal distribution can be used as a model for a probability distribution. This lesson connects the content of Key Concepts 1.1 and 1.2.

- To begin, have students explore problem 1 on **Handout 1.11.A: Modeling Probability Distributions with the Normal Distribution**. In this problem, students are tasked with completing a diagram of normally distributed data relating to diastolic blood pressure. The problem also has students use what they know about probability to convert data values to probabilities.
- You can allow students to work with a partner to answer the questions on the handout. This should encourage some academic conversations between pairs of students. The handout includes some higher-order questions that help students build on their understanding of the normal distribution as a model of a data distribution to understand the normal distribution as a model for a probability distribution.
- It is important to debrief the handout with the class after students have had about 15 minutes to work on it.
- To debrief problem 1(a): Students should use the fact that the mean is 80 mm Hg and the standard deviation is 10 mm Hg to determine the values along the horizontal axis. They should use the empirical rule (68-95-99.7) to determine the number of people in each partition of the graph. The total number of people in the study, 40,000, was selected so students do not have to round any values. Expect students to explain that the normal distribution is symmetric, which means that the percent of the data and the number of people in each range are equal in corresponding intervals above and below the mean.



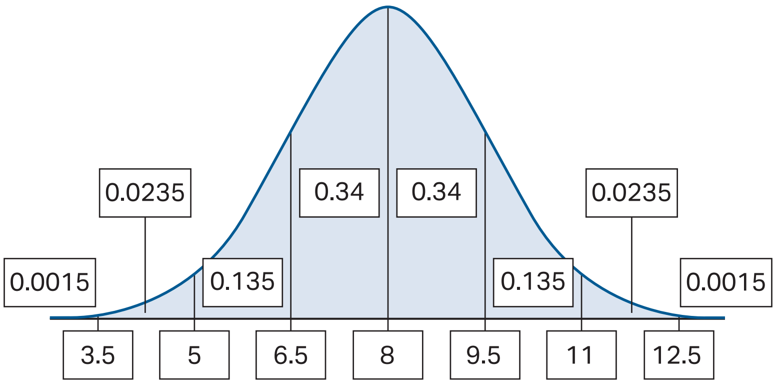
- To debrief problem 1(b): Students should identify that the probability that a randomly selected person from the study had a diastolic blood pressure higher than 80 mm Hg is equal to the probability that a randomly selected person from the study had a diastolic blood pressure lower than 80 mm Hg. They both have a probability of 0.50, because half of the values are above the mean and half are below the mean.
- To debrief problems 1(c), 1(d), 1(e), and 1(f), students should identify that the probability that a randomly selected person has a blood pressure within 1 standard deviation of the mean is 0.68, the probability that a randomly selected person has a blood pressure within 2 standard deviations of the mean is 0.95, and the probability that a randomly selected person has a blood pressure within 3 standard deviations of the mean is 0.997. Students should identify that the probabilities they determined in parts (c), (d), and (e) are a different representation of the empirical rule. Encourage students to continue to express probabilities as numbers between 0 and 1 instead of percentages.
- To summarize this part of the lesson, you can have students construct a vocabulary organizer (see Handout 3.2.C) for the normal curve. Students should identify that when a normal distribution is used as a model of a probability distribution, the probability of data occurring within one standard deviation of the mean is approximately 0.68, the probability of data occurring within two standard deviations of the mean is approximately 0.95, and the probability of data occurring within three standard deviations of the mean is approximately 0.997.

PART 2: MODELING SHOE SIZE WITH THE NORMAL DISTRIBUTION

In this part, students continue to use the empirical rule (68-95-99.7) and the symmetry of the normal curve to answer probability questions about normally distributed data sets. Unlike in Part 1, the problems in this section do not provide the total number of data values, so students rely on probabilities to complete them.

- To begin this part of the lesson, you can direct students' attention to problem 2 on Handout 1.11.A. This problem requires students to reason only with probabilities rather than with frequencies, as they did in previous lessons. It is important to debrief the handout answers to make sure that students understand how to reason using probabilities.

- To debrief problem 2(a), students should fill in the horizontal labels using the given information that women's shoe size has a mean of 8 and a standard deviation of 1.5. They should also reason that because shoe sizes are normally distributed, they can make informed decisions about the probabilities in each partition of the normal curve diagram.



- To debrief 2(b), have students explain how they determined that the probability that a randomly selected woman in the United States has a shoe size between 6.5 and 8 is 0.34. They should articulate how they used the symmetry of the normal distribution and the empirical rule to find this value.
- To debrief 2(c), have students explain how they determined that the probability that a randomly selected woman in the United States has a shoe size less than 6.5 is 0.16. Encourage students to share multiple methods for arriving at the answer. Some students might have summed the individual probabilities below size 6.5. Other students might have summed the probabilities above size 6.5 and subtracted that value from 1.

Guiding Student Thinking

Some students may not understand why you can subtract the sum of the probabilities of a shoe size greater than 6.5 from 1 to find the probability of a shoe size less than 6.5. This is a good opportunity to remind students that the sum of all of the individual probabilities in a probability distribution is 1.

- To debrief 2(d), have students explain how they determined the probability that a randomly selected woman in the United States has a shoe size greater than size 6.5. The value is 0.84. Encourage students to share multiple methods for arriving at their answer, but also to connect the solution methods for problems 2(c) and 2(d). That is, since the probabilities must sum to 1, students can reason that $P(\text{shoe size} \geq 6.5) = 1 - P(\text{shoe size} < 6.5)$.

- To debrief 2(e), have students explain how they determined that the probability that a randomly selected woman in the United States has a shoe size between 9.5 and 11 is 0.135. Students should articulate how they used the symmetry of the normal distribution and the empirical rule to determine the probability.
- For 2(f), discuss with students that they have to compare $P(\text{shoe size} > 12.5)$ and $P(\text{shoe size} < 5)$. They can reason informally that because a shoe size of 12.5 is three standard deviations away from the mean and a shoe size of 5 is only two standard deviations away from the mean, that $P(\text{shoe size} > 12.5) < P(\text{shoe size} < 5)$. Students may also determine the probabilities directly. Either way, it is more likely for a randomly selected woman in the United States to have a shoe size less than size 5 than to have a shoe size greater than 12.5. Students may also point out that either situation is relatively unlikely, because $P(\text{shoe size} > 12.5) = 0.0015$ and $P(\text{shoe size} < 5) = 0.025$.
- At this point, some students may need some additional practice using the normal distribution as a model of a probability distribution. **Handout 1.11.B: Calculating Probabilities with the Normal Distribution** can be used during class time or as an out-of-class assignment. Solutions are given in the Assess and Reflect section of the lesson. An important part of completing the assignment is allowing students to share their answers with classmates and defend them when appropriate. A debrief of this handout provides an excellent opportunity for students to engage in mathematical argumentation and academic conversation.

UNIT 1

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

An airline has a flight scheduled for 9 a.m. every morning. The flight departs early on some mornings and late on others. The airline has analyzed past departure data and determined that the distribution of departure times in minutes after 9 a.m. is approximately normally distributed with a mean of 7 minutes and a standard deviation of 4 minutes.

(a) What is the probability that on a randomly selected day, the flight will depart after 9:11 a.m.?

Since the standard deviation is 4 minutes, 9:11 a.m. is one standard deviation above the mean departure time of 9:07 a.m. From the normal distribution, 50% of flights depart at 9:07 a.m. or earlier and 34% of flights depart between 9:07 a.m. and 9:11 a.m., so 84% of all flights depart before 9:11 a.m. This means that the probability that the flight will depart before 9:11 a.m. is 0.84. Therefore, the probability that a flight will depart after 9:11 a.m. is 0.16.

(b) What is the probability that on a randomly selected day, the flight will depart between 8:59 a.m. and 9:11 a.m.?

Based on the distribution, 9:11 a.m. is one standard deviation above the mean departure time of 9:07 a.m. and 8:59 a.m. is two standard deviations below the mean departure time. Approximately 81.5% of all flights take off between 8:59 a.m. and 9:11 a.m., so the probability that the flight will depart between 8:59 a.m. and 9:11 a.m. is 0.815.

(c) If a random sample of 40 days is selected, how many flights would you expect to depart between 8:59 a.m. and 9:11 a.m.? Justify your response.

The expected number of flights departing between 8:59 a.m. and 9:11 a.m. would be the product of the sample size and the probability of departing between those times. In this case, we would expect (40)(0.815), or 32.6, planes out of 40 planes to depart between 8:59 a.m. and 9:11 a.m.

Guiding Student Thinking

Some students may not know an approach for calculating part (c). Ask students how they could apply their answer to part (b) to the information given in this part of the problem. They can use the probability they calculated from the normal distribution model to predict the number of flights they would expect to depart in a given range of times.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 1.11.A: Modeling Probability Distributions with the Normal Distribution

See lesson for answers.

Handout 1.11.B: Calculating Probabilities with Normal the Distribution

1. (a) The probability that a randomly selected fully charged cell phone battery lasts longer than 14 hours is 0.50, because the mean is 14 hours and the distribution is symmetric.

   (b) The probability that a randomly selected fully charged cell phone battery lasts between 12 and 14 hours is 0.475. Because the distribution is symmetric and roughly follows the empirical rule, the probability that the battery lasts between 12 hours, which is two standard deviations below the mean, and 14 hours, which is the mean, is $0.50 - 0.025 = 0.475$.

   (c) The probability that a randomly selected fully charged cell phone batteries lasts less than 15 hours is 0.84. Since the mean is 14 hours and the standard deviation is 1 hour, the probability that the battery lasts between 14 hours and 15 hours is 0.34. Since the mean is 14 hours, the probability that the battery lasts fewer than 14 hours is 0.50, so the probability that the battery lasts fewer than 15 hours is the sum of the these probabilities, $0.34 + 0.50 = 0.84$.

   (d) It would be surprising if a randomly selected fully charged cell phone battery lasted more than 17 hours, which is three standard deviations above the mean. Based on the empirical rule, the probability that a battery lasts between 11 and 17 hours is 0.997, so the probability that a battery lasts longer than 17 hours is $(1 - 0.997)/2 = 0.0015$.

2. (a) The probability that a baby born in the United Kingdom weighs more than 3.39 kg is 0.50. Because the birth weight data can be modeled by a normal curve, 50% of the babies will weigh more than the mean, 3.39 kg.

   (b) The probability that a baby born in the United Kingdom weighs between 2.29 kg and 4.49 kg is 0.95, because 2.29 kg is two standard deviations below the mean and 4.49 is two standard deviations above the mean.

   (c) P(baby weight < 3.94 kg) = 0.84. P(baby weight > 2.29 kg) = 0.975. Therefore, it is more likely that a baby born in the United Kingdom weighs more than 2.29 kg.

3. (a) The probability that a randomly selected teenager sleeps more than 8.5 hours per night is 0.16. Because 8.5 hours is one standard deviation above the mean, the probability based on the empirical rule is 0.16.

   (b) The probability that a randomly selected teenager sleeps less than 4 hours per night is 0.025. Because 4 hours is two standard deviations below the mean, the probability based on the empirical rule is 0.025.

   (c) The probability that a randomly selected teenager gets between 8 and 10 hours of sleep is more than 0.135. Because 8.5 hours is one standard deviation above the mean and 10 is two standard deviations above the mean, the probability of a teenager sleeping between 8.5 and 10 hours is 0.135. Because the range would be a little larger, the probability is a greater than 0.135.

PRACTICE PERFORMANCE TASK

Are Grades and Homework Connected?

OVERVIEW

DESCRIPTION

In this practice performance task, students determine if events are independent by calculating probabilities using a Venn diagram and a two-way contingency table, and use independent events to complete a probability distribution and calculate an expected value.

CONTENT FOCUS

This task is designed to assess students' understanding of different ways to calculate probabilities and to determine if they understand important concepts such as independence and expected value. It is intended to be used after students complete the lessons in Unit 1 that address Key Concept 1.2. In this task, students calculate probabilities within two events, calculate probabilities within two variables, and use a discrete random variable to find the expected value.

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~45 minutes

HANDOUT

Unit 1 Practice Performance Task: Are Grades and Homework Connected?

MATERIALS

- scientific calculator or graphing utility

UNIT 1

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ Probabilistic reasoning allows us to anticipate patterns in data.	
Learning Objectives	**Essential Knowledge**
1.2.1 Create or analyze a data display for a categorical data set.	**1.2.1a** Venn diagrams and contingency tables are common displays of categorical data and are useful for answering questions about probability. **1.2.1b** The intersection of two categories is the set of elements common to both categories. **1.2.1c** The union of two categories is the set of elements found by combining all elements of both categories. **1.2.1d** For categorical data, variability is determined by comparing relative frequencies of categories.
1.2.2 Determine the probability of an event.	**1.2.2a** The sample space is the set of all outcomes of an experiment or random trial. An event is a subset of the sample space. **1.2.2b** Probabilities are numbers between 0 and 1 where 0 means there is no possibility that an event can occur, and 1 means the event is certain to occur. The probability of an event occurring can be described numerically as a ratio of the number of favorable outcomes to the number of total outcomes in a sample space. **1.2.2c** A probability distribution is a function that associates a probability with each possible value or interval of values for a random variable. The sum of the probabilities over all possible values of the independent variable must be 1.

Learning Objectives	Essential Knowledge
1.2.3 Calculate relative frequencies, joint frequencies, marginal frequencies, or conditional probabilities for a categorical data set.	**1.2.3a** Relative frequencies are the number of times an event occurs divided by the total number of observations. They can be used to estimate probabilities of future events occurring. **1.2.3b** Joint frequencies are events that co-occur for two or more variables. They are the frequencies displayed in cells in a two-way contingency table. **1.2.3c** Marginal frequencies are events that summarize the frequencies across all levels of one variable while holding the second variable constant. They are the row totals and column totals in a two-way contingency table. **1.2.3d** The conditional probability of B, given A has already occurred, is the proportion of times B occurs when restricted to events only in A.
1.2.4 Determine if two events are independent.	**1.2.4a** Two events, A and B, are independent if the occurrence of A does not affect the probability of B. **1.2.4b** Two events, A and B, are independent if the probability of A and B occurring together is the product of their probabilities.

SUPPORTING STUDENTS

BEFORE THE TASK

In this practice performance task, students are expected to use their understanding of probability to answer a multipart question. As a warm-up, you might choose to have students answer some questions involving a Venn diagram and/or a two-way contingency table, both of which are prominent features of this practice performance task.

- To prepare students to engage in the task, you could begin the class by posing some questions to students. These questions could be part of a warm-up exercise or could be asked verbally.
    - What is a Venn diagram? What do we use a Venn diagram to display?

      A Venn diagram is a display for categorical data. Venn diagrams are helpful for showing the quantities of objects in multiple categories when they intersect and when they do not.
    - What is a two-way (or contingency) table? What do we use a contingency table to display?

      A two-way table is a display for categorical data. It is helpful for presenting the frequencies of categorical variables.
    - How do you calculate the probability of an event?

      When outcomes are equally likely, the probability of an event can be determined by calculating the ratio of the number of favorable outcomes to the number of total outcomes.

DURING THE TASK

Because this is a practice performance task, you could have students engage in the task differently than they would in a conventional assessment. The following are some possible implementation strategies:

- Students could work in pairs to complete the task. It is not recommended that students work in small groups. There is ample work and enough potential discussion areas for two students, but in groups of more than two, some students may not have an opportunity to meaningfully engage in the task.
- You could chunk the task into its four parts and have students complete one part at a time. Students could check their solutions for each part with you or the scoring guidelines before moving on to the next part. During the check, spend a few moments discussing the solution with students. Focus on what changes, if any, they could make to their solution to craft a more complete response the next time they engage in a performance task.

- Have students complete the task individually. Then distribute the scoring guidelines to students to have them score their own responses or those of a classmate. Finally, have students reflect on their solutions and the scoring guidelines and make recommendations to themselves about what they could do to craft a more complete response the next time they engage in a performance task.

AFTER THE TASK

Whether you decide to have students score their own solutions, have them score a classmate's solution, or score the solutions yourself, the results of the practice performance task should be used to inform instruction.

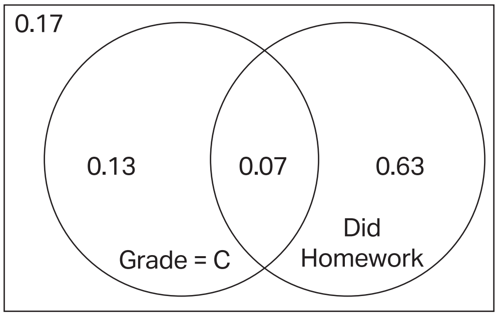
Students should understand that converting their score into a percent does not provide a good measure of how they performed on the task. You can use the suggested scoring conversion guide to discuss with students

SCORING GUIDELINES

There are 12 possible points for this performance task.

Student Stimulus and Part (a)

(a) Construct a Venn diagram of the probabilities from the teacher's survey. Based on the probabilities, are the events "course grade was a C" and "did at least 30 minutes of homework" independent? Explain your answer.

Sample Solutions	Points Possible
The following Venn diagram represents the probabilities described in the stem of the question. Since the probability that a randomly selected student's course grade was a C and the student did at least 30 minutes of homework is 0.07, the overlapping circle should be labeled 0.07. This means that the remaining part of the "Grade = C" circle is 0.13 and the remaining part of the "Did Homework" circle is 0.63. This means that 0.17 represents the probability that a student did not have either a grade of a C or did not do at least 30 minutes of homework. 	**3 points maximum** 1 point for drawing and labeling the Venn diagram correctly 1 point for concluding that the events "grade is a C" and "did at least 30 minutes of homework" are not independent 1 point for using probabilities to explain why the events are not independent *Scoring note*: Only one of the explanations of independence is necessary in order for the students to get the point for an explanation. Also, if a student incorrectly labels the Venn diagram, but uses those values correctly to find the probabilities and draw a conclusion about independence, the student should not get the first point but can still earn the remaining two points.

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The events "grade is a C" and "did at least 30 minutes of homework" are not independent, because the probability that a randomly selected student had a grade of a C and did at least 30 minutes of homework is 0.07, which is not equal to the product of P(C) and P(H), which is (0.20)(0.70) = 0.14. Also, $P(\text{C}\mid\text{H}) = P$(C and H) / P(H), which is 0.07/0.70, or 0.10. For the events to be independent, $P(\text{C}\mid\text{H}) = P(\text{C})$, but here $P(\text{C}\mid\text{H}) = 0.10$ while $P(\text{C}) = 0.20$.	

Targeted Feedback for Student Responses

If several students struggled with part (a), it could mean they need additional assistance creating a Venn diagram from a description. You could provide a labeled Venn diagram for students to complete. They could also have struggled with the analysis of the Venn diagram. For this, you can help students see that the probabilities for all parts of the diagram must sum to 1.

TEACHER NOTES AND REFLECTIONS

Student Stimulus and Part (b)

(b) The table below shows some of the relative frequencies of student responses to the teacher's survey.

	Course Grade			
	A	B	C	Total
Did at least 30 minutes of homework		0.21	0.07	0.70
Did not do at least 30 minutes of homework	0.08	0.09		0.30
Total	0.50		0.20	1.00

(i) If a student's course grade was a B, what is the probability that a randomly selected student did at least 30 minutes of homework a night?

(ii) Are the events "did at least 30 minutes of homework" and "course grade is a B" independent? Explain your answer.

Sample Solutions	Points Possible
(i) The probability that a randomly selected student did at least 30 minutes of homework given that the student's course grade is a B is the relative proportion of students that did at least 30 minutes of homework among only the students who earned a B. This is 0.21/0.30 = 0.70. (ii) From part (b)(i), P(30 minutes of homework|grade is a B) = P(30 minutes of homework). In other words, knowing a student had a grade of a B did not change the probability that the student did at least 30 minutes of homework. Therefore, the events are independent.	**3 points maximum** 1 point for calculating the probability of doing at least 30 minutes of homework given that the student's grade was a B 1 point for concluding that the events "grade is a B" and "did at least 30 minutes of homework" are independent 1 point for using probabilities to explain why the events are independent *Scoring note*: If a student incorrectly calculates the conditional probability but uses that value in the correct manner to discuss independence, the student does not get the first point but can still earn the remaining two points.

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Targeted Feedback for Student Responses
If several students make mistakes in part (b), it could mean students need help interpreting two-way contingency tables. They may need help determining which columns and rows to attend to when answering the questions.

TEACHER NOTES AND REFLECTIONS

Student Stimulus and Part (c)

(c) Based on the results from parts (a) and (b), what should the teacher conclude about the association between a student's course grade and whether or not the student did at least 30 minutes of homework?

Sample Solutions	Points Possible
For students who had a course grade of a C or an A, there appears to be an association between their course grade and doing at least 30 minutes of homework, but for students who had a course grade of a B, there does not seem to be an association between course grade and doing at least 30 minutes of homework. The proportion of students who had a course grade of C and did at least 30 minutes of homework (0.07) is half of the proportion (0.14) if the events were independent. The proportion of students who had a course grade of B and did at least 30 minutes of homework (0.21) is exactly the proportion (0.21) if the events were independent. Furthermore, the proportion of students	**3 points maximum** 1 point for indicating an association between students with a grade of a C and doing at least 30 minutes of homework 1 point for indicating there is no association between students with a grade of a B and doing at least 30 minutes of homework 1 point for concluding that not doing homework is the variable that seems to affect low-performing students. *Scoring note*: If a student incorrectly calculates the probabilities in parts (a) or (b) but interprets them correctly in part (c), the student can still earn full credit for this part.

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who had a course grade of A and did at least 30 minutes of homework (0.42) is larger than the proportion (0.35) if the events were independent. It seems that there is an association between having a course grade of an A or a C and doing at least 30 minutes of homework but there does not seem to be an association between having a course grade of a B and doing at least 30 minutes of homework.	
Targeted Feedback for Student Responses	
If several students make mistakes in part (c), it could mean students need help with the multiplication rule for independent events.	

TEACHER NOTES AND REFLECTIONS

Student Stimulus and Part (d)

(d) The school's principal is conducting some focus groups and will randomly choose 3 students from math classes. Assume that the 3 students will be chosen from the group of students who answered the survey above.

(i) What is the probability that all 3 students selected had a course grade of an A and did at least 30 minutes of homework each night?

(ii) Suppose that you know these probabilities:

- The probability that 1 student out of the 3 had a course grade of an A and did at least 30 minutes of homework each night is 0.4239.
- The probability that 2 students out of the 3 had a course grade of an A and did at least 30 minutes of homework each night is 0.3069.

What is the probability that none of the 3 students selected had a course grade of an A and did at least 30 minutes of homework each night? Explain the work leading to your answer.

Sample Solutions	Points Possible
(i) The probability that a randomly selected student's course grade is an A and the student did at least 30 minutes of homework is 0.42. Therefore, the probability that all 3 randomly selected students had a course grade of an A and did 30 minutes of homework is (0.42)(0.42)(0.42) = 0.074088. (ii) The sum of all possible probabilities for a random variable is 1. From the table, the value of none of the three students must be 1 – 0.42386 – 0.30694 – 0.074088 = 0.195112. Alternatively, the complement of the probability that a randomly selected student did not both have a course grade of an A and did 30 minutes of homework is 0.58. Therefore, the probability that no students in a group of 3 scored a grade of an A and did at least 30 minutes of homework is (0.58)(0.58)(0.58) = 0.195112.	**3 points maximum** 1 point for calculating the probability that all three students had a grade of an A and did at least 30 minutes of homework 1 point for a correct explanation of the work that leads to their answer in part (ii) 1 point for recognizing that the sum of the probabilities must be 1 and then calculating the probability that none of the students had a course grade of an A and did at least 30 minutes of homework OR 1 point for calculating the probability that none of the three students had a grade of an A and did at least 30 minutes of homework *Scoring note*: If a student arrives at an incorrect value for the probability in part (i) but uses it properly in part (ii), they can receive the second and third points. If a student incorrectly calculates the probability value in part (ii) but has a correct explanation, they can receive the third point.
Targeted Feedback for Student Responses	
If several students make mistakes in part (d), it could mean students need help with the multiplication rule for independent events.	

TEACHER NOTES AND REFLECTIONS

Points Received	Appropriate Letter Grade (If Graded)	How Students Should Interpret Their Score
11 or 12 points	A	"I know all of these probability calculations really well."
8 to 10 points	B	"I know all of these probability calculations well, but I made a few mistakes."
5 to 7 points	C	"I know some of these probability calculations well, but not all of them."
2 to 4 points	D	"I only know a few of these probability calculations."
0 or 1 points	F	"I don't know most of these probability calculations at all."

LESSON 1.12

Accuracy and Precision

OVERVIEW

LESSON DESCRIPTION

Part 1: Sampling a Text

In this part of the lesson, students explore how a sample of a population can be used to answer measurement questions about the population, especially when the population is too large for each individual to be surveyed. Some intentionally biased sampling methods are demonstrated to help students develop an understanding of the importance of randomness in sampling. Students observe that random sampling yields samples that are representative of the population.

Part 2: Defining Accuracy and Precision

In this part of the lesson, students learn about accuracy and precision and use their dot plots from Part 1 to evaluate differing levels of accuracy and precision.

AREA OF FOCUS

- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~90 minutes

HANDOUTS

Lesson

- 1.12: Sampling the Gettysburg Address

MATERIALS

- calculator
- access to **Desmos.com** or other graphing utility

CONTENT FOCUS

Students learn about three closely related concepts in this lesson: bias, accuracy, and precision. In statistics, the term *bias* describes any systematic error in data collection. One example is a weight scale that is miscalibrated, so it adds a few ounces with each measurement. Bias can also occur in a survey if the survey respondents are not representative of the entire population. Accuracy is the degree to which the data collected cluster around a central value that is close to the true value of the attribute being measured. Accuracy can be improved by increasing the number of unbiased samples from the population. Precision is a measure of statistical variability. Samples that are precise will closely cluster around each other, but their center may or may not be close to the true value. Deeply understanding these three concepts thoroughly takes time. This lesson lays some groundwork for more advanced understanding in later courses.

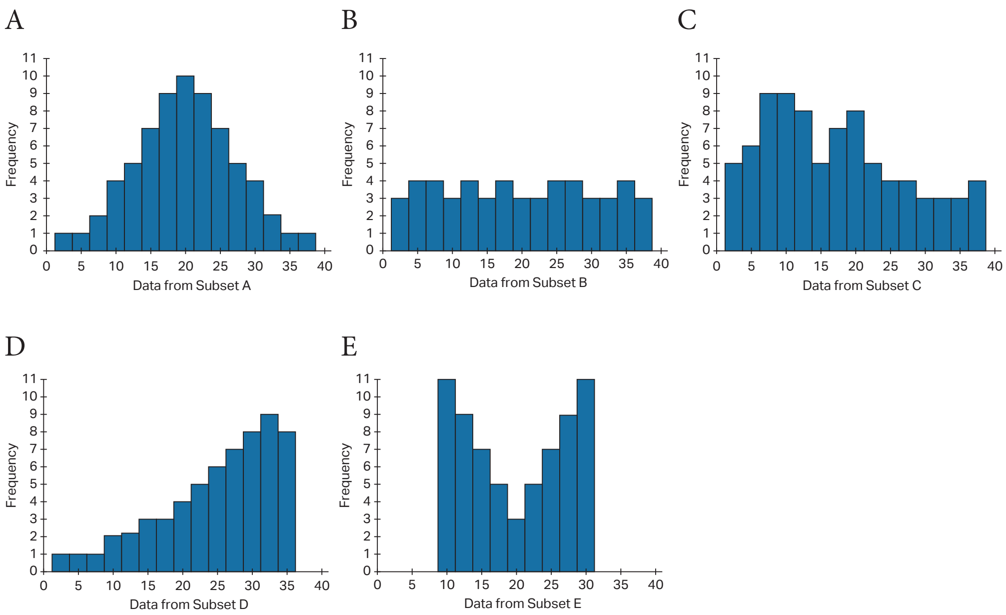
COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ The method by which data are collected influences what can be said about the population from which the data were drawn, and how certain those statements are.	
Learning Objectives	**Essential Knowledge**
1.3.1 Distinguish between accuracy and precision as measures of statistical variability and statistical bias in measurements.	**1.3.1a** Accuracy is how close the measurements in a measurement process are to the true value being estimated. Accuracy is determined by comparing the center of a sample of measurements to the true value of the measure. **1.3.1b** Precision is how close the measurements in a measurement process are to one another. Precision is determined by examining the variability of a sample of measurements. **1.3.1c** Bias is the tendency of a measurement process to systematically overestimate or underestimate the true measure of a phenomenon. Bias is an indication of the inaccuracy of the measurement process.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

In order to estimate the mean time that students spend traveling to school, a school district administrator used five different sampling techniques to create samples from the population of all students in the district. Each sampling technique was used to create multiple samples, and then the administrator calculated the average for each sample. The graphs of the averages for each sampling technique are shown below.



A district-wide survey reveals that the mean commute time for all students is 20 minutes.

(a) Which sampling techniques appear to most accurately estimate the mean commute time for all students in the school district? Explain.

(b) Which sampling techniques appear to overestimate the mean commute time for all students in the school district? Explain.

(c) Which sampling technique, B or E, is more precise when estimating the mean commute time for all students in the school district? Explain.

PART 1: SAMPLING A TEXT

In this part of the lesson, students explore how a sample of a population can be used to answer measurement questions about the population, especially when the population is too large for each individual to be surveyed. Some intentionally biased sampling methods are demonstrated to help students develop an understanding of the importance of randomness in sampling. Students observe that random sampling yields samples that are representative of the population.

- Begin this part of the lesson by letting students know the task:
  We're going to try to figure out the average length of a word in the Gettysburg Address without actually calculating the lengths of every word and finding the mean word length. This involves sampling the text.
- For students who are unfamiliar with the Gettysburg Address, you can explain that it was a famous speech delivered by President Lincoln on November 19, 1863, dedicating the National Cemetery of Gettysburg, Pennsylvania, at the site of an important battle in the American Civil War. Distribute a copy of the text of the speech to each student, which is provided on **Handout 1.12: Sampling the Gettysburg Address**.
- Split the class into small groups of three students each. Then assign each group a different sampling method. Make sure students understand that the length of a word is the number of letters it contains. There are tables on the handout that students can use to keep track of their word lengths. Sampling methods can include:
  1. Find the length of the shortest word in each line.
  2. Find the length of the longest word in each line.
  3. Find the length of the first word in each line.
  4. Find the length of the last word in each line.
  5. Find the length of the middle word in each line.
  6. Find the length of a typical length word in each line.
  7. Find the lengths of any 18 words in the speech.
  8. Find the lengths of the first nine words and the last nine words of the speech.
  9. Find the lengths of 18 words in the middle of the speech.
  10. Find the lengths of two words in every other line of the speech.
- For methods 5, 6, and 9, students should explain how they decided what "middle" or "typical" means in their sample.

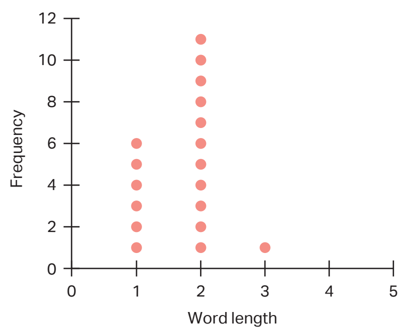
Instructional Rationale

Many of the sampling techniques listed for students to use are intentionally biased. The goal is for students to see that some methods of collecting data systematically create errors that will either over- or underrepresent the true value of the measure they want to know. The data gathered through these biased techniques will be used to generate different dot plots to help students explore accuracy and precision.

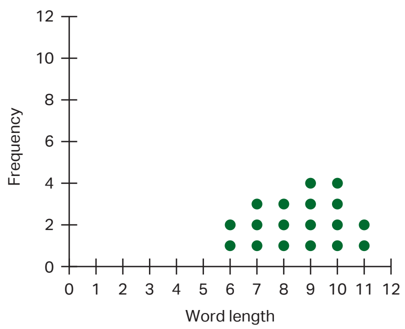
- Once students have collected their data, ask each group to create a dot plot of the word lengths. Students can create their dot plots by hand or with technology, such as using **Desmos.com**. The groups of students who used sampling method 1 will create a dot plot clustered around length 2. The groups of students who used sampling method 2 will create a dot plot clustered around length 9. The groups of students who have sampling methods 3, 4, and 5 will create dot plots that have some spread. The groups of students who used sampling method 6 will create a dot plot clustered around 4 with very little spread.

> **Meeting Learners' Needs**
> Some students may benefit from watching a short video clip or listening to an audio recording of the Gettysburg Address.

Dot Plot for Sampling Method 1

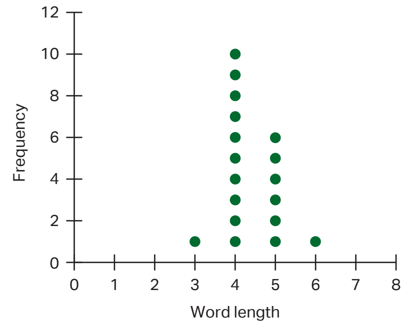


Dot Plot for Sampling Method 2



UNIT 1

Sample Dot Plot for Sampling Method 6



- Have students share their dot plots with the class. Then remind them that the task of the lesson is to find the average length of a word in the Gettysburg Address without counting the length of every word.
- Let students choose which dot plot they think shows a "good" estimate of the average word length in the Gettysburg Address and have them defend their claim. Many students may choose the dot plot from sampling method 6, but some students may have different choices and adequate defenses of them.
- Give students a chance to discuss with their groups whether any of the dot plots seem less representative of the average word length and why the sampling methods used to derive those plots might have made them less representative. It is likely that students will choose dot plots in which clusters of word lengths are low or high. For such plots, make sure that students can articulate that they resulted from sampling methods that had intentionally focused on only short words or only long words in each line. This makes their data biased. You can define the statistical meaning of *bias* for students at this point:

  Bias is the tendency of a measurement process to systematically overestimate or underestimate the true measure of a phenomenon.
- You can ask students how they could get a better estimate of the average word length without finding the lengths of every word in the speech. Students might suggest pooling several of the samples, but be sure that they are aware that including the biased samples in their data will skew the results and not allow them to have confidence in their conclusions.
- If time allows, let students pool some of the less biased samples to create a dot plot of the word length. When the plot of pooled data is complete, have them use it to estimate the average word length. You can compare their estimate to the actual average word length of about 4.3.

Guiding Student Thinking

Some students might suggest that they should choose some quantity of words from the speech at random, find the average length, and repeat that process many times. That is an excellent suggestion and describes the method that statisticians would use to sample the text.

- Students will return to these dot plots in the next part of the lesson, after they have learned the definitions of accuracy and precision. These new terms will provide students with an expanded vocabulary for evaluating the results of their sampling methods.

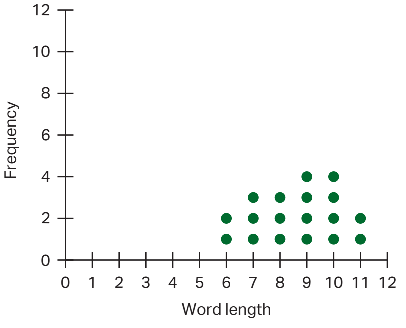
PART 2: DEFINING ACCURACY AND PRECISION

In this part of the lesson, students learn about accuracy and precision and use their dot plots from Part 1 to evaluate differing levels of accuracy and precision.

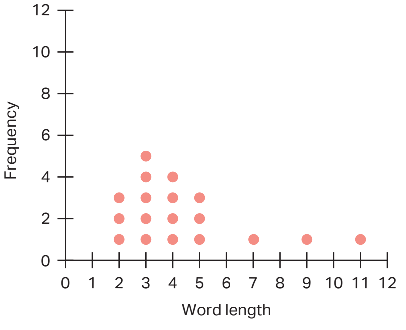
- Display the terms *accuracy* and *precision*. You can have students briefly discuss what they think the difference between the two terms is, or if they think there is no difference between them.
- After they've had some time to think and share, explain that in statistics, these terms have different definitions:
  Accuracy is how close measurements are to the true value being estimated. Precision is how close the measurements in a measurement process are to one another.
- At this point, assign students to groups of four. The groups must be able to see the dot plots that they made with samples from the Gettysburg Address. Give students their task:
  Identify which of the dot plots you made for the Gettysburg Address samples demonstrate low accuracy and which demonstrate high accuracy. Then identify which dot plots demonstrate low precision and which demonstrate high precision.
- Give students time to work on the task. As you circulate around the room, you may need to help students who are struggling with the difference between high and low accuracy and high and low precision. Plots with high accuracy will have a tight cluster of dots around the approximate mean of 4.3. Plots with low accuracy will not have many dots near the mean at all. Plots with high precision will have tight clusters of dots, but not necessarily near the mean. Plots with low precision have no clusters of dots.

Low Accuracy, Low Precision

Longest word method

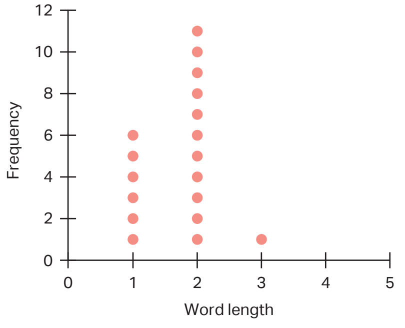


Middle 18 words method



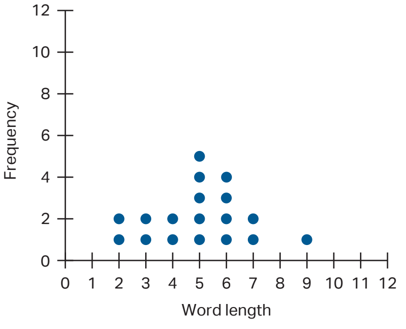
Low Accuracy, High Precision

Shortest word method

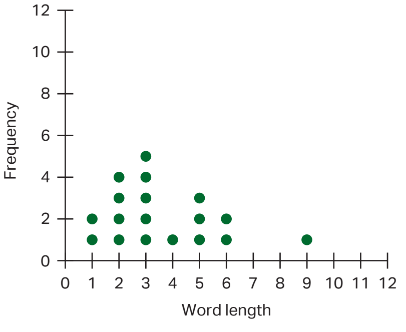


High Accuracy, Low Precision

First word method

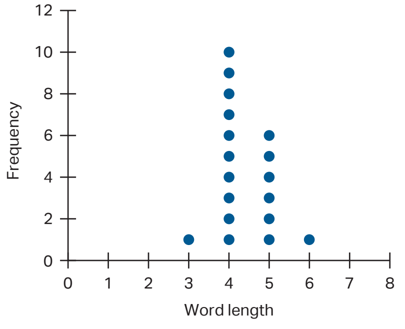


Last word method

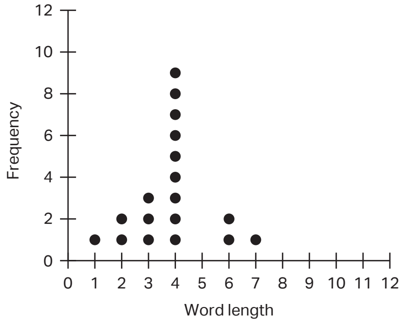


High Accuracy, High Precision

Average word method



Middle of each line method



- If students do not have examples of dot plots with high accuracy, low accuracy, high precision, and/or low precision, you can have students draw hypothetical dot plots with these characteristics.
- Next, ask students several questions and encourage them to improve their definitions on their papers during the discussion. If needed, be sure to direct students' thinking that accuracy is related to the center of the distribution and that precision is related to the variability of sample measurements.
  - What is the difference between accuracy and precision?

    Accuracy describes how close the sample measurements are to the actual value of the attribute being measured. If your sampling technique yields accurate samples, then the more samples you have, the closer the measurements are to the actual population value. Precision describes how the samples vary from each other. If your sampling method yields precise samples, then the more samples you have, the closer the measurements are to each other.
  - How could bias affect accuracy?

    Bias can overestimate or underestimate whatever you are measuring, so it could cause a sample measurement to be far from its actual value.
  - How can you improve the accuracy of samples of the attribute you are measuring?

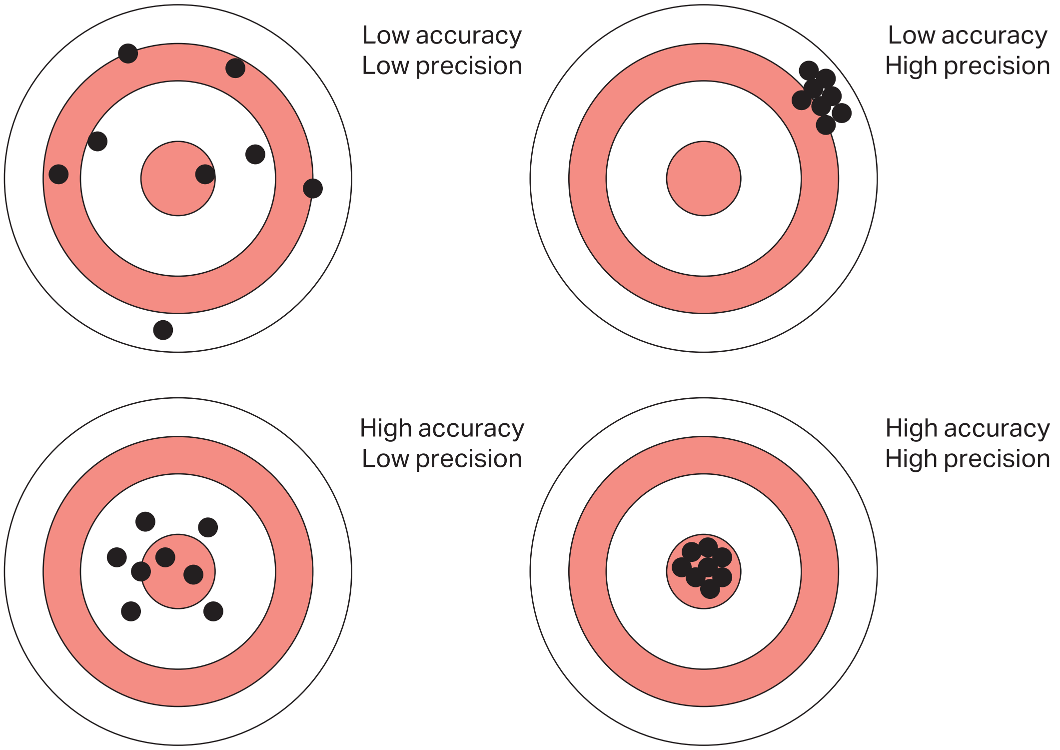
    The average of samples needs to be the average of the measurement of interest for the entire population. For example, in the Gettysburg Address problem, the average number of letters in the words from the sample should be as close as possible to the average number of letters for words in the entire speech. If it isn't, then the sample doesn't reflect the population.
  - How can you improve the precision of samples of what you are measuring?

    Precision can be improved by increasing the number of measurements in the sample if the sampling procedure is not biased. In the Gettysburg Address example, you can be more precise by increasing the number of words sampled.

- To conclude the lesson, you can share the image below. It can help students make sense of accuracy and precision.

Instructional Rationale

It is important to wait until the end of the lesson to share the image, so students have an opportunity to make sense of accuracy and precision on their own.

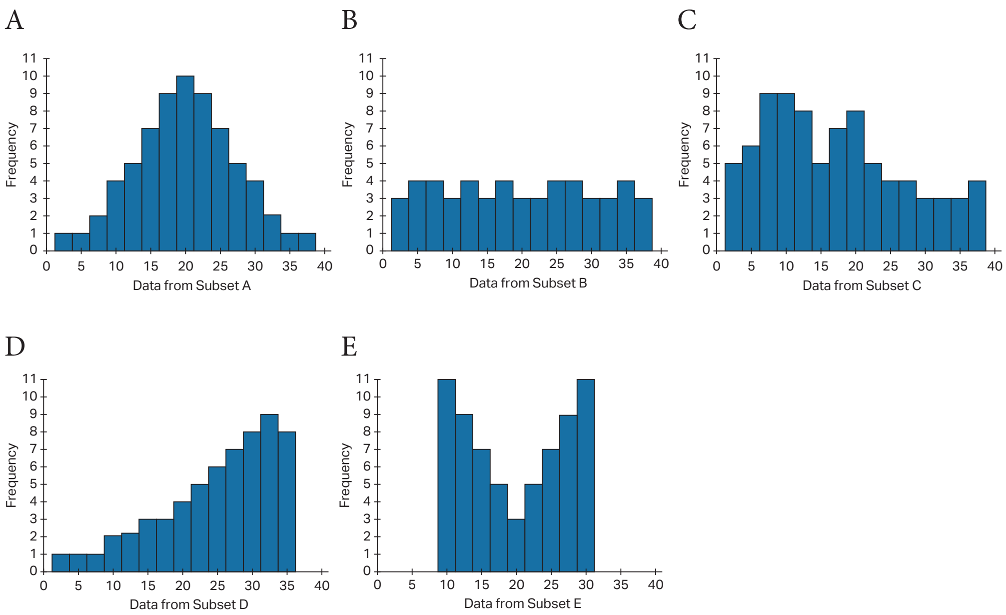


ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

In order to estimate the mean time that students spend traveling to school, a school district administrator used five different sampling techniques to create samples from the population of all students in the district. Each sampling technique was used to create multiple samples, and then the administrator calculated the average for each sample. The graphs of the averages for each sampling technique are shown below.



A district-wide survey reveals that the mean commute time for all students is 20 minutes.

(a) Which sampling techniques appear to most accurately estimate the mean commute time for all students in the school district? Explain.

Sampling techniques A, B, and E appear to accurately estimate the mean commute time for students in the school district because the centers for those graphs are approximately 20.

(b) Which sampling techniques appear to overestimate the mean commute time for all students in the school district? Explain.

The graph for sampling technique D is skewed left with most values above 20 minutes, so sampling technique D appears to overestimate the mean commute time for residents in the city.

(c) Which sampling technique, B or E, is more precise when estimating the mean commute time for all students in the school district? Explain.

The range of sample averages for sampling technique B is greater than the range of sample averages for sampling technique E. Therefore, sampling technique E provides a more precise estimate for the mean commute time for students in the school district.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 1.12: Sampling the Gettysburg Address

See lesson for answers.

Unit 1

Performance Task

PERFORMANCE TASK

Designing a Study

OVERVIEW

DESCRIPTION

In this performance task, students conduct a survey or simulation to investigate either a given research question or a research question of their own design. They analyze the results and present their conclusions to the class.

CONTENT FOCUS

In this performance task, students start by learning the features of a good research question. Then they design and execute a survey or simulation to answer a question that meets those criteria. The main goal of this performance task is for students to use what they have learned about statistics and probability to analyze data they collect and form a conclusion about the question.

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Connections Among Multiple Representations
- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~180 minutes

MATERIALS

- calculator
- graphing utility

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ Statistics are numbers that summarize large data sets by reducing their complexity to a few key values that model their center and spread. ▪ The method by which data are collected influences what can be said about the population from which the data were drawn, and how certain those statements are.	
Learning Objectives	**Essential Knowledge**
1.3.2 Describe how the size of a sample impacts how well it represents the population from which it was drawn.	**1.3.2a** The law of large numbers states that the mean of the results obtained from a large number of trials will tend to become closer to the true value of the phenomenon being measured as more trials are performed. This means we can trust larger samples more than smaller ones. **1.3.2b** The law of large numbers assumes that there is no systematic error of measurement in the sample.
1.3.3 Design a method for gathering data that is appropriate for a given purpose.	**1.3.3a** An experiment is a method of gathering information about phenomena where the independent variable is manipulated by the researcher. **1.3.3b** An observational study is a method of gathering information about phenomena where the independent variable is not under the control of the researcher. **1.3.3c** A survey is a method of gathering information from a sample of people using a questionnaire.
1.3.4 Identify biases in sampling methods for experiments, observational studies, and surveys.	**1.3.4a** Experiments can be subject to systematic bias if the experiment does not sample from the population randomly and does not randomly assign sampling units to experimental and control conditions. **1.3.4b** Observational studies can be subject to sampling bias if the sampling unit being observed is not selected randomly. **1.3.4c** Surveys can be subject to bias from several factors, including sampling bias and response bias.

BEFORE THE TASK

In preparation for the task, students spend some time exploring the components of a good research question so they are prepared to conduct a small research experiment.

Instructional Rationale

Due to the difficulty of writing a good research question and the limited amount of time available, students are offered two types of research questions to choose from for the performance task. The first type is an association question that will allow students to apply their statistical knowledge, and the other type is a simulation problem that will provide them with an opportunity to apply their understanding of simulations.

- First, let students know that it is challenging to design a good research question. There are many factors that need to be considered, and it takes time and practice. Before students begin the performance task, they will analyze why certain questions are considered good research questions.
- There are two types of questions that students can choose from for their performance task:

1. Association: Is there an association between two categorical variables measured from each participant? Some examples are:
   (a) Is there an association between the type of cereal someone eats and whether or not they drink the milk left in their cereal bowl after they have finished the cereal?
   (b) Is there an association between the age of a high school student and their response to a survey question based on its wording? For example, would students respond to the questions *Do you watch cartoons?* and *Do you still watch cartoons?* in the same way?
2. Simulation: Use a simulation to decide the likelihood that a complex event will occur. Some examples are:
   (a) Everyone who buys a raffle ticket to support the marching band will randomly receive one of three pieces of marching band merchandise: a T-shirt, a hat, or a pair of shorts. How many raffle tickets would you have to buy to get all three pieces of merchandise?
   (b) Assume you have 3,000 songs downloaded to your mobile phone. Of these songs, 50 songs are by your favorite artist. If you play the songs on random shuffle, how many songs would you have to listen to in a row before you heard a song by your favorite artist?

- For each of the examples provided on the previous page, ask students some questions like the ones below to have them analyze how it can serve as a research question:
  - Why is this a good research question?

    A good research questions is one for which we can collect or simulate data that can be interpreted and/or used to make a statistical argument about a claim, or proposed answer to the question. The answer to the question should not just be a simple statement of fact; there needs to be space to discuss and interpret the findings. For example, "Do teenagers eat cereal for breakfast?" wouldn't be a good research question because it can be answered with one survey question. A better question is, "Are teenagers more likely to drink the milk in the cereal bowl than adults?" because this requires you to survey different types of people and think about different times of day when a person might eat cereal.
  - How might you collect evidence to attempt to answer this?

    For an association question, you can conduct research surveys. For a simulation question, you can create a real-world or computer simulation.
  - Identify any type of bias that might be present in these research questions.

    Bias can occur if the questions in a survey are posed in such a way that it affects how people respond, or if the researcher does not collect data in a manner that would accurately represent the population of interest.

DURING THE TASK

In this part of the performance task, students plan and design their experiment and then complete the data collection, analysis, and presentation. This part of the task should take several days to complete.

- Students can work in pairs to complete the performance task.
- There are several parts to the performance task. Each part is described on the following page and also on the handout.

- To make sure that students know what is expected of them, have them review the scoring rubric included with the performance task.
  A. To complete Part A, students choose between a survey and a simulation and then select one of two different research questions to answer.
  B. To complete Part B, students create the instrument they will use to answer the research question. This means they design the survey questions they will ask or they design the simulation they will use. They should examine their survey instrument for potential sources of bias both before and after implementation.
  C. To complete Part C, students should collect their data by administering the survey or performing the simulation.
  D. To complete Part D, students need to analyze their data. They should determine at least three summary statistics, include an appropriate data display, discuss any symmetry or skew identified in the quantitative data, and discuss independence of probabilistic categorical data.
  E. To complete Part E, students should answer their research question by citing their results and analysis and present the results in a report. The report could be in the form of a poster, slide show, paper, or something else that you determine to be appropriate. Students should create and deliver a very brief (five minutes or less) presentation to share the results of their work. Students should ask each other whether the appropriate statistics were used and whether the evidence supports the conclusions made in the presentation.

Meeting Learners' Needs

Although there are two questions provided, you could allow students to write their own research questions. However, if students are designing their own questions, you will need to make sure the questions are of sufficient complexity for this performance task.

Classroom Ideas

After the groups have made their presentations and students have commented on each other's work, you could choose to provide each group with time to make adjustments to their analysis before their projects are scored using the rubric below.

AFTER THE TASK

After the students have presented their results and have provided verbal feedback on each other's presentations, you can score their work using the following rubric.

Rubric for Statistics Projects	Points Possible	Points Earned
Parts A and B: Research Question/Design	**3**	
There is an explanation of the research question.	1	
The instrument is designed in a way that has potential to answer the research question.	1	
Potential sources of bias are considered and addressed.	1	
Part C: Data Collection	**3**	
The data collection method is clearly described.	1	
The data collection method includes measures to reduce bias.	1	
The quantity of data collected is appropriate.	1	
Part D: Analysis	**6**	
Appropriate and correct summary statistics are used to address the research question.	2	
Appropriate and correct analysis related to symmetry/skew or independence is included.	2	
An appropriate and correct data display is included in the analysis.	2	
Part E: Conclusion and Presentation	**3**	
The conclusion clearly and correctly addresses the question of interest.	1	
Appropriate conclusions are made with supporting evidence.	1	
The report and presentation are clear and all questions are answered appropriately.	1	
TOTAL	**15**	

Points Received	Appropriate Letter Grade (If Graded)	How Students Should Interpret Their Score
13 to 15 points	A	"I know how to design a study and analyze the results."
10 to 12 points	B	"I mostly know how to design a study and analyze the results, but I made a few mistakes."
7 to 9 points	C	"I know a lot about how to design a study and analyze the results, but not all of it."
4 to 6 points	D	"I only know a little bit about how to design a study and analyze the results."
0 to 3 points	F	"I don't know much about how to design a study and analyze the results."

PART A: RESEARCH QUESTION

Choose either a survey or a simulation.

Survey options

1. Is there an association between the type of cereal someone eats and whether or not they drink the milk left in their cereal bowl after they have finished the cereal?
2. Is there an association between the age of a high school student and their response to a survey question based on its wording? For example, would students respond to the questions *Do you watch cartoons?* and *Do you still watch cartoons?* in the same way?

Simulation options

1. Everyone who buys a raffle ticket to support the marching band will randomly receive one of three pieces of marching band merchandise: a T-shirt, a hat, or a pair of shorts. How many raffle tickets would you have to buy to get all three pieces of merchandise?
2. Assume you have 3,000 songs downloaded to your mobile phone. Of these songs, 50 songs are by your favorite artist. If you play the songs on random shuffle, how many songs would you have to listen to in a row before you heard a song by your favorite artist?

PART B: DESIGN

Create a survey or simulation that you will use to answer the question. Make sure to analyze your survey or your simulation for potential sources of bias. You should assess possible biases both before and after you carry out your survey or simulation.

PART C: DATA COLLECTION

Administer your survey or perform your simulation.

PART D: DATA ANALYSIS

Analyze your data using at least three statistics. For quantitative data, you should describe the center and spread of the distribution as well as identify any skew or symmetry present in a display of the data. For categorical data, you should analyze the independence variables. You should analyze whether or not the results are biased, and examine why.

PART E: CONCLUSION AND PRESENTATION

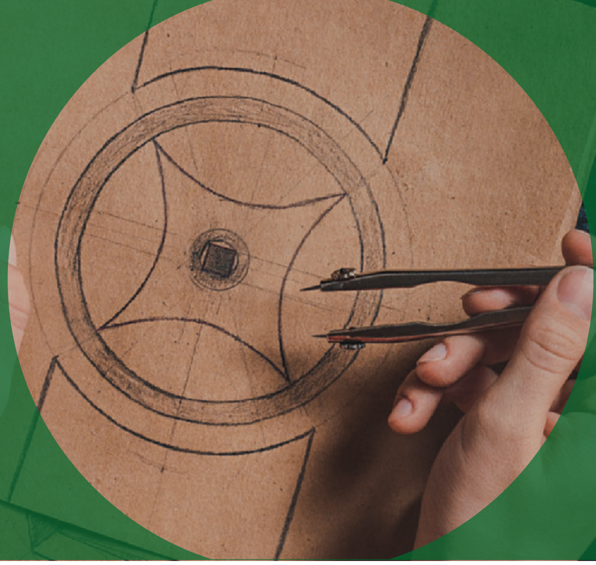
Using the evidence you gathered, provide an answer to your question. You must prepare a report that summarizes your research and its results. The report could be in the form of a poster, slide presentation, written paper, or other form approved by your teacher. You should prepare a brief presentation of your report that you can deliver to your class.

Rubric for Statistics Projects	Points Possible	Points Earned
Parts A and B: Research Question/Design	**3**	
There is an explanation of the research question.	1	
The instrument is designed in a way that has potential to answer the research question.	1	
Potential sources of bias are considered and addressed.	1	
Part C: Data Collection	**3**	
The data collection method is clearly described.	1	
The data collection method includes measures to reduce bias.	1	
The quantity of data collected is appropriate.	1	
Part D: Analysis	**6**	
Appropriate and correct summary statistics are used to address the research question.	2	
Appropriate and correct analysis related to symmetry/skew or independence is included.	2	
An appropriate and correct data display is included in the analysis.	2	
Part E: Conclusion and Presentation	**3**	
The conclusion clearly and correctly addresses the question of interest.	1	
Appropriate conclusions are made with supporting evidence.	1	
The report and presentation are clear and all questions are answered appropriately.	1	
TOTAL	**15**	

Unit 2

Unit 2

Tools and Techniques of Geometric Measurement



Overview

SUGGESTED TIMING: APPROXIMATELY 7 WEEKS

This unit introduces students to the basic objects of geometry and the tools used to explore these objects throughout the remainder of the course. The basic objects students investigate in this unit include lines, rays, segments, and angles. These figures serve as the building blocks of more complex objects that students explore in later units. Students continue to expand their understanding of measurement by developing techniques for quantifying and comparing the attributes of geometric objects. The tools they use to analyze objects include straightedges, compasses, rulers, protractors, dynamic geometry software, the coordinate plane, and right triangles. In addition, students use an informal understanding of transformations throughout the unit to justify whether two basic objects are congruent. They formalize transformations and define congruence and similarity through transformations in Unit 3. This unit culminates with an introduction to right triangle trigonometry, which integrates the tools and techniques of the unit into an investigation of new ways to express the relationship between angle measures and side lengths.

Throughout Units 2–4, specific learning objectives require students to prove geometric concepts. Students' proofs can be organized in a variety of formats, such as two-column tables, flowcharts, or paragraphs. The format of a student's proof is not as important as their ability to justify a mathematical claim or provide a counterexample disproving one. They should develop an understanding that a mathematical proof establishes the truth of a statement by combining previously developed truths into a logically consistent argument.

ENDURING UNDERSTANDINGS

This unit focuses on the following enduring understandings:

- A formal mathematical argument establishes new truths by logically combining previously known facts.

- Measuring features of geometric figures is the process of assigning numeric values to attributes of the figures, which allows the attributes to be compared.
- Pairs of lines in a plane that never intersect or that intersect at right angles have special geometric and algebraic properties.
- Right triangles are simple geometric shapes in which we can relate the measures of acute angles to ratios of their side lengths.

KEY CONCEPTS

This unit focuses on the following key concepts:

- 2.1: Measurement in Geometry
- 2.2: Parallel and Perpendicular Lines
- 2.3: Measurement in Right Triangles

UNIT RESOURCES

The tables below outline the resources provided by Pre-AP for this unit.

Lessons for Key Concept 2.1: Measurement in Geometry				
Lesson Title	**Learning Objectives Addressed**	**Essential Knowledge Addressed**	**Suggested Timing**	**Areas of Focus**
Lesson 2.1: Measuring Segments and Angles	2.1.3, 2.1.4, 2.1.5, 2.1.6	2.1.3a, 2.1.3b, 2.1.4a, 2.1.5a, 2.1.5b, 2.1.6a, 2.1.6b	~90 minutes	Engagement in Mathematical Argumentation
Lesson 2.2: Copying Line Segments and Angles	2.1.7	2.1.7a, 2.1.7b, 2.1.7c	~90 minutes	Engagement in Mathematical Argumentation, Greater Authenticity of Applications and Modeling
Lesson 2.3: Measuring Distance in the Coordinate Plane	2.1.8	2.1.8a, 2.1.8b	~45 minutes	Connections Among Multiple Representations, Engagement in Mathematical Argumentation

The following Key Concept 2.1 learning objectives and essential knowledge statements are not addressed in Pre-AP lessons. Address these in teacher-developed materials.

- Learning Objectives: 2.1.1, 2.1.2, 2.1.9
- Essential Knowledge Statements: 2.1.1a, 2.1.1b, 2.1.1c, 2.1.1d, 2.1.2a, 2.1.2b, 2.1.4b, 2.1.4c, 2.1.9a, 2.1.9b, 2.1.9c, 2.1.9d, 2.1.10d

Learning Checkpoint 1: Key Concept 2.1 (~45 minutes)

This learning checkpoint assesses learning objectives and essential knowledge statements from Key Concept 2.1. For sample items and learning checkpoint details, visit Pre-AP Classroom.

Lessons for Key Concept 2.2: Parallel and Perpendicular Lines

Lesson Title	Learning Objectives Addressed	Essential Knowledge Addressed	Suggested Timing	Areas of Focus
Lesson 2.4: Parallel and Perpendicular Lines in the Coordinate Plane	2.2.1, 2.2.2, 2.2.3, 2.2.5, 2.2.7	2.2.1a, 2.2.1b, 2.2.2a, 2.2.3a, 2.2.3b, 2.2.5a, 2.2.5b, 2.2.7a, 2.2.7b	~90 minutes	Engagement in Mathematical Argumentation, Connections Among Multiple Representations
Lesson 2.5: The Perpendicular Bisector Theorem	2.1.10, 2.2.6	2.1.10a, 2.1.10b, 2.1.10c, 2.2.6a, 2.2.6b, 2.2.6c	~90 minutes	Engagement in Mathematical Argumentation, Greater Authenticity of Applications and Modeling, Connections Among Multiple Representations

The following Key Concept 2.2 learning objectives and essential knowledge statements are not addressed in Pre-AP lessons. Address these in teacher-developed materials.

- Learning Objectives: 2.2.4
- Essential Knowledge Statements: 2.2.2b, 2.2.2c, 2.2.4a, 2.2.7c

Practice Performance Task for Unit 2 (~45 minutes)
This practice performance task assesses learning objectives and essential knowledge statements addressed up to this point in the unit.

Lessons for Key Concept 2.3: Measurement in Right Triangles

Lesson Title	Learning Objectives Addressed	Essential Knowledge Addressed	Suggested Timing	Areas of Focus
Lesson 2.6: Using Right Triangles in the Coordinate Plane	2.3.1, 2.3.2	2.3.1a, 2.3.1b, 2.3.1c, 2.3.2a	~45 minutes	Engagement in Mathematical Argumentation, Connections Among Multiple Representations
Lesson 2.7: Similarity and the Pythagorean Theorem	2.3.2, 2.3.3	2.3.2a, 2.3.3a, 2.3.3b	~90 minutes	Engagement in Mathematical Argumentation, Greater Authenticity of Applications and Modeling
Lesson 2.8: Introducing the Tangent Ratio	2.3.4, 2.3.5, 2.3.7	2.3.4c, 2.3.5a, 2.3.5b, 2.3.7a, 2.3.7b	~90 minutes	Greater Authenticity of Applications and Modeling, Engagement in Mathematical Argumentation

Lesson 2.9: The Sine and Cosine Ratios	2.3.4, 2.3.5, 2.3.7	2.3.4a, 2.3.4b, 2.3.5a, 2.3.5b, 2.3.7a, 2.3.7b	~135 minutes	Greater Authenticity of Applications and Modeling, Engagement in Mathematical Argumentation

The following Key Concept 2.3 learning objectives and essential knowledge statements are not addressed in Pre-AP lessons. Address these in teacher-developed materials.

- Learning Objectives: 2.3.6
- Essential Knowledge Statements: 2.3.6a, 2.3.6b

Learning Checkpoint 2: Key Concepts 2.2–2.3 (~45 minutes)

This learning checkpoint assesses learning objectives and essential knowledge statements from Key Concepts 2.2 and 2.3. For sample items and learning checkpoint details, visit Pre-AP Classroom.

Performance Task for Unit 2 (~45 minutes)

This performance task assesses learning objectives and essential knowledge statements from the entire unit.

UNIT 2

LESSON 2.1

Measuring Segments and Angles

OVERVIEW

LESSON DESCRIPTION

Part 1: Measuring Segments
Students engage in an activity in the synthetic (coordinate-free) plane to identify congruent segments using standard and informal rulers to measure lengths.

Part 2: Measuring Angles
Students utilize informal rigid motion transformations to determine whether two angles are congruent. Students also learn how to measure an angle in degrees with a protractor.

Part 3: Defining Congruence
Students synthesize their working definitions of congruence and continue to explore methods for determining whether two segments or two angles are congruent.

CONTENT FOCUS

This lesson provides students with their first exposure to the concept of the congruence of two geometric objects. Students should have a basic understanding of the building blocks of geometry (lines, line segments, rays, and angles), be comfortable identifying these objects, and be able to use the appropriate notation to label them. Students develop two equivalent methods for determining if two objects are congruent: (1) comparing the corresponding measures of the objects; and (2) finding a rigid motion transformation that maps one object to coincide with the other. The definition of congruence that students develop in this lesson permeates the remainder of the course. The informal work with rigid motion transformations foreshadows deeper work with transformations that students will undertake in Unit 3.

AREA OF FOCUS

- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~90 minutes

HANDOUTS

Lesson

- 2.1.A: Congruent Line Segments
- 2.1.B: Congruent Angles

Practice

- 2.1.C: Practice with Congruent Segments and Angles

MATERIALS

- unmarked straightedges or rulers
- pieces of string
- compasses
- patty paper or squares of wax paper
- protractors

COURSE FRAMEWORK CONNECTIONS

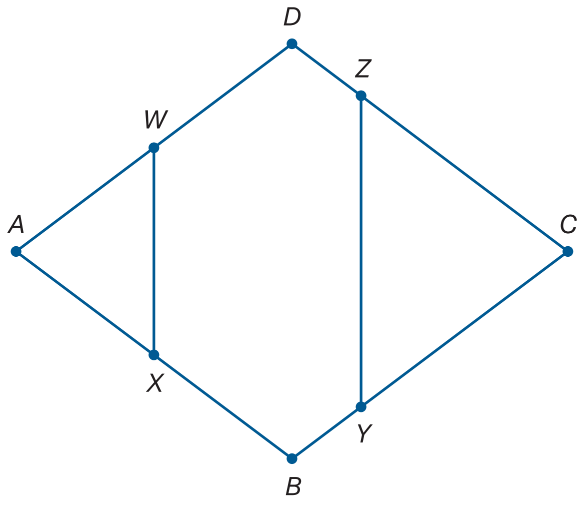
Enduring Understandings	
▪ Measuring features of geometric figures is the process of assigning numeric values to attributes of the figures, which allows the attributes to be compared.	
Learning Objectives	**Essential Knowledge**
2.1.3 Measure a line segment.	**2.1.3a** The length of a line segment is the distance between its endpoints. **2.1.3b** The length of a line segment is measured using a specified unit of measure. Units of measure can be formal or informal.
2.1.4 Measure an angle.	**2.1.4a** An angle can be measured by determining the amount of rotation one ray would make about the vertex of the angle to coincide with the other ray. The amount of rotation is measured as a fraction of the rotation needed to rotate a full circle.
2.1.5 Prove whether two or more line segments are congruent.	**2.1.5a** Two line segments are congruent if and only if one segment can be translated, rotated, or reflected to coincide with the other segment without changing the length of either line segment. **2.1.5b** Two line segments are congruent if and only if they have equal lengths.
2.1.6 Prove whether two or more angles are congruent.	**2.1.6a** Two angles are congruent if and only if one angle can be translated, rotated, or reflected to coincide with the other angle without changing the measure of either angle. **2.1.6b** Two angles are congruent if and only if they have equal measures.

UNIT 2

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

Determine which line segments and which angles in the diagram below are congruent. Explain how you know they are congruent.



PART 1: MEASURING SEGMENTS

Parts 1 and 2 of this activity invite students to consider what it means for two geometric objects to be considered "the same." In this first part of the lesson, students participate in an activity involving line segments in the synthetic plane. Throughout this course, *synthetic* refers to relationships studied in a plane without coordinates, while *analytic* refers to relationships studied in the coordinate plane.

The goal of this activity is for students to deduce that two line segments are congruent if they have the same length. Students develop this reasoning by measuring and comparing the lengths of different line segments. To encourage a robust discussion at the conclusion of the first activity, the class should use a variety of different types of measuring tools, such as unmarked straightedges, index cards, pieces of string, or compasses.

- To begin, let students know that the goal of the lesson is to determine if two geometric objects are the same. You can ask students a motivating question for the lesson:
  - What do you think it means for two objects to be the same?

    Students may suggest that two objects are the same if they have equal size or equal shape. Some students may say that the shapes have to be equal in every way. Other students may suggest that the objects must have similar features but not be identical. For example, all tables could be considered "the same" in some sense, even though most types of tables are distinct in many ways.

- If possible, before distributing **Handout 2.1.A: Congruent Line Segments**, project an image of the line segments shown on the handout and ask students to individually think of at least one thing they notice and one thing they wonder about these line segments.
- Have each student pair up with a classmate to share what they notice and what they wonder. Invite a few pairs to share their thoughts with the whole class.
- Record what students notice and what they wonder in a central location that the entire class can see. Listen for instances in which students use the word "same" in their descriptions. For example, they

Meeting Learners' Needs

Some students may need to have their own copy of the handout. If so, distribute the handout before asking them to notice and wonder.

Classroom Ideas

You can have students write what they notice and what they wonder on individual whiteboards so everyone can quickly share their thoughts with the entire class.

might suggest, "Some of the segments seem to be the same length," or "I wonder if they have the same slope." You can return to their statements later as students refine their definitions.

- Ask the class questions like the ones below to prompt them to think specifically about properties of the line segments:
  - If you were to group these line segments into pairs, what properties of the line segments could you use?

    Students may suggest properties such as the lengths of the segments, the orientations of the segments, or how close the segments are to each other.
  - How could you use the lengths of the line segments to group them into pairs?

    Student may suggest comparing the lengths of all of these line segments and seeing which have the same lengths.
  - What tools would be helpful in pairing the line segments according to length?

    Student may suggest rulers, tape measures, or other measuring tools.
- With students still in pairs, provide each pair with one measuring tool, such as an index card, a piece of string, a protractor with ruler markings, a ruler with different units, or a compass, as well as **Handout 2.1.A: Congruent Line Segments**. Make sure to distribute several different types of measuring tools throughout the class. Ask each pair of students to closely observe and analyze the line segments by sorting them into pairs in which both line segments have the same length. Students can record their line segment pairings on the handout.

Meeting Learners' Needs

One way to differentiate this activity is to pair students intentionally and assign them a particular measuring tool to match their level of comfort with these concepts. A standard ruler will be the most direct method of engaging in the activity (if students know how to use a ruler), while the compass will be the most challenging, as it can only be used to indirectly measure the length of the segments.

Instructional Rationale

It is important to have a variety of different measuring tools in use around the room. Most students will conclude that if segments have the same length, then they are "the same." A more subtle observation to make in this activity is that the segments don't need to be measured with a standard unit of measure to be compared. Students who mark off the length of a line segment on an index card to compare it to other segments can arrive at the same conclusions as students who use a standard ruler to directly measure the segments.

- Suggest that students find a way to record the lengths of the line segments they paired based on the specific tools they were assigned. For example, students might describe the length of one segment as "three-quarters of this piece of string." Students who have the compass or blank index card may struggle to define a unit of measure, so they may not have a good way to record the lengths. Let these students know their efforts are acknowledged and their findings are acceptable. This will be beneficial to the class when students share their results. Through the activity, students should understand that line segments can be identified as congruent regardless of how their lengths are measured or whether or not they are measured using standard units.
- Once students have made their pairings, ask them to share their line segment pairs and to explain why they paired them.

  $\overline{A_1B_1} \cong \overline{Q_1R_1}$; $\overline{C_1D_1} \cong \overline{E_2F_2}$; $\overline{G_2H_2} \cong \overline{M_1N_1}$; $\overline{I_2J_2} \cong \overline{U_2V_2}$; $\overline{K_1L_1} \cong \overline{S_1T_1}$; $\overline{O_2P_2} \cong \overline{W_1X_1}$; $\overline{Y_1Z_1} \cong \overline{E_1F_1}$; $\overline{A_2B_2} \cong \overline{O_1P_1}$; $\overline{G_1H_1} \cong \overline{I_1J_1}$; $\overline{C_2D_2} \cong \overline{Y_2Z_2}$; $\overline{K_2L_2} \cong \overline{W_2X_2}$; $\overline{M_2N_2} \cong \overline{U_1V_1}$; $\overline{Q_2R_2} \cong \overline{S_2T_2}$

- This is a good opportunity to reinforce proper notation for line segments and to introduce notation for congruence. A line segment with endpoints A and B is denoted by $\overline{AB}$. The measure of a line segment is also referred to as its length. Pre-AP materials will use the notation $m\overline{AB}$ to refer to the measure, or length, of line segment $\overline{AB}$. This notation might be different than the notation you have used in the past. Pre-AP materials use it to be consistent with angle measure notation.
- If students disagree about any of their segment pairings, allow them to engage in a brief academic conversation to debate and discuss why they reached different conclusions. You may need to steer the conversation toward the correct congruent segment pairs. Some students may have decided that several segments appear to be nearly congruent and decided that "close enough" was acceptable. Reinforce to students that in this activity, segment pairs need to be exactly, not approximately, the same length. Once students reach a consensus about the congruent segment pairs, they can move on to developing a definition of congruence for line segments.
- At this point, you can provide the following working definition to students:

  Two line segments are congruent if they have the same measure. When line segments $\overline{AB}$ and $\overline{CD}$ are congruent, we write $\overline{AB} \cong \overline{CD}$.

Guiding Student Thinking

It is important for students to understand that $\overline{AB}$ is a geometric object, and that $m\overline{AB}$ is a number. The following two statements are equivalent and can be used interchangeably: $\overline{AB} \cong \overline{CD}$ and $m\overline{AB} = m\overline{CD}$.

PART 2: MEASURING ANGLES

In this part of the lesson, students determine whether two angles are congruent by comparing the angles' measures. Previously, students used various measuring tools to determine the lengths of line segments and then compared those lengths to determine which segments were congruent.

The goal of this activity is for students to compare the measures of angles without using a protractor. They can trace the angles on patty paper and physically manipulate the traced angles until one angle is on top of the other. This activity makes use of informal understandings about rigid motion transformations that students developed in middle school. Specifically, students learned that if one shape can be moved so that it coincides exactly with another shape—without bending, stretching, or otherwise deforming either shape—then the two shapes are congruent. At the end of this part of the lesson, you can introduce students to a protractor as a tool for measuring angles.

Classroom Ideas
If you have difficulty obtaining patty paper, this activity can be completed with any type of translucent paper with which students can trace the angle.

- Let students know that they will be examining angles rather than segments. If possible, you can project an image of **Handout 2.1.B: Congruent Angles** for the class to see. Ask questions like the ones below to prompt students to think about angles. Through this discussion, you should steer students toward the understanding that the measure of an angle is a measure of the relationship between the sides of the angle.
  - Suppose that we wanted to sort these angles based on their measures. What feature or property should we measure for each angle?

    Students may suggest measuring the lengths of the sides or the space between the sides.
  - If we don't have a tool for measuring an angle, how could we figure out which ones are congruent?

    Students may suggest using a protractor (even though the question specifies not using a tool) or finding a method for matching up the angles.

Instructional Rationale

Students might be reluctant to engage in this part of the activity if they already know how to use a protractor. The reason to refrain from giving students protractors to directly measure the angles is to encourage them to continue to develop some informal understandings about rigid motion transformations. The understandings they begin to cultivate in this lesson will be further developed in Lesson 2.4 when they use transformations of slope triangles to justify the relationships between the slopes of parallel and perpendicular lines in the coordinate plane. These informal investigations will support students in learning the formal definitions of rigid motion transformations in Unit 3.

- If students are not still paired up from the line segment activity, have them reform their pairs. Distribute **Handout 2.1.B: Congruent Angles** and patty paper to each pair of students. Ask each pair of students to closely observe and analyze the angles by sorting them into congruent pairs. Students can record their angle pairings on the handout.
- Students may need a hint for how to use the patty paper to indirectly measure the angles. You could recommend that they trace each angle. If they need further hints, guide them to try placing their tracings of an angle on top of another.

Guiding Student Thinking

Several of the angles are congruent even though the sides of the angles have different lengths. Encourage students to consider the relationship between the sides, not the lengths of the sides, as they match up the angles.

- Once students have made their pairings, ask them to share their angle pairs and to explain why they paired them.

  $\angle A \cong \angle J$; $\angle B \cong \angle L$; $\angle C \cong \angle E$; $\angle D \cong \angle F$; $\angle H \cong \angle N$; $\angle G \cong \angle I$; $\angle M \cong \angle K$
- This is a good opportunity to reinforce proper angle notation. An angle can be denoted using an angle symbol and its vertex. An angle with vertex A would be notated as $\angle A$. However, this method can create confusion if more than one angle shares the same vertex. An angle can also be identified using points that lie on each side of the angle and the vertex. An angle with vertex B and points A and C on either side of the angle would be notated as $\angle ABC$.
- If students disagree about any of their congruent angle pairs, allow them to engage in a brief academic conversation to debate and discuss why they reached different conclusions. You may need to steer the conversation toward the correct congruent

angle pairs. Some students may have decided that several angles appear to be nearly congruent and decided that "close enough" was acceptable. Reinforce to students that angle pairs need to be exactly the same measure for this activity. Once students reach a consensus about the congruent angle pairs, they can move on to developing a definition of congruence for angles.

- At this point, you can provide the following working definition to students:

  Two angles are congruent if one angle can be moved so it coincides exactly with the other without bending or stretching the angle. When angles $\angle A$ and $\angle B$ *are congruent, we write* $\angle A \cong \angle B$.

PART 3: A DEFINITION FOR CONGRUENCE

In this part of the lesson, students try to work with and reconcile the two working definitions of congruence they learned in parts 1 and 2. By the end of this part of the lesson, students should understand that these definitions are equivalent and can be used for both segments and angles. That is, if two segments have the same measure, it is also true that one of the segments can be moved so it coincides with the other segment without stretching or bending either segment. Therefore, the segments are congruent. Likewise, if two angles can be moved so either coincides with the other angle without bending or stretching, the two angles have the same measure. Therefore, the angles are congruent.

- Have students reflect on the two working definitions of congruence that they have so far by asking questions such as the ones below. They should discuss their answers to each question with a partner.

  1. *Two line segments are congruent if they have the same measure. When two line segments* $\overline{AB}$ *and* $\overline{CD}$ *are congruent, we write* $\overline{AB} \cong \overline{CD}$.
  2. *Two angles are congruent if one angle can be moved so it coincides exactly with the other without bending or stretching either angle. When two angles* $\angle A$ and $\angle B$ *are congruent, we write* $\angle A \cong \angle B$.

  - Are the two working definitions of congruence the same? Can you use aspects of the line segment definition of congruence for angles? Can you use aspects of the angle definition of congruence for line segments? Why or why not?

- As students discuss their answers with a partner, listen for students who think that the definitions are the same and also for students who do not think the definitions are the same. When you bring the class together for a debrief, make sure to ask students on both sides to share their reasoning. You only have to let them discuss their positions for a minute or two.

- After the student discussion, lead a whole-group debrief about the two working definitions. The important mathematical understanding to elicit from the students is that the two working definitions are the same and can be combined into a single definition of congruence:

  If two segments or angles are congruent, then they have the same measure and one of the segments or angles can be moved so it coincides with the other segment or angle without bending or stretching. If one segment or angle can be moved so it coincides with another segment or angle, then the segment pair or angle pair are congruent and have the same measure.

- At this point, some students may still be unconvinced, because they do not have a standardized tool for measuring angles. As a way to convince students that two angles have the same measure, you can introduce the protractor to students.
- If you choose to teach students about protractors now, you can also introduce the notation for the measure of an angle. The notation for the measure of $\angle A$ is $m\angle A$.

Guiding Student Thinking

It is important for students to understand that $\angle A$ is a geometric object, but $m\angle A$ is a quantity. The following two statements are equivalent and can be used interchangeably: $\angle A \cong \angle B$ and $m\angle A = m\angle B$.

- For some practice, you can provide students with more complicated geometric shapes and ask them to determine which segments and angles (if any) in the figures are congruent. **Handout 2.1.C: Practice with Congruent Segments and Angles** has some examples of shapes. At this point in the course, students may already know the names of several of the shapes, but you do not have to dwell on the names. It would be appropriate to let students know the names if they ask, or to correct them if they use incorrect terminology.

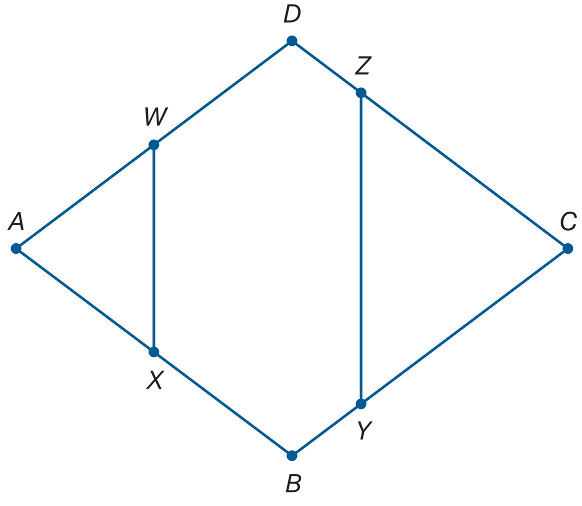
UNIT 2

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

Determine which line segments and which angles in the diagram below are congruent. Explain how you know they are congruent.



The congruent angles are $\angle A \cong \angle C$; $\angle B \cong \angle D$; and $\angle CZY \cong \angle CYZ \cong \angle AWX \cong \angle AXW$. Students should explain that they measured the angles and compared the measures or that they traced and matched the angles.

The congruent segments are $\overline{AD} \cong \overline{AB} \cong \overline{BC} \cong \overline{CD}$. Students should explain that they measured the segments and compared their lengths or that they traced and matched the segments.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information in the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 2.1.A: Congruent Line Segments

See lesson for answers.

Handout 2.1.B: Congruent Angles

See lesson for answers.

Handout 2.1.C: Practice with Congruent Segments and Angles

1. $\overline{IK} \cong \overline{JL}$; $\overline{KL} \cong \overline{IJ}$; $\angle I \cong \angle L$, $\angle IKL \cong \angle LJI$; $\angle IKJ \cong \angle LJK$; $\angle LKJ \cong \angle IJK$
2. $\overline{EF} \cong \overline{GH}$; $\angle A \cong \angle C$
3. $\overline{PQ} \cong \overline{PT} \cong \overline{ST}$; $\overline{QR} \cong \overline{RS}$; $\angle TPQ \cong \angle QRS \cong \angle STP$; $\angle PQR \cong \angle RST$

UNIT 2

LESSON 2.2

Copying Line Segments and Angles

OVERVIEW

LESSON DESCRIPTION

Part 1: Understanding that Rulers and Protractors Are Not Precise
Students extend their work with measuring tools from the previous lesson to discover that standard rulers and protractors are not the best tools for creating a congruent copy of a line segment or an angle.

Part 2: Exploring the Distance Copier and the Point Connector
Students engage in an activity to explore the compass as a distance copier and the straightedge as a point connector.

CONTENT FOCUS

This lesson is the first opportunity for students to learn about and practice with the basic synthetic construction tools: the straightedge and the compass. A major component of this lesson is highlighting the functionality of these tools. Some students may know that a compass can be used to draw a circle. This is because the primary use of a compass in geometry is to copy a distance, and all points on a circle are the same distance from its center. A straightedge is used to construct a straight line, often by connecting two points.

Many students are already familiar with using a ruler as a straightedge, but they should also understand that a straightedge does not include any measurement markings. For example, students might have used the edge of a book to draw a straight line in Algebra 1.

AREAS OF FOCUS

- Engagement in Mathematical Argumentation
- Greater Authenticity of Applications and Modeling

SUGGESTED TIMING

~90 minutes

HANDOUTS

Lesson

- 2.2.A: Copying Segments and Angles

Practice

- 2.2.B: Practice Copying Segments and Angles

MATERIALS

- rulers or straightedges
- protractors
- patty paper
- compasses
- index cards

Students should understand that a ruler is a type of straightedge that includes measurement markings.

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ A formal mathematical argument establishes new truths by logically combining previously known facts. ▪ Measuring features of geometric figures is the process of assigning numeric values to attributes of the figures, which allows the attributes to be compared.	
Learning Objectives	**Essential Knowledge**
2.1.7 Construct a congruent copy of a line segment or an angle.	**2.1.7a** A synthetic geometric construction utilizes only a straightedge and a compass to accurately draw or copy a figure. **2.1.7b** A straightedge is a tool for connecting two distinct points with a line segment. **2.1.7c** A compass is a tool for copying distances between pairs of points.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

> Use your straightedge to create an angle formed by two line segments. Then use your straightedge and compass to create an exact copy of the line segments and angle you created.

PART 1: UNDERSTANDING THAT RULERS AND PROTRACTORS ARE NOT PRECISE

In this part of the lesson, students explore using rulers and protractors as tools for creating exact copies of segments and angles. By the end of Part 1, they should discover that marked rulers and protractors are not the best tools for creating a congruent copy of a line segment or an angle because neither tool provides the necessary level of precision.

Guiding Student Thinking

How successful students are at creating congruent copies of segments and angles will depend largely on the precision of measuring tools that the students use. Many classroom rulers have inch markings every $\frac{1}{8}$ inch or centimeter markings every $\frac{1}{2}$ centimeter. Likewise, many protractors only have markings every 5°. If your classroom rulers or protractors are more precise than that, students will experience greater success in creating congruent segments and may not be convinced that a ruler or protractor is an inadequate tool for copying segments and angles. In this part of the activity, it is better for students to fail at creating congruent copies of the segments and angles, because that motivates the need for a more precise method of creating copies. If you need more imprecise tools, there are printable rulers and protractors available at **https://www.printablerulers.net/**.

- Let students know that the goal of this activity is to try to create congruent line segments and congruent angles using rulers and protractors. You should have students first focus on the segments and then on the angles.
- Pair up the students, and provide each pair with one copy of **Handout 2.2.A: Copying Segments and Angles**, two rulers, two protractors, and some patty paper. One student's role will be the "measurer," and the other's role will be the "copier." The measurer will determine the length of a given segment and tell the copier what length to make the copy of that segment. The copier should not look at the ruler and should rely only on the value that the measurer reports. Then, the copier should try to create a congruent segment on the patty paper. When the copier thinks they have created a congruent segment, they should try to match up the copied segment with the handout. Encourage students to use the conventional notation of $\overline{A'B'}$ to name their attempted copy of $\overline{AB}$.
- There are four line segments on **Handout 2.2.A: Copying Segments and Angles**. You could have students alternate roles so that each student has a chance to measure and copy two segments.

- Students should experience varying levels of success copying the segments. While students are working, ask them some questions like the ones below to get them to reflect on the process of attempting to copy a line segment. You do not need to debrief as a class right now, but you will after the students try to copy the angles.
  - Why do you think that your "congruent" line segments are not exactly congruent? What would help you create an exact copy of the segment?

    Students should identify that it was difficult to figure out the precise length of the segments because the ruler did not have enough markings, so it was not possible to communicate the exact length to the copier. Students might suggest that a more precise ruler would help them figure out the correct length of the original segment.

Classroom Ideas
To make sure the copier doesn't look at the ruler or protractor themselves, you can have students sit back-to-back or place a textbook or folder vertically as a barrier between the students.

- Once students have tried to copy the line segments exactly, ask them to try to copy the angles exactly. Use the same process as before with students in pairs, with one serving as measurer and the other serving as copier. Again, you can have students switch roles so each measures and attempts to copy two angles.
- Continue to ask students about their methods for trying to copy the angles while they are working. Encourage students to use the conventional notation of $\angle A'$ to name their attempted copy of $\angle A$.
  - Why do you think that your pairs of "congruent" angles are not exactly congruent? What would help you create an exact copy of the angle?

    Students should identify that it was difficult to figure out the precise measure of the angle because the protractor did not have enough markings, so it was not possible to communicate the exact degree measure to the copier. Students might suggest that a more precise protractor would help them figure out the correct measure of the original angle.
- Once students have tried without success to copy the angles and have thought about their method for trying to copy an angle, bring the class back together. Lead a whole-class discussion about the successes and failures of trying to copy both segments and angles.

- Through the discussion, try to elicit from students that the tools they had were not precise enough to make congruent copies of these geometric figures. You may need to emphasize for students that congruent segments are exactly the same length, not almost the same length. Likewise, congruent angles are exactly the same measure, not almost the same measure. Often, in the real world, constructing only almost the same measurement is not good enough and can have serious consequences. You could cite examples where cost or safety depends on exact measurements, like landing a spacecraft on another planet, or building a bridge or overpass where construction must begin on both sides simultaneously and meet in the middle.

Once students identify that the tools they had were not precise enough to create congruent segments or angles, it will be time to introduce them to straightedge and compass constructions.

PART 2: EXPLORING THE DISTANCE COPIER AND THE POINT CONNECTOR

In Part 1, students concluded that it is difficult to exactly copy line segments and angles with rulers and protractors because these tools lack the necessary precision. In this part of the lesson, students start working with two simple tools that can be used to copy distances. Many students probably know one of these tools is called a compass, but they may think it is only used to construct circles and arcs. In this part of the lesson, we define this tool by what it does: copy distances. Students also work with a straightedge to connect points, rather than use the tick marks of a ruler to measure the length.

Instructional Rationale

A standard compass has an arm with a sharp point used to anchor it on paper and an arm with a pencil used to draw arcs and circles. The sharp end may cause concern for some teachers and students. It is possible to find compasses with a rounded tip instead of a pointed tip through standard school or office supply companies. Such compasses do not cost much more than standard compasses. Some classrooms may be equipped with “safety compasses” that look more like rulers that can be rotated. These kinds of safety compasses do not copy distances the same way a standard (or rounded-tip) compass does and cannot be used for this activity. For this reason, Pre-AP strongly encourages the use of standard or rounded-tip compasses and discourages the use of safety compasses.

- Let students know that the goal for this activity is to learn how to use two new tools to create congruent copies of line segments and angles.

- Introduce a device called a "distance copier." Some students may know that this tool is commonly called a compass, and later in the lesson you will refer to it as such. Explain that the arms of the distance copier can be adjusted so the distance between the tips of its arms is the same as the length of a line segment.
- Also introduce the "point connector," which students already know as a ruler or straightedge. If students are using a ruler, let them know that in this part of the lesson, they will not use any of its markings but will use its "straight edge" to draw line segments that connect two points.

Classroom Ideas
At this point in the lesson, you could explain to students that the compass and straightedge have been used to investigate geometric relationships for thousands of years. There is more information about classical constructions available here: **mathworld.wolfram.com/GeometricConstruction.html.**

- At this point, you could model how to use the distance copier on one of the line segments from **Handout 2.2.A: Copying Segments and Angles** or could show a short instructional applet or video, like **mathopenref.com/constcopysegment.html**, to help students figure out how to use the distance copier. Alternatively, you could have students explore this tool without much explanation.
- Have students pair up again and make sure each pair has a compass, a straightedge, and more patty paper. Just like in the first part of the lesson, one student should be the measurer and one student should be the copier. This time, the measurer should use the arms of the compass to indirectly measure the length of the segment from **Handout 2.2.A: Copying Segments and Angles**. Instead of telling the copier the length of the segment, the measurer should hand the compass to the copier, who should use it, along with a straightedge, to draw a congruent line segment on the patty paper. When the copier thinks they have a congruent segment, they should match the endpoints of the copy to the original. Encourage students to use the conventional notation for the endpoints of the copied line segment, which includes the original endpoint names and prime symbols.
- Allow students enough time to copy all four segments. Have students trade roles so each student measures and copies two segments.
- While students are working, ask them some questions to get them thinking about the differences between determining the length of a line segment with a compass and with a ruler.
    - How is the "distance copier" different than a ruler? How is it the same as a ruler? Which one will help you create a congruent copy of a line segment? Why?
    - What are the steps you used to create a congruent segment?

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- After they copy all the line segments successfully, spend a few minutes debriefing as a large group. Ask students to share their experiences with the "distance copier" and their answers to the questions you asked.

  Students should identify that a ruler was too imprecise to exactly copy the length of the given segments, but the arms of the "distance copier" could be opened to the precise distance and then used to mark how far away the endpoints of the copied segments should be.

- Ask some students to share the steps involved in copying a line segment. Display these steps prominently in the classroom.

  (1) Mark a point not on the segment.

  (2) Set the ends of the compass arms to span the length of the given line segment.

  (3) Move the fixed end of the compass to the point you marked.

  (4) Make an arc of any length.

  (5) Mark a point on the arc.

  (6) Connect the two points you marked.

- At this point, you can let students know that the "distance copier" is commonly called a compass.

Classroom Ideas

If you have a word wall, you can create a section specifically for geometric constructions so all students have access to these steps at all times. You could also have students set aside a section of their notebook for listing the steps of the constructions they learn. If you are in a one-to-one school where all students have a laptop or tablet, you could have students bookmark the instruction applet.

Meeting Learners' Needs

The word *compass* has several definitions. Some students might know a compass as device used to indicate direction. The geometric construction definition of the word is different and might cause confusion.

- After you debrief with the class about the congruent segment construction, you should model how to use the compass and straightedge to copy an angle, or you can show a short instructional applet or video, like **mathopenref.com/constcopyangle.html**. Copying an angle is more complex than copying a segment, so some direct instruction will probably be necessary for this task.

- Have students pair up again to practice copying the angles on **Handout 2.2.A: Copying Segments and Angles**. Instead of telling the copier the angle measure, the measurer will have to pass the compass back and forth to the copier, because two different distances must be measured for a congruent angle construction. When the copier thinks they have constructed a congruent angle, they should use patty paper to match it to the original. Encourage students to use the conventional notation for the copied angle, which includes the original vertex name and a prime symbol.

- Allow students enough time to copy all four angles. Have students trade roles so each gets an opportunity to measure and copy two angles.
- While students are working, ask them some questions to get them thinking about the difference between determining the angle measure with a protractor and copying an angle with a straightedge and compass.
  - How is the "distance copier" different than a protractor? How is it the same as a protractor? Why do you think it is possible to create a congruent copy of an angle without knowing its measure?
  - What are the steps you used to create a congruent angle?
- After students copy all the angles successfully, spend a few minutes debriefing as a large group. Ask students to share their experiences with the "distance copier" and their answers to the questions you asked.

  Students should indicate that a protractor was too imprecise to exactly determine the angle measure, but the "distance copier" can be used to copy the angle without knowing its measure.

- Ask some students to share the steps involved in copying an angle. Display these steps prominently in the classroom.

  (1) Given $\angle ABC$ to copy, mark a ray, $\overrightarrow{XY}$, to be one side of the copied angle.

  (2) Set the fixed end of the compass at the vertex of $\angle ABC$, and set the compass opening to any distance.

  (3) Mark an arc that intersects each side of $\angle ABC$, and label the points of intersection M and N.

  (4) Without changing the compass opening, move the fixed end of the compass to point X, and mark an arc that intersects $\overrightarrow{XY}$ at a point to be labeled S.

  (5) Move the fixed end of the compass to point M, and set the compass opening so it spans from point M to point N.

  (6) Move the fixed end of the compass to point S, and make another arc that intersects the arc made in step 4 at a point to be labeled T.

  (7) Draw a ray from point X through point T.

Guiding Student Thinking

The two constructions students do in this part of the lesson are very different. The reason the segment copying construction creates a congruent segment is because that is exactly what a compass does: copy distances. However, the reason the angle copying construction creates a congruent angle is because the construction marks vertices of congruent isosceles triangles. At this point in the course, students should not be expected to formally justify the reason why the construction works; they should do that in Unit 3. Right now, it is enough for them to argue that because they figured out how wide to make the opening of the angle, they can be sure that the two angles are congruent.

- At this point, you can share with students that the term *construction* refers to a procedure that uses tools, such as a compass and straightedge, to allow us to build a specific geometric object based on properties of another object.

Students need some time and purposeful practice to become fluent with the functions of the compass and straightedge. Use **Handout 2.2.B: Practice Copying Segments and Angles** as either an in-class or an out-of-class assignment to help students hone these skills.

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

Use your straightedge to create an angle formed by two line segments. Then use your straightedge and compass to create an exact copy of the line segments and angle you created.

Students' copies will vary depending on their original angles.

Guiding Student Thinking

If students are struggling with the steps for copying a segment or an angle, you can point them to the videos or applets you used during class. Some students will benefit from repeated viewings of the applet and working slowly through the steps.

HANDOUT ANSWERS AND GUIDANCE

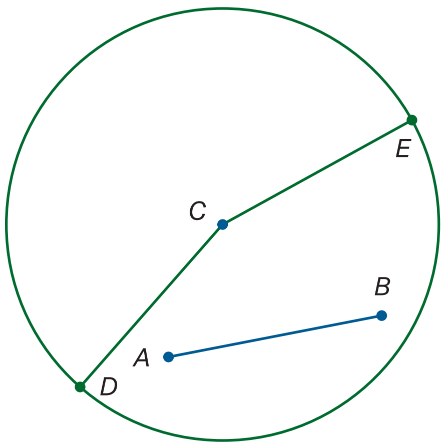
To supplement the information in the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 2.2.A: Copying Segments and Angles

This handout is used for students to learn about copying segments and angles.

Handout 2.2.B: Practice Copying Segments and Angles

1. Points D and E should lie on a circle whose center is C and whose radius is $\overline{AB}$, as shown below. Students do not need to construct the circle to construct the congruent segments.



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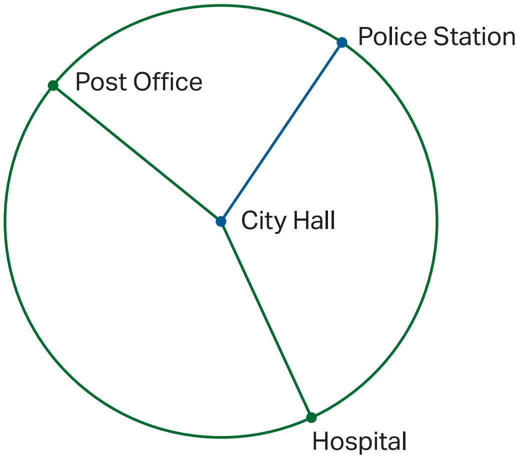
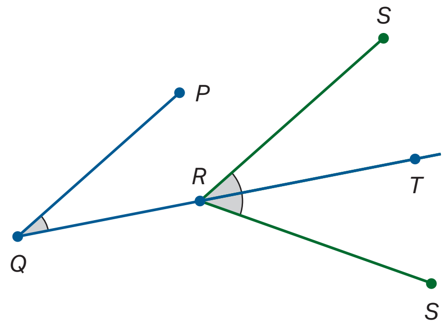
It is true that $\overline{CD} \cong \overline{CE}$. Because $\overline{CD}$ and $\overline{CE}$ were constructed to be congruent to the same segment, $\overline{AB}$, they must be congruent to each other as well.

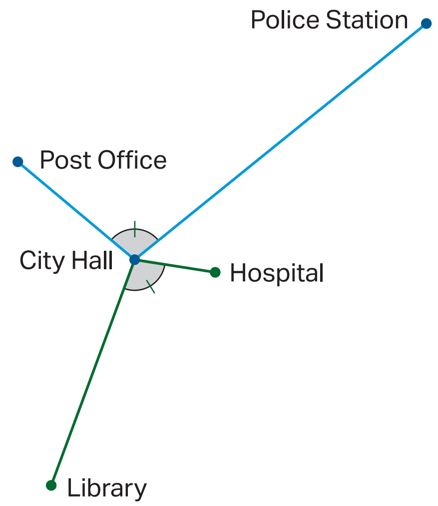
2. The post office and hospital can be located anywhere on a circle whose radius is set by the distance between city hall and the police station. Every point on the circle will be equidistant from city hall.



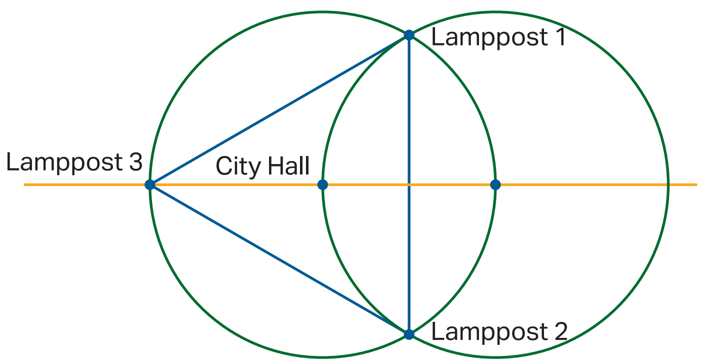
3. There are at least two possible angles $\angle SRT$ that can be constructed so that $\angle SRT \cong \angle PQR$. Students only have to construct one of these angles.



4. The angle formed by the hospital, city hall, and the library must be congruent to the angle formed by the post office, city hall, and the police station. One possible solution is shown below. Note that the distances from city hall to the other locations do not have to be the same.



5. The lampposts must be in the formation of an equilateral triangle. The vertices of the triangle must lie on a circle whose center is city hall.



This can be constructed in the following way:

(1) Draw a line with a straightedge, and label one point on it as city hall.

(2) Open a compass to any opening, and draw a circle whose center is city hall.

(3) With the same compass opening, draw a second circle whose center is either of the intersections of the first circle and the line.

(4) Use the straightedge to connect the two points where the circles intersect.

(5) Use the straightedge to connect each of the two circle intersection points from step 4 to the other intersection point of the line and the first circle (that is, the one not used as the center of the second circle). The lampposts are equidistant from each other and from city hall.

Guiding Student Thinking

Problem 5 is intended to be an extension of the concept of constructing congruent segments. Students should try the problem on their own first but may need to collaborate with a peer or seek assistance from you before being able to complete the construction.

LESSON 2.3

UNIT 2

Measuring Distance in the Coordinate Plane

OVERVIEW

LESSON DESCRIPTION

Part 1: Finding Exact Distances without a Ruler
Students investigate a scenario that utilizes their understanding of the Pythagorean theorem in context.

Part 2: Calculating Distance in the Coordinate Plane
Students calculate the distance between points in the plane using an auxiliary right triangle and build the distance formula by examining the structure of their own calculations.

Part 3: Practice with the Distance Formula
Students have an opportunity to use the Pythagorean theorem and the distance formula in problem-solving scenarios.

AREAS OF FOCUS

- Connections Among Multiple Representations
- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~45 minutes

HANDOUTS

Lesson

- 2.3.A: Running Through the Park
- 2.3.B: Finding Distance in the Coordinate Plane

Practice

- 2.3.C: Problem Solving with the Distance Formula

MATERIALS

- graph paper

CONTENT FOCUS

The most important understanding for students to develop about the distance formula, $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$, is that it is an application of the Pythagorean theorem in the coordinate plane. Students determine that the length of a horizontal or vertical line segment can be calculated as the difference of the x-coordinates or y-coordinates respectively. However, any nonvertical, nonhorizontal line segment in the coordinate plane can be thought of as the hypotenuse of a right triangle with legs that are horizontal and vertical. In such cases, the quantities (x_2-x_1) and (y_2-y_1) are the lengths of the horizontal and vertical legs of the triangle. Therefore, the length of this line segment—the hypotenuse of this triangle—is the square root of the sum of the squares of the lengths of the legs of the right triangle.

COURSE FRAMEWORK CONNECTIONS

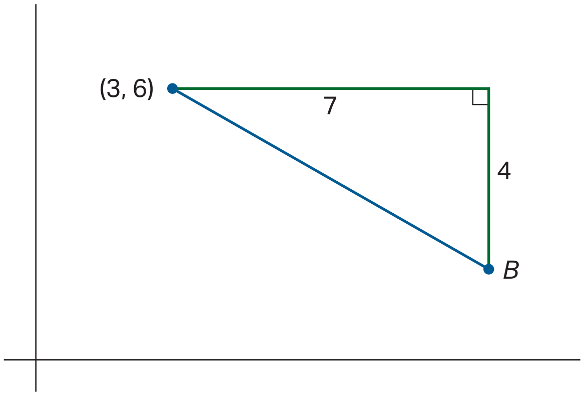
Enduring Understandings	
▪ Measuring features of geometric figures is the process of assigning numeric values to attributes of the figures, which allows the attributes to be compared. ▪ Right triangles are simple geometric shapes in which we can relate the measures of acute angles to ratios of their side lengths.	
Learning Objectives	**Essential Knowledge**
2.1.8 Calculate the distance between two points.	**2.1.8a** The distance between two points in the plane is the length of the line segment connecting the points. **2.1.8b** The distance between two points in the coordinate plane can be determined by applying the Pythagorean theorem to a right triangle whose hypotenuse is a line segment formed by the two points and whose sides are parallel to each axis.

UNIT 2

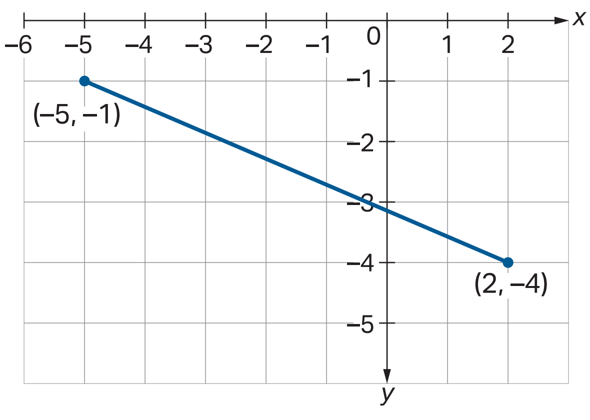
FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

1. A line segment that is the hypotenuse of a right triangle is shown below. The lengths of the vertical and horizontal legs are also shown. What are the coordinates of point *B*? Explain how you know.



2. Determine the length of the line segment shown below. Draw a right triangle to help you find the length.



3. The distance between points *P* and *Q* is 17 units. Point *P* is located at $(5,-5)$, and point *Q* is located at $(-3,y)$. What could be the value of *y*?

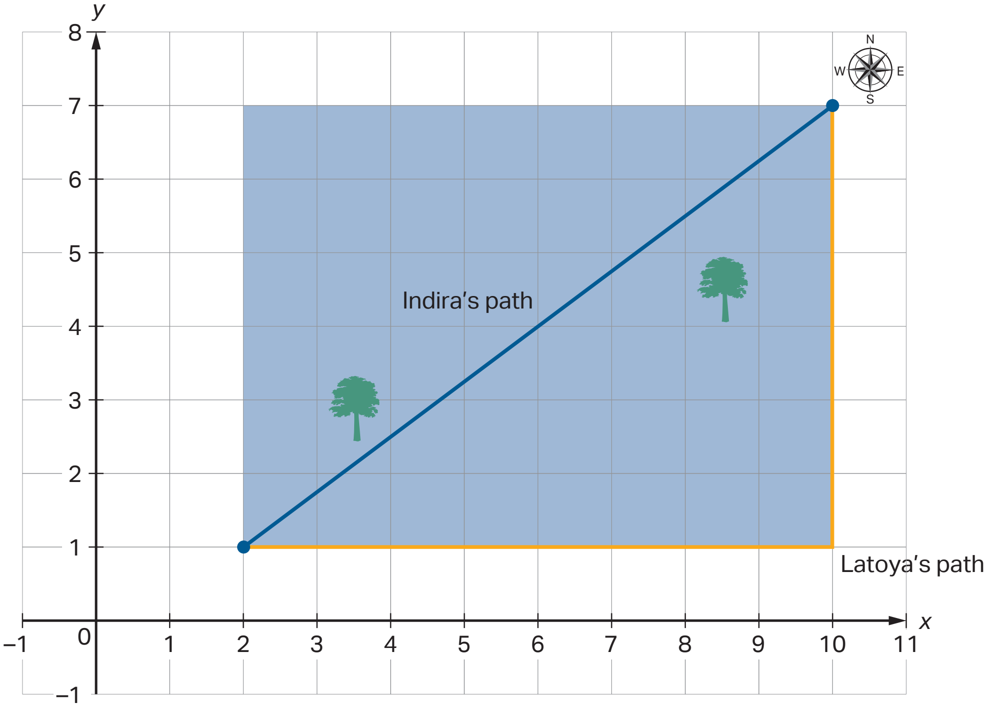
PART 1: FINDING EXACT DISTANCES WITHOUT A RULER

In this part of the lesson, students are presented with a scenario in which two people run between the same two locations but use different routes. Students model this scenario by finding the distance between two points in a coordinate plane without using rulers. This investigation leverages students' understanding of measuring distance without a ruler and their knowledge of the Pythagorean theorem. The first part of the lesson sets up the second part, in which students develop the distance formula.

- Begin by presenting the scenario below to students. The scenario and its graph are on **Handout 2.3.A: Running Through the Park**. To allow you to control the pace of the investigation, the handout does not have all the questions on it.

Running Through the Park

Indira and Latoya are going to run from the southwest corner of a park to the northeast corner, but each will take a different route. Indira decides to run through the park along a straight path, while Latoya decides to run along the edge of the park. A scale drawing of the park is shown here. Each unit represents 10 yards.



Handout 2.3.A

- Allow some time for the students to closely observe and analyze the prompt and the diagram. Ask them what they notice and what they wonder. Then you can ask them to make a conjecture.
    - Who do you think is going to run the longer distance, Indira or Latoya? Why do you think that?
- Let students have a few moments to make hypotheses about who will run the longer distance. Make sure to have students give a reason for their hypotheses to distinguish their reasoning from pure guesswork.
- Listen for students to say things like, "It's always shorter to cut through than to run around," or "The shortest distance between two points is a straight line." These responses will help you to identify which students already have a good sense of spatial relationships. Also listen for students who think that running around the edge will be shorter to identify which students will need to be convinced of the correct answer.
- Let students know that they will now actually determine who runs the longer distance. You can choose to pair students or have them work individually.
- While students are working, take note of the methods they use to find the distances that Indira and Latoya run. Many students will recognize that Latoya's path includes horizontal and vertical segments, making it possible to determine the length of Indira's path because it is the hypotenuse of a right triangle whose legs are the segments of Latoya's path. This means they will be able to use the Pythagorean theorem to determine the distance between the southwest and northeast corners of the park.

Meeting Learners' Needs

Some students may remember the formula for the Pythagorean theorem but not recall that the order in which its operations are performed is important. This is a good opportunity to review the order of operations and why we use it.

- After a few minutes of work time, have one or two students share their solutions and methods. It is important to have a whole-class debrief to make sure there is consensus about the solution and acceptable solution methods.

On the scaled map of the park, Latoya runs $6 + 8 = 14$ units, which converts to 140 yards. Indira runs $\sqrt{6^2+8^2}=\sqrt{100}=10$ units, which converts to 100 yards. Therefore, Latoya runs the longer distance, 140 yards, which is 40 yards more than Indira.

UNIT 2

- Continue to debrief with the class using some higher-order questions to encourage students to reflect on the process of solving the problem. Questions like the ones below will help prepare students for the next part of the lesson.
  - Whose distance did you figure out first? Why?

    I figured out Latoya's distance because it was easy to count.
  - How did you use Latoya's distance to figure out Indira's distance?

    If the path Latoya followed formed the legs of a right triangle, then Indira's path was its hypotenuse. So I used the Pythagorean theorem to find the distance Indira ran.
  - Suppose that we only wanted to find the distance Indira ran. Would you have figured it out the same way?

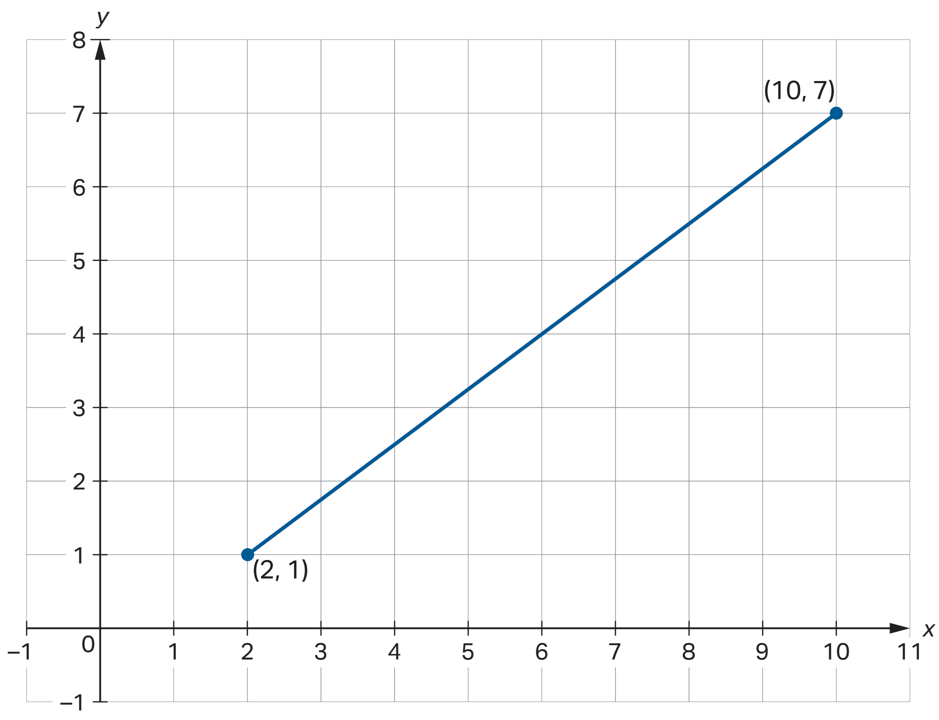
    Probably. Indira's distance isn't vertical or horizontal, but you can see how it is part of a right triangle. So, you can use the Pythagorean theorem.

PART 2: CALCULATING DISTANCE IN THE COORDINATE PLANE

In this part of the lesson, students formalize the ideas they started to develop in Part 1 about calculating distance in the coordinate plane using the coordinates of points. The culmination of this part of the lesson is the distance formula. While it is easier and faster to provide students with the formula and have them practice using it, that approach often leaves students with a fleeting and/or incomplete understanding of the concepts underlying the distance formula. Having students engage in an investigation in which they build the formula themselves, using their understanding of the Pythagorean theorem, is more likely to lead to durable knowledge about distance.

- Begin by having students closely observe a line segment in the plane on **Handout 2.3.B: Finding Distance in the Coordinate Plane**. Give students some time to notice and wonder about the image.

Finding Distance in the Coordinate Plane



Handout 2.3.B

Instructional Rationale

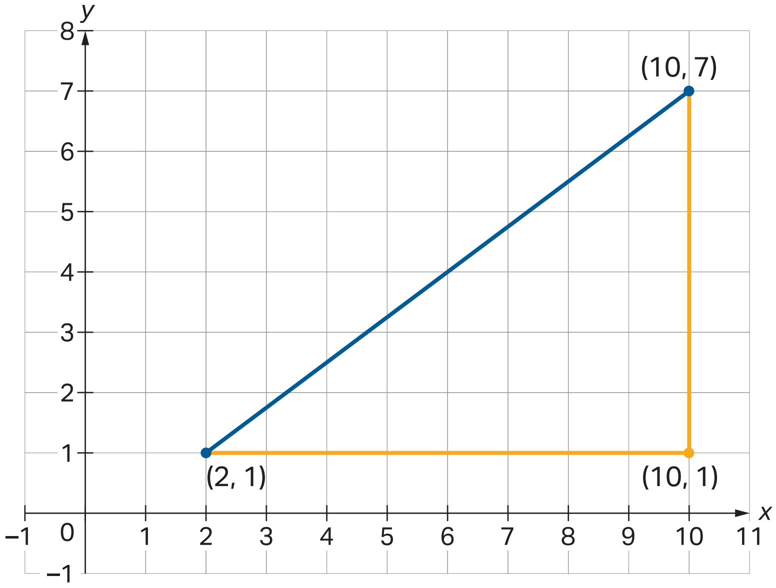
The goal is for students to use what they learned in Part 1 to build an understanding of distance. It is intentional that the segment shown here is the same as Indira's path. Students already know the distance between these points, so they can focus their attention on how to determine the length of the segment when they are not also given segments that serve as legs of a right triangle.

- If no students notice that the line segment is the same as Indira's path from Part 1, you may need to prompt them to look back at **Handout 2.3.A: Running Through the Park**.
  - How is this diagram the same as the diagram in Part 1? How is it different from the diagram in Part 1?

    The coordinates of the endpoints of Indira's path are the same, but the park is not shown.

- What did we do to find the distance between these points? Could we do the same thing again?

- Students might struggle to answer the second question, but they don't have to be able to answer it correctly at this point. Students may not know that they can draw additional lines or segments into a diagram to help them answer questions. These helpful additions to the drawing are called *auxiliary lines*.
- Ask students to draw in the horizontal and vertical legs of a right triangle whose hypotenuse is the segment shown. Their drawing should look like the one plotted below. Some students might draw the legs of the triangle above the segment such that the right angle is at (2, 7). That is a congruent alternative.



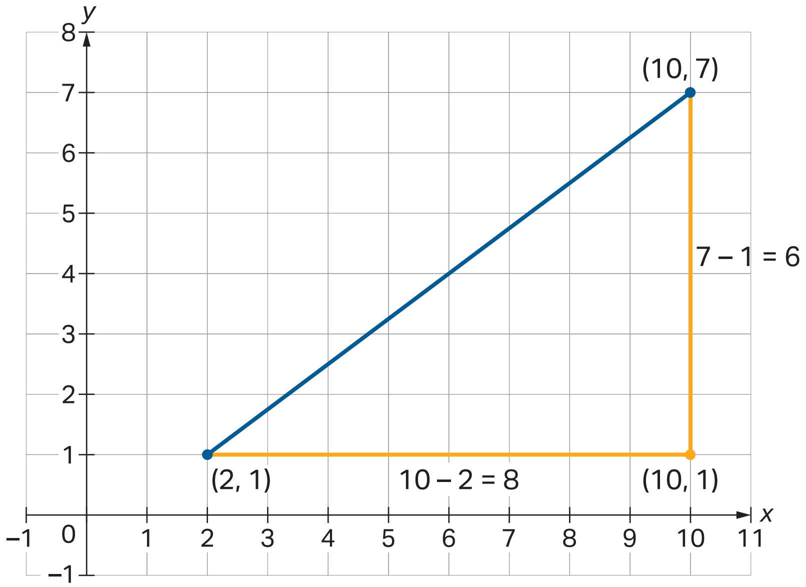
- Ask students to determine the length of the horizontal leg and to figure out a way to calculate its length using the coordinates of its endpoints.

  The length is 8 units. You can find its length by subtracting the x-coordinates of its endpoints: $10 - 2 = 8$.

- Ask students to determine the length of the vertical leg and to figure out a way to calculate its length using the coordinates of its endpoints.

  The length is 6 units. You can find its length by subtracting the y-coordinates of its endpoints: $7 - 1 = 6$.

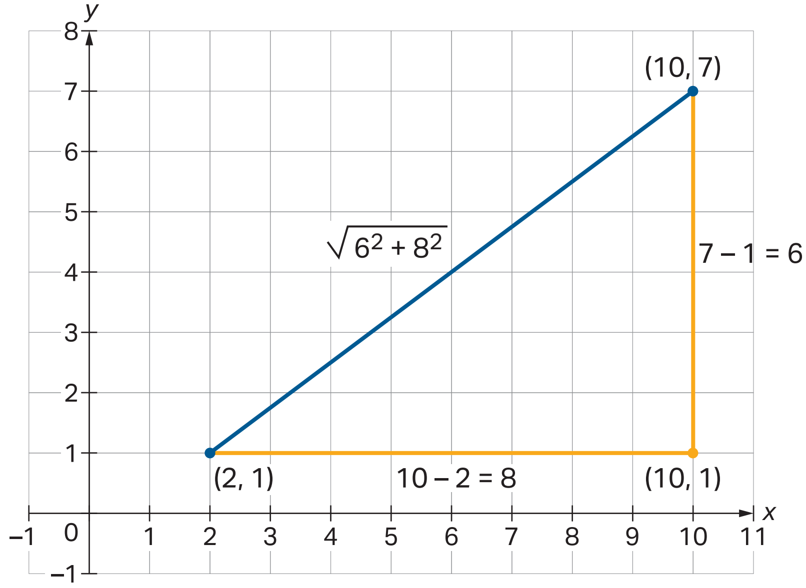
- Encourage students to show the relationships between these values by labeling the diagram as shown.



- Now ask students to use the information they have about the legs of the triangle to determine the length of the hypotenuse of the triangle. Make sure to remind students that their goal is to find the length of the segment, which is also the hypotenuse of the right triangle formed with the two legs.

  The length of the hypotenuse is $\sqrt{6^2+8^2}=\sqrt{100}=10$

- Continue to label the diagram so students continue to make connections between the coordinates, the triangle, and the calculations. Then ask students the following questions to summarize what they did.



- What did we do to find the distance between the points (2, 1) and (10, 7)?

  We drew in a right triangle where the segment was the hypotenuse. Then we found the length of the legs of the triangle to use the Pythagorean theorem.

- How can we represent the length of the hypotenuse using all of the calculations we did in a single equation?

  If we let h be the length of the hypotenuse, then $h = \sqrt{(10-2)^2 + (7-1)^2}$

Guiding Student Thinking

Having students figure out how to combine all of their work into one equation is a vital step toward constructing a general distance formula, but it will probably take them some time. You can help students understand where its components come from by building the equation piece by piece. One way to support student thinking is by writing out a sequence of actions in words. For each action, ask students to write out the corresponding expression or equation without simplifying it. It may seem strange to students to not perform the operations, but if they perform the operations they will not see the structure of the distance formula emerge in the examples.

UNIT 2

- Help students develop a single equation for the hypotenuse by having them write out a list of the operations they performed. You may have to demonstrate for them how to write their operations in mathematical language and how to combine the operation expressions into a single equation for the hypotenuse. You can use a table like the one shown here to organize the steps.

Actions	Expressions/Equations
Subtract the x-coordinates to find the horizontal leg length.	$10 - 2$
Subtract the y-coordinates to find the vertical leg length.	$7 - 1$
Write the equation for the Pythagorean theorem.	(horizontal leg length)^2 + (vertical leg length)^2 = (hypotenuse length)^2
Substitute your expressions for the leg lengths into the Pythagorean theorem equation.	$(10 - 2)^2 + (7 - 1)^2 = h^2$
Solve for h by taking the square root of both sides of the equation.	$h = \sqrt{(10-2)^2 + (7-1)^2}$

- Most students will need to see at least one or two more examples before being able to write a general formula for the distance between two coordinate pairs. You can do several more worked examples with your students before working through a generalization with them.
- You can have students work through additional examples individually or in pairs. If you have time, you can assign different sets of coordinate pairs to students so there is some variation in solutions while still sharing the same method.
- Before moving on to a generalized solution, debrief with the class by having several students share their solutions and methods with the class. An important part of this sharing is having each student conclude

Meeting Learners' Needs

Some students may benefit from a brief refresher on how to simplify the square root of a real number. This would be an appropriate time to provide students with an opportunity to practice this skill.

by writing the distance as a single equation. Students should be able to observe patterns in a few examples of these distance equations and make a generalization.

- After students share their work, ask them to think about a general method they could use in each case. As a class, write out the sequence of steps along with the corresponding general mathematical expression or equation, just as you did with the first example.

Classroom Ideas

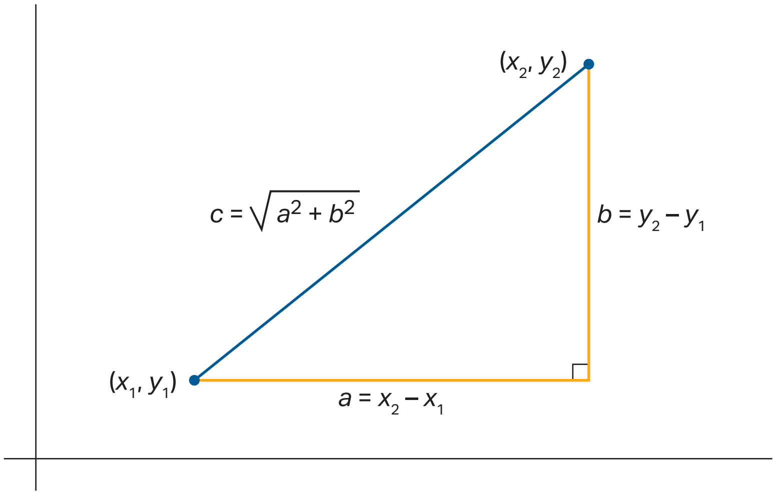
One way to get students to share their solutions and equations quickly is to have them do their work on a personal whiteboard or on large chart paper. Having several examples displayed at the same time will help students observe patterns common to all solutions.

Actions	Expressions/Equations
Subtract the *x*-coordinates to find the horizontal leg length.	$x_2 - x_1$
Subtract the *y*-coordinates to find the vertical leg length.	$y_2 - y_1$
Write the equation for the Pythagorean theorem.	(horizontal leg length)^2 + (vertical leg length)^2 = (hypotenuse length)^2
Substitute your expressions for the leg lengths into the Pythagorean theorem equation.	$h^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$
Solve for *h* by taking the square root of both sides of the equation.	$h = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Guiding Student Thinking

Writing out the steps involved in finding the distance between two general coordinate pairs will help demystify the distance formula for students. Sometimes they do not understand that while the formula may look complicated, it is in fact just a sequence of operations shown together.

- It can be helpful to draw a diagram like this one to help students make connections between the lengths, the coordinates, the calculations, and the triangle.



- At this point, formally define the distance formula for students.

  The distance, d, between two points in the coordinate plane, (x_1, y_1) and (x_2, y_2), can be calculated with the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Challenge students to consider whether using the distance formula requires them to plot the points or draw a right triangle to determine the distance between two coordinate pairs. Some students will see that the distance formula does not require them to plot the points or draw a right triangle. Other students may need more time and practice before coming to this realization. Part 3 of the lesson is an opportunity for some of that practice.

PART 3: PRACTICE WITH THE DISTANCE FORMULA

This part of the lesson provides students with an opportunity to practice using the distance formula in some problem-solving situations. It is not the goal of these problems for students to simply practice the raw skill of using the distance formula; such problems can be found in other places, like in a textbook or online. Instead, the problems on **Handout 2.3.C: Problem Solving with the Distance Formula** are designed to deepen students' understanding of the connection between the Pythagorean theorem and the distance formula. You could choose to use these problems in class or assign them to be completed out of class.

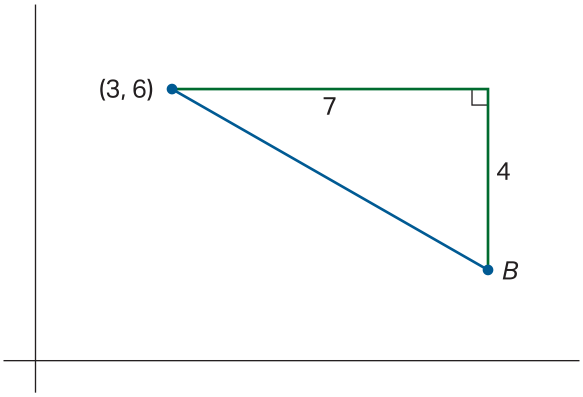
UNIT 2

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

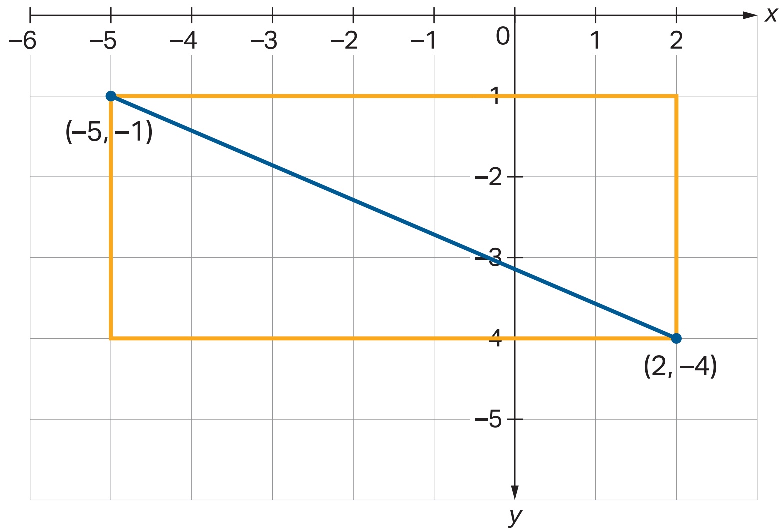
When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

1. A line segment that is the hypotenuse of a right triangle is shown below. The lengths of the vertical and horizontal legs are also shown. What are the coordinates of point *B*? Explain how you know.



The coordinates of *B* are (10, 2). I know because the horizontal distance from the upper left point to *B* is 7 units, so I added 3 + 7 to find the *x*-coordinate of *B* because it is to the right of (3, 6). The vertical distance from the upper left point to *B* is 4, so I subtracted 6 – 4 to find the *y*-coordinate of *B* because it is below (3, 6).

2. Determine the length of the line segment shown below. Draw a right triangle to help you find the length.



The length of the line segment is $\sqrt{(-5-2)^2+(-1-{}^-4)^2}=\sqrt{49+9}=\sqrt{58}$.

3. The distance between points P and Q is 17 units. Point P is located at $(5,-5)$, and point Q is at $(-3,y)$. What could be the value of y?

The value of y could be either 10 or −20. Either value will produce a coordinate pair for Q such that the distance between points P and Q is 17 units.

Guiding Student Thinking

Using the distance formula in problem 3 in the formative assessment goal may require students to solve a quadratic equation. Some students will need some guidance to work through that example. A similar problem is given as question 4 on **Handout 2.3.C: Problem Solving with the Distance Formula**. If students know Pythagorean triples, then both problems could be solved using their knowledge of right triangles instead.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 2.3.A: Running Through the Park

See lesson for solutions.

Handout 2.3.B: Finding Distance in the Coordinate Plane

See lesson for solutions.

Handout 2.3.C: Problem Solving with the Distance Formula

1. (a) Point B' is $\sqrt{85}$ units from point A.

   (b) Point B' is $\sqrt{13}$ units from point B.

2. (a) Segment $\overline{AD}$ is $\sqrt{128}$ or $8\sqrt{2}$ units, which is approximately 11.314 units. Segment $\overline{BC}$ is $\sqrt{98}$ or $7\sqrt{2}$ units, which is approximately 9.899 units.

   (b) The perimeter of quadrilateral $ABCD$ is $2\sqrt{29}+2\sqrt{109}$, which is approximately 31.651 units.

3. The distance between the plotted points is $\sqrt{(9.9)^2+(-9.6)^2}=\sqrt{190.17}\approx 13.790$ units. Each unit represents 500 feet, so City Hall and the Philadelphia Museum of Art are approximately 500(13.790) ≈ 6,895 feet apart.

4. The value of x could be –66 or 54. Either value of x will give a distance between (–6, 8) and (x, –3) of 61 units.

5. There are infinitely many coordinate pairs possible for N, because there are infinitely many points (located on a circle) that are 13 units from M (–2, 7). Any point (x, y) that satisfies the equation $\sqrt{\left(x-{}^{-}2\right)^2+\left(y-7\right)^2}=13$ will be 13 units from M.

LESSON 2.4

UNIT 2

Parallel and Perpendicular Lines in the Coordinate Plane

OVERVIEW

LESSON DESCRIPTION

Part 1: Reviewing Slopes of Parallel and Perpendicular Lines
Students engage in two card sorting activities to activate their prior knowledge about the slopes of parallel and perpendicular lines.

Part 2: Justifying the Relationships between Slopes of Parallel and Perpendicular Lines
Students explore the relationship between the slopes of parallel and perpendicular lines in the coordinate plane through patty paper manipulations.

Part 3: Extending the Slope Relationship to Line Segments
In the final part of this lesson, students use their understanding of parallel and perpendicular lines in the coordinate plane to determine if two line segments are parallel, perpendicular, or neither.

CONTENT FOCUS

In Algebra 1, students learned that parallel lines in the coordinate plane have equal slopes or are both vertical lines. They also learned that perpendicular lines in the coordinate plane have opposite reciprocal slopes, or one line is horizontal and the other is vertical. In this lesson, students utilize this prior knowledge to confirm and informally justify these relationships using transformations.

Earlier in the course, students used translations, reflections, and rotations to argue that two line segments or two angles are congruent. Students now use the concept of translation to analytically construct a line

AREAS OF FOCUS

- Engagement in Mathematical Argumentation
- Connections Among Multiple Representations

SUGGESTED TIMING

~90 minutes

HANDOUTS

Lesson

- 2.4.A: Card Sort Activity—Parallel and Perpendicular Lines

Practice

- 2.4.B: Parallel and Perpendicular Line Segments

MATERIALS

- graph paper
- patty paper or squares of wax paper
- straightedge
- access to **Desmos.com**

parallel to a given line. Students observe that this parallel line and its slope triangle have the same orientation as the original line and its slope triangle. Students use the concept of rotation to analytically construct a line perpendicular to a given line and observe that the horizontal and vertical legs of their slope triangles have switched orientations. From this, students confirm that the slopes of perpendicular lines are opposite reciprocals.

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings

- Measuring features of geometric figures is the process of assigning numeric values to attributes of the figures, which allows the attributes to be compared.
- A formal mathematical argument establishes new truths by logically combining previously known facts.

Learning Objectives	Essential Knowledge
2.2.1 Justify the relationship between the slopes of parallel or perpendicular lines in the coordinate plane using transformations.	**2.2.1a** The relationship between the slopes of parallel lines can be justified by comparing their slope triangles using translation. **2.2.1b** The relationship between the slopes of perpendicular lines can be justified by comparing their slope triangles using rotation by 90°.
2.2.2 Solve problems involving two or more parallel lines, rays, or line segments.	**2.2.2a** Two distinct lines, rays, or line segments in the coordinate plane are parallel if and only if they have the same slope or are both vertical.
2.2.3 Construct a line, ray, or line segment parallel to another line, ray, or line segment that passes through a point not on the given line, ray, or line segment.	**2.2.3a** Given a line and a point not on the given line, there is exactly one line through the point that will be parallel to the given line. **2.2.3b** Two parallel lines, rays, or line segments in the coordinate plane will have equal slopes and contain no common points.
2.2.5 Solve problems involving two or more perpendicular lines, rays, or line segments.	**2.2.5a** A line, ray, or line segment is perpendicular to another line, ray, or line segment if and only if they form right angles at the point where the two figures intersect. **2.2.5b** A line, ray, or line segment is perpendicular to another line, ray, or line segment in the coordinate plane if and only if the two figures intersect and their slopes are opposite reciprocals of each other, or if one is vertical and other is horizontal.

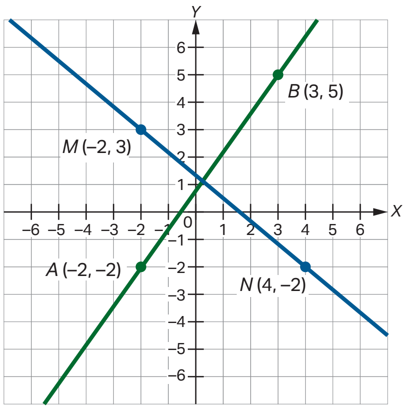
Learning Objectives	Essential Knowledge
2.2.7 Construct a line, ray, or line segment perpendicular to another line, ray, or line segment.	**2.2.7a** A horizontal line, ray, or line segment in the coordinate plane is perpendicular to a vertical line, ray, or line segment if they intersect. **2.2.7b** Two perpendicular lines, rays, or line segments in the coordinate plane will intersect and have slopes that are opposite reciprocals of each other, or one will be vertical and the other will be horizontal.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

Answer the questions below about parallel and perpendicular lines:

1. Are the lines whose equations are $y = 2x - 5$ and $y = -2x - 5$ parallel, perpendicular, or neither? Explain your answer.
2. Would the two lines shown below be considered parallel, perpendicular, or neither? Explain your answer.



3. $\overline{PQ}$ has endpoints at (–4, 6) and (3, 4). $\overline{RS}$ has endpoints at (–8, 4) and (13, –2). Are these two line segments parallel, perpendicular, or neither? Explain your answer.

UNIT 2

PART 1: REVIEWING SLOPES OF PARALLEL AND PERPENDICULAR LINES

In this part of the lesson, students engage in two variants of a card sorting activity. Students first match cards to identify pairs of parallel lines. Then students match cards to identify pairs of perpendicular lines. One major goal of this activity is to activate students' prior knowledge from Algebra 1 that parallel lines have the same slope (or are both vertical lines) and that perpendicular lines have opposite reciprocal slopes (or one line is horizontal and the other is vertical).

In this activity, students also match the graph of a line with its slope. A secondary goal of this activity is to remind students how to use a slope triangle to determine the slope of a line, or to introduce the concept of slope triangles if students did not encounter them in Algebra 1.

- To begin, assign student pairs and provide each pair with a set of cards. Have students match the cards of lines that are parallel. Because the purpose of this activity is to activate students' prior knowledge of properties of parallel lines, try to refrain from reminding students that the slopes of parallel lines are equal.
- While students are working, ask them to identify the slopes of the lines. Later in the lesson, students will use this information to make conjectures about the relationship between the slopes of parallel lines.
- As you circulate around the room, look at how students are determining the slopes of the lines. Some students may start by sorting the graphs into lines with positive and negative slopes. Some students may use a formula to find the slope. Some students may draw a right triangle whose hypotenuse lies on the line, called a slope triangle, and use it to determine the horizontal and vertical differences between the corresponding endpoint coordinates of the hypotenuse. You can formatively assess students' understanding of slope by asking questions like the following:

Meeting Learners' Needs

This activity is a good opportunity to identify students struggling with slope. Some students may invert the horizontal and vertical changes in the slope formula. You can help these students recall the correct order by reminding them that slope is a measure of the steepness of a line. The steepness of a road, for example, is determined by how much the road rises or falls over a corresponding section of the road's length. If you find most students are still struggling with the concept of slope, this would be a good moment to pause and engage students in a few practice problems on finding the slope of a line.

- How do you know that those lines have positive (or negative) slopes?

  Students may talk about the line rising (or falling) from left to right.

- What method did you use to find the slope?

  Students may directly or indirectly reference the slope formula, describe finding the horizontal and vertical changes between coordinate pairs, or describe constructing a slope triangle.

- After student pairs complete the matching activity, have them share their matches with another pair of students and identify the slopes of the pairs of parallel lines. Solutions for the matching activity and the correct slopes are given here:

  Cards 1 and 10: slope = 2

  Cards 2 and 16: slope = $-\frac{5}{3}$

  Cards 3 and 13: slope = $-\frac{1}{2}$

  Cards 4 and 15: slope = $\frac{5}{3}$

  Cards 5 and 14: slope = $-\frac{3}{5}$

  Cards 6 and 12: slope = $\frac{1}{2}$

  Cards 7 and 9: slope = –2

  Cards 8 and 11: slope = $\frac{3}{5}$

- If there is a disagreement among your students about any matching pairs or their slopes, you can encourage an academic conversation by asking the students who disagree to explain to each other and the class the reasons for their choices. Students can engage in a brief debate to reach a consensus about how to correctly determine the slope of a line.

- Once there is a consensus about matching pairs and their slopes, it's important to continue to probe student understanding to ensure that they understand the concepts of slope and parallel lines. You can do this by asking students to make some observations and conjectures about the slopes of parallel lines:

  - What do you know about the slopes of parallel lines? Why do think that is true?

    Parallel lines have the same slope. If the lines had different slopes then they would eventually intersect, because the lines would not rise (or fall) in the same way.

- Now ask students to re-sort the same cards, this time matching cards that show perpendicular lines. To give the students some context, let them know that while they are matching the perpendicular lines, they should be thinking about the relationship between the slopes of the lines. At this point, try to refrain from reminding them that the slopes of perpendicular lines are opposite reciprocals. A little later in the lesson, students will confirm this relationship.
- As you circulate around the room, listen to what students are saying about the slopes. Their conversations during this part of the lesson provide valuable insight into students' understanding and allow you to target feedback more meaningfully. Some students may only pay attention to whether the slopes are positive or negative. Other students might remember that the relationship between perpendicular lines involves reciprocals and start to match those cards. Some students may focus on the angles the lines form with a gridline or an axis and not consider the slope.
- After students complete this phase of the matching activity, you can have pairs of students share their matches with another pair of students and identify the slopes of the lines. Because there are pairs of parallel lines in the original card sort, each card has two possible perpendicular matches. Solutions for the matching activity and the correct slopes are provided here:

  Cards 1 and 10 with Cards 3 and 13: slopes = 2 and $-\frac{1}{2}$

  Cards 2 and 16 with Cards 8 and 11: slopes = $-\frac{5}{3}$ and $\frac{3}{5}$

  Cards 4 and 15 with Cards 5 and 14: slopes = $\frac{5}{3}$ and $-\frac{3}{5}$

  Cards 6 and 12 with Cards 7 and 9: slopes = $\frac{1}{2}$ and -2

- If there is a disagreement among your students about matching pairs and their slopes, you can encourage academic conversation by asking students who disagree to explain to each other and the class the reasons for their choices. Students can engage in a brief debate to reach a consensus about how to correctly determine the slope of a line.

Meeting Learners' Needs

If students have trouble getting started on the activity because they cannot remember what it means for two lines to be perpendicular, you can prompt them by drawing a few examples of each category of line pairs: parallel lines, intersecting but not perpendicular lines, and perpendicular lines. Ask students to identify which pairs of lines are perpendicular and to explain what makes them perpendicular. A visual cue will often be all students need to remember the definition of perpendicular.

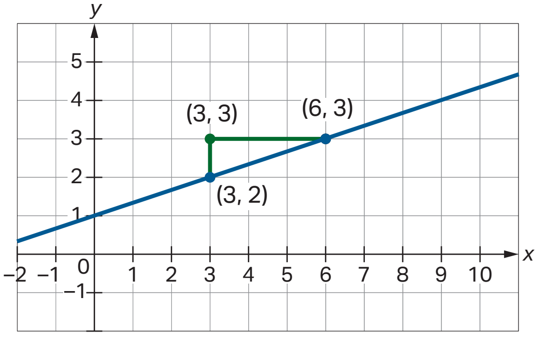
- Once there is a consensus about the matching pairs and their slopes, ask students to make some observations and conjectures about the slopes of perpendicular lines:
    - How do you think the slopes of perpendicular lines are related? Why do think this is true?

      Perpendicular lines have slopes that are opposite reciprocals. Students may suggest that these slopes have different signs because the lines go in opposite directions. However, students should recognize from drawing a few examples that having slopes with opposite signs only means that the direction of the line changes; it does not guarantee a perpendicular line.
- It is not necessary for students to correctly explain why the slopes of perpendicular lines are opposite reciprocals. The primary goal at this point is to have students think about the relationship. They will justify the relationship in the next part of the lesson.

Instructional Rationale

A critical aspect of this lesson is the following discussion about slope triangles. If students took Pre-AP Algebra 1, they will be familiar with slope triangles. Slope triangles figure prominently in the second part of the lesson, because students will compare the slope triangles of parallel and perpendicular lines and use translations and rotations to justify the slope relationships.

- Next, students will explore slope triangles more deeply. You can display the equation of a line to students and ask them to generate the graph of the line. In the image below, the line is $y=\frac{1}{3}x+1$, but you can use any nonvertical or nonhorizontal line.
- To engage students in thinking about the line, have them pick two points that lie on the line. Then use segments of gridlines that include these two points to generate a slope triangle, as shown below. Because gridlines are perpendicular, the slope triangle is a right triangle whose hypotenuse lies on the line. The triangle can be above or below the line. Students will probably choose points with integer coordinates, but they do not have to select points that make the smallest possible slope triangle with whole-number length legs.



- Once students construct a slope triangle, you can ask them some higher-order questions so they focus on the properties of the triangle:
  - Why would we call this a "slope triangle"? What is the relationship between the slope of the line and the triangle?

    Students should state that the triangle can be used to determine the slope of the line because the lengths of the legs of the triangle are the vertical and horizontal distances between the two selected points. Dividing the vertical change by the horizontal change will give the slope of the line.

  - Do we have to draw this particular slope triangle? Will every slope triangle for a line give us the same slope?

    Any slope triangle drawn on a line will give the same slope. The ratio of the vertical and horizontal differences is the same between any pair of points on the line.

  - How could the slope triangle help you figure out if the line has a positive or negative slope?

    The direction of the triangle is important. You have to follow the triangle from one point on the line to the other point on the line, but you can move either vertically or horizontally first. If you move up then right—or down then left—the line has a positive slope. If you move up then left—or down then right—the line has a negative slope.

  - Does it matter if the slope triangle is above or below the line? Why or why not?

    It does not matter if the triangle is above or below the line. What matters is moving from one point on the line to another point on the line. Corresponding leg lengths of the slope triangles are the same, so the ratios from both triangles will have the same value. The corresponding differences between vertex coordinates in the slope triangles have opposite signs, so the ratios of either will have the same sign.

- Students may need some convincing that any slope triangle can be used to determine the slope of the line. If they do, you can ask students to draw two or three additional slope triangles of different sizes on the line and to calculate the ratios of the vertical and horizontal differences. Students should see that even though the triangles are different sizes, any ratio formed by signed lengths of the vertical and horizontal legs will be equal.

Once students understand how to construct a slope triangle and what it represents, they are ready to move on to Part 2. Students do not need total mastery of slope triangles. Rather, they only need to know that they can calculate the slope of the line using any right triangle drawn with its hypotenuse on the line and with one vertical leg and one horizontal leg.

PART 2: JUSTIFYING THE RELATIONSHIPS BETWEEN SLOPES OF PARALLEL AND PERPENDICULAR LINES

In this part of the lesson, students connect the card sorting activity in Part 1 with their slope triangle investigations to craft a geometrically convincing argument justifying the slope relationships of parallel and perpendicular lines. The goal of the activity is for students to observe that translating a line results in a congruent slope triangle with the same orientation as the slope triangle drawn for the original line, while rotating a line 90° changes the direction of the line and inverts the horizontal and vertical legs of its slope triangle.

- You can start this part of the lesson by asking a slightly revised version of a question posed in Part 1:
    - Why do parallel lines have the same slope? Why do perpendicular lines have opposite reciprocal slopes?
- Let students know that their task for this part of the lesson is to develop a convincing mathematical argument that helps explain why the slopes of parallel lines and slopes of perpendicular lines have their characteristic relationships.

Guiding Student Thinking

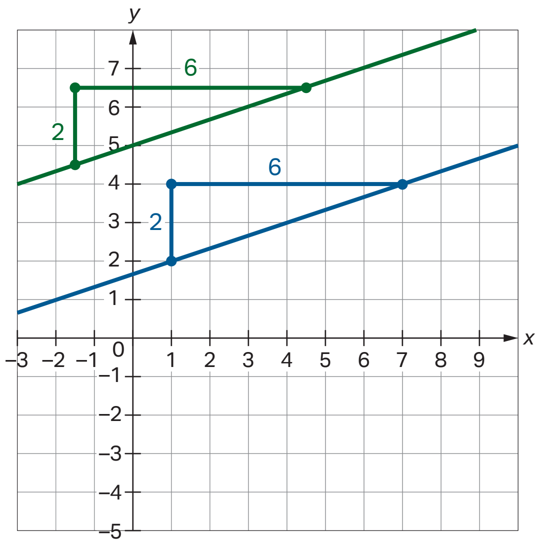
This activity might be one of the first times in this course that students will be expected to create a mathematical argument. Because reasoning with transformations will be a focus of future units, and because proof is incredibly important in mathematics in general, it is vital to let students use this activity as an opportunity to see how convincing mathematical arguments can be generated with simple tools and logic.

- You can have students work in pairs or individually. Distribute graph paper and ask students to draw a line on the graph paper with a straightedge. Students should then determine the slope of the line and draw a slope triangle for the line. Discourage students from drawing horizontal or vertical lines, because those lines do not have meaningful slope triangles.
- While students are drawing their lines and the associated slope triangles, you can distribute patty paper to everyone. Once they have drawn the line and its slope triangle, ask students to trace their line and slope triangle on the patty paper. They can use a straightedge to make sure they have correctly traced their lines.
- Now ask the students to move the traced lines to any other location on the graph so that their original lines and the traced lines are parallel. While they are working, you can ask them some questions to focus their attention on the slope triangles of the parallel lines:
  - What do you notice about the slopes of the lines? What do you notice about the slope triangles of the lines?

    The slopes of the lines are equal. The slope triangles have the same size, shape, and orientation, but they are in different locations.
- Once they have made observations about the parallel lines, you can ask the students to move the patty paper again so the traced line forms another parallel line. Ask students to make additional observations about the slope triangles of parallel lines:
  - Are all the slope triangles for your parallel lines the same shape, size, and orientation? Do you think that will be true for every parallel line you could make? Why do you think that has to be true?

    The slope triangles have to have the same shape, size, and orientation, because moving the line so it is parallel would also move its slope triangle up or down or right or left (or any combination of these) without changing its shape, size, or orientation.
- An important aspect of mathematical argumentation is peer-to-peer evaluation and, possibly, critique. So it is important to have students share their observations and record them in the room so everyone can see them. You can point out to the class that students are making a mathematical argument because they are using known information about mathematical concepts (properties of lines, slopes, slope triangles, and transformations) to generate a new conjecture about related

concepts ("The slopes of parallel lines are the same because translating a line does not change the slope triangle.") that answers a question ("Why are the slopes of parallel lines equal?"). Students might use the term "slide" instead of "translate," but it isn't necessary to introduce the term "translate" at this time. However, if students already know it and are comfortable with it, then you should use it.



- At this point, you can remind students that they are also going to explore why the slopes of perpendicular lines are opposite reciprocals. Students will use their traced line and slope triangle on the patty paper to create a line perpendicular to the original line they drew.
- Now ask students to move the patty paper so the traced line is perpendicular to the original line.
- When students are reasonably confident that they have a perpendicular line, you can ask them questions to focus their attention on the slope triangles of the perpendicular lines:
  - What do you notice about the slopes of the perpendicular lines? What do you notice about the slope triangles of these lines?

    The slopes of the perpendicular lines are opposite reciprocals. One of the slope triangles looks like it's been turned.

Meeting Learners' Needs
If students need more scaffolding to get started on this task, you could suggest that they rotate the line without telling them how much to rotate the line.

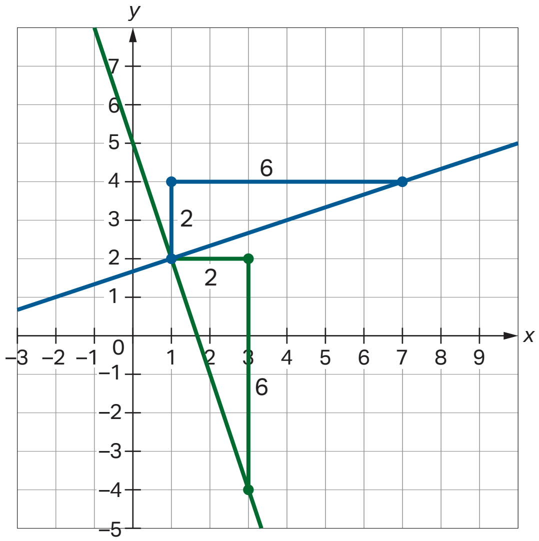
Guiding Student Thinking

Students may be confused about how to know when lines form a 90° angle. They may want to use a protractor or even the corner of a piece of paper to measure the angle. You could use this opportunity to ask students about the relationship between the slopes of perpendicular lines that they explored in Part 1 of the lesson. Encourage them to use the opposite reciprocal slopes to figure out when the paper has been rotated exactly 90°.

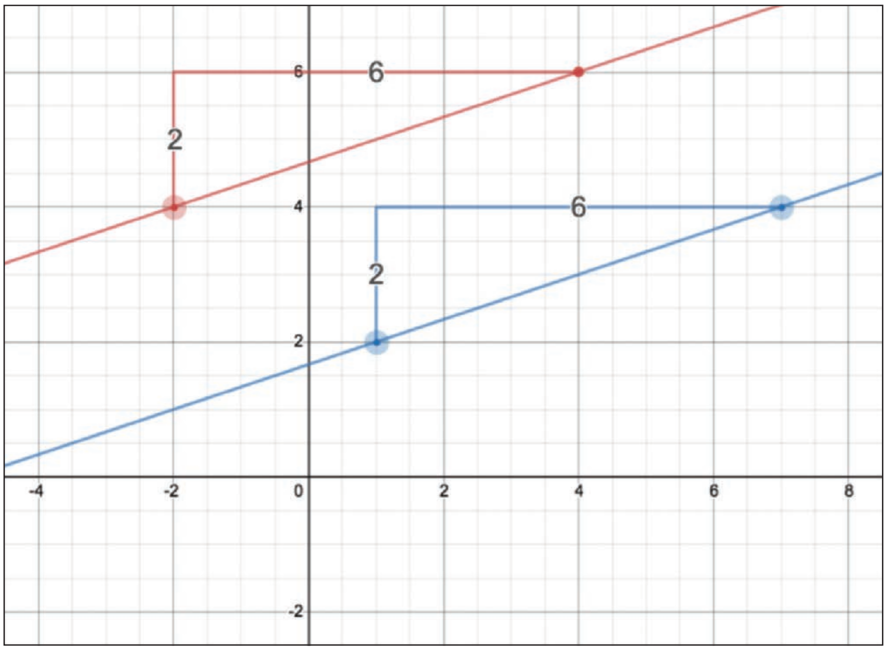
- Have students make a conjecture about the slope triangles of perpendicular lines. Students might want to create a second line perpendicular to the original to test their conjecture. While they are working, ask students questions to help them develop their conjectures:
  - Why do you think that slopes of perpendicular lines have that relationship?
  - If one line has a positive slope, why must the other line have a negative slope (and vice versa)?
- Give students time to make conjectures and to develop their reasoning, and then have students share their results. Listen for conjectures and justifications such as the following:

  The slope triangles have been turned 90°. This means that the vertical leg became a horizontal leg, and the horizontal leg became a vertical leg. If a line has a positive slope, then as it is being rotated 90°, it becomes horizontal (or vertical) first, before the slope becomes negative.
- It is important to have students share their observations and record them so everyone in the class can see them. You can point out to the class that they are making another mathematical argument, as they are using known information about mathematical concepts (properties of lines, slopes, slope triangles, transformations) to generate a new conjecture about related concepts ("The slopes of perpendicular lines are opposite reciprocals because rotating the line changes the slope from positive to negative, or vice versa, and inverts the horizontal and vertical legs of the slope triangle.") that answers a question ("Why are the slopes of perpendicular lines opposite reciprocals?"). Students might use the term "turn" instead of "rotate," but it isn't necessary to introduce the term "rotate" at this time. However, if students already know it and are comfortable with it, then you should use it.

UNIT 2



- To end this part of the lesson, you can use the Desmos demonstration, **preap.org/slope-proof**, which consists of a moveable line and its associated slope triangle. These figures can be animated with either a translation or a rotation to confirm the arguments students made during this part of the lesson. See the image below:



By the end of this part of the lesson, students should understand why parallel lines have equal slopes and why perpendicular lines have opposite reciprocal slopes. In the next section of the lesson, students will extend this knowledge to line segments.

PART 3: EXTENDING THE SLOPE RELATIONSHIP TO LINE SEGMENTS

The goal of the final part of this lesson is for students to use what they learned in parts 1 and 2 about the slopes of parallel and perpendicular lines in the coordinate plane to determine whether two line segments in the coordinate plane are parallel, perpendicular, or neither.

- Have students work on the problems on **Handout 2.4.B: Parallel and Perpendicular Line Segments**. Some of the problems ask students to confront the idea that parallel lines "don't intersect." Several line segments that do not intersect have unequal slopes, so the line segments would intersect if they were extended. Other problems have line segments that do not intersect but have slopes that are opposite reciprocals, so students consider whether such line segments can be called perpendicular if they do not intersect to form right angles.
- This part of the lesson is designed to promote problem-solving skills and support productive struggle in learning mathematics. Students could work individually or in pairs. Groups of three or more students are unlikely to be effective, because there are not enough tasks in each problem for more than two people to complete. It is also possible to assign this part of the lesson to be completed outside of the classroom and to follow up with students once they have finished it.
- Take some class time for your students to present their solutions to the problems and to debrief about the problem set. You may find that some students worked to solve the problem by extending the lines and tried to determine if the lines are parallel by visual inspection only. You can validate this approach as a good first step to developing some insight into the problem. However, students should confirm any purely geometric hypotheses with algebraic calculations and reasoning.
- During the class debrief, ask some questions like the ones below to make sure that students develop a deeper understanding of parallel and perpendicular lines:
  - If two lines in the same plane do not intersect, will they be parallel? How could you tell if they are parallel?

    Two coplanar lines that do not intersect must be parallel. If they were not parallel, then they would intersect. In the coordinate plane, lines that do not intersect must have equal slopes or both be vertical. If the lines have equal slopes or are both vertical, then they will be parallel.

- If two line segments in the same plane do not intersect, will they be parallel? How could you tell if they are parallel?

  Two coplanar line segments that do not intersect might be parallel, but they might not be parallel. If the line segments are in the coordinate plane, you could compare their slopes. If the slopes of the line segments are equal, then the segments would be parallel. If the slopes of the line segments are not equal, then the segments would not be parallel, because if the segments were extended into lines, those lines would intersect somewhere.

- If two lines in the same plane have slopes that are opposite reciprocals of each other, will they be perpendicular? Why or why not?

  Two coplanar lines that have opposite reciprocal slopes must be perpendicular. The lines will intersect at right angles. The slope triangles of the lines are 90° rotations of each other, and the lines have different orientations. If the lines did not have opposite reciprocal slopes, then they would not be perpendicular.

- If two line segments in the same plane have slopes that are opposite reciprocals of each other, will they be perpendicular? Why or why not?

  Two coplanar line segments that have opposite reciprocal slopes might be perpendicular. The line segments must also intersect to be considered perpendicular.

- Solutions for the problems, as well as suggested guidance for students who struggle with problems, are included in the Assess and Reflect on the Lesson section below.

UNIT 2

ASSESS AND REFLECT ON THE LESSON

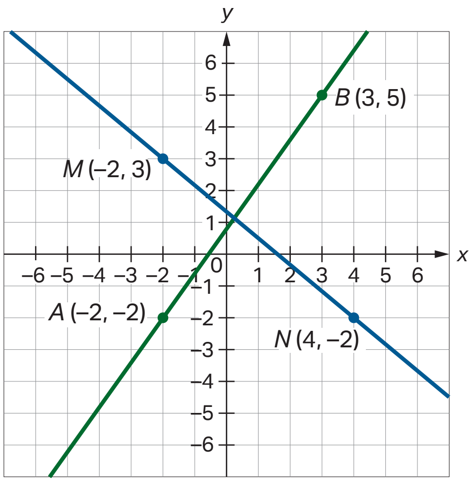
FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

1. Are the lines whose equations are $y = 2x - 5$ and $y = -2x - 5$ parallel, perpendicular, or neither? Explain your answer.

   The lines are neither parallel nor perpendicular. The slopes are not equal, so the lines are not parallel. The slopes are not opposite reciprocals, so the lines are not perpendicular.

2. Would the two lines shown below be considered parallel, perpendicular, or neither? Explain your answer.



   The lines are neither parallel nor perpendicular. The slope of the line that contains points A and B is $\frac{7}{5}$, and the slope of the line that contains points M and N is $-\frac{5}{6}$. Because the slopes are not opposite reciprocals, the lines are not perpendicular.

3. $\overline{PQ}$ has endpoints at (–4, 6) and (3, 4). $\overline{RS}$ has endpoints at (–8, 4) and (13, –2). Are these two line segments parallel, perpendicular, or neither? Explain your answer.

   The line segments are parallel. The slope of both line segments is $-\frac{2}{7}$, so they are parallel.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 2.4.A: Card Sort Activity—Parallel and Perpendicular Lines

See lesson for answers.

Handout 2.4.B: Parallel and Perpendicular Line Segments

1. The segments are parallel because they have the same slope, $-\frac{3}{4}$.
2. The segments are perpendicular because they intersect and have opposite reciprocal slopes, $-\frac{5}{4}$ and $\frac{4}{5}$.
3. The segments are neither parallel nor perpendicular. Their slopes are $\frac{5}{6}$ and -1.
4. Even though the line segments do not intersect, they are not parallel, because they have different slopes, $\frac{8}{9}$ and $\frac{7}{8}$.

Guiding Student Thinking

Students might struggle with the observation that the line segments in question 4 do not intersect while also accepting that they are not parallel. This kind of thinking is an important precursor for determining whether a quadrilateral is a parallelogram in Unit 3. You can help students develop a more sophisticated understanding of parallelism by asking them to think about what would happen to the line segments if they were extended indefinitely in both directions. The extended lines would eventually intersect on their left extensions and get noticeably farther apart on their right extensions.

5. Because the line segments do not intersect, they would not be considered perpendicular, even though their slopes are opposite reciprocals, $\frac{1}{3}$ and -3.

Guiding Student Thinking

Students might struggle with the idea that the line segments in question 5 are not perpendicular, given that the focus of the lesson has been on slope. You can help students develop a more complete understanding of perpendicularity by asking them what it really means for two lines to be perpendicular. A critical feature of perpendicular lines or line segments is that they need to form a right angle at the point of intersection.

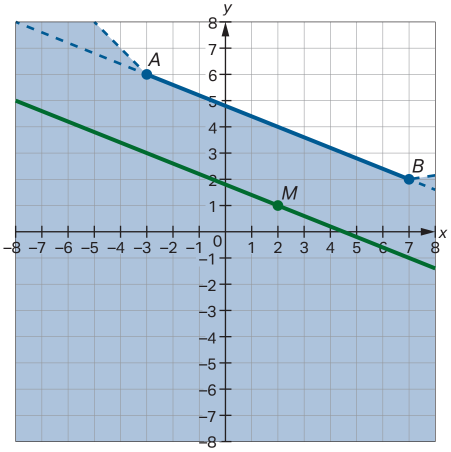
6. (a) Answers will vary. One possible choice is $(-3, 3)$, but any point (x, y) that lies on the line $y=-\frac{2}{5}x+\frac{9}{5}$ would satisfy the criteria.

(b) Answers will vary. One possible choice is (4, 6), but any point (x, y) that lies on the line $y = \frac{5}{2}x - 4$ where $y \geq -\frac{2}{5}x + \frac{24}{5}$.

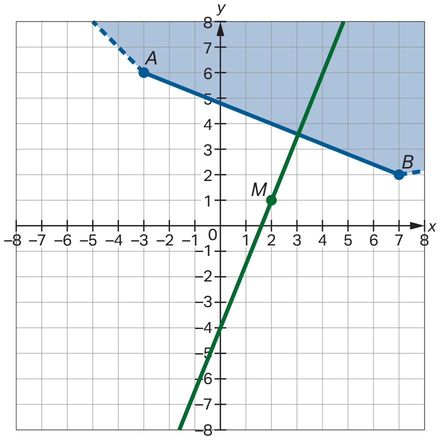
(c) Answers will vary. One possible choice is (−2, −1). Any point (x, y) that lies in the region shown but does not lie on the line $y = -\frac{2}{5}x + \frac{9}{5}$ would satisfy the criteria.

Meeting Learners' Needs

Problems 6a through 6e do not require students to draw the segments or to use a coordinate plane. However, many students will benefit from plotting the coordinates on graph paper.



(d) Answers will vary. One possible choice is (7, 4). Any point (x, y) that lies in the region shown but does not lie on the line $y = \frac{5}{2}x - 4$ would satisfy the criteria.



(e) Answers will vary. One possible choice is (0, −4). Any point (x, y) that satisfies $y > -\frac{2}{5}x + \frac{24}{5}$ would be acceptable.

LESSON 2.5

UNIT 2

The Perpendicular Bisector Theorem

OVERVIEW

LESSON DESCRIPTION

Part 1: Placing the Brothers' Fence
Students explore a scenario in which two brothers have separate houses on the same plot of land. Students determine how to split the property between the brothers using given criteria.

Part 2: Constructing the Perpendicular Bisector
In this part of the lesson, students develop the perpendicular bisector construction by investigating properties of the perpendicular bisector. The section concludes with a formal statement of the perpendicular bisector theorem.

Part 3: Connecting the Perpendicular Bisector and the Midpoint Formula
Students return to the scenario of the brothers' fence and synthetically construct the perpendicular bisector of the path between the houses. They analytically verify that the coordinates of the intersection point of the perpendicular bisector and the path and the point indicated by the midpoint formula are the same.

CONTENT FOCUS

Students learn to construct the perpendicular bisector of a given segment and develop the perpendicular bisector theorem. This lesson reinforces students' idea of a compass as a "distance copier" by helping them observe that any intersection points of two equal compass arcs are equidistant from the centers of both arcs. If these arcs are centered at the endpoints of a segment, then their intersection points lie along the

AREAS OF FOCUS

- Engagement in Mathematical Argumentation
- Greater Authenticity of Applications and Modeling
- Connections Among Multiple Representations

SUGGESTED TIMING

~90 minutes

HANDOUTS

Lesson

- 2.5.A: Placing the Brothers' Fence
- 2.5.B: Placing the Brothers' Fence in the Coordinate Plane

MATERIALS

- access to **Desmos.com**
- patty paper
- compass
- straightedge

perpendicular bisector of the segment. Through these activities, students expand their reasoning skills, which they can apply throughout the course.

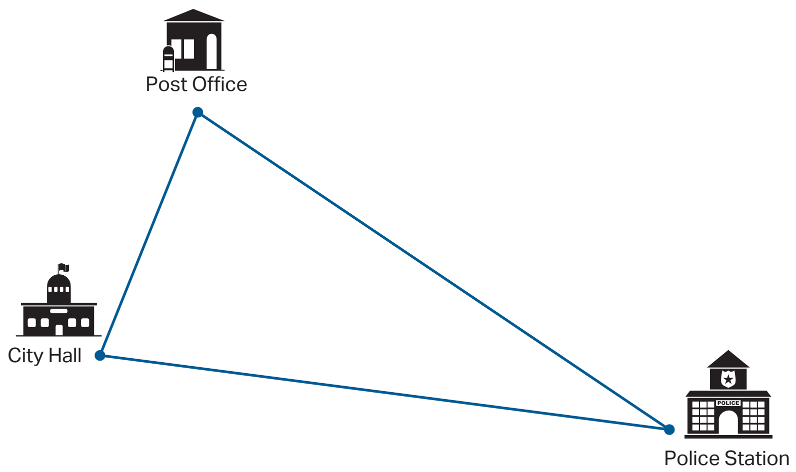
COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ A formal mathematical argument establishes new truths by logically combining previously known facts. ▪ Pairs of lines in a plane that never intersect or that intersect at right angles have special geometric and algebraic properties.	
Learning Objectives	**Essential Knowledge**
2.1.10 Solve problems involving a segment bisector or an angle bisector.	**2.1.10a** The midpoint of a line segment is the point located on the line segment equidistant from the endpoints. **2.1.10b** In the coordinate plane, the *x*- and *y*-coordinates of the midpoint of a line segment are the arithmetic means of the corresponding coordinates of the endpoints. **2.1.10c** A bisector of an angle is a line, ray, or line segment that contains the vertex of the angle and divides the angle into two congruent adjacent angles.
2.2.6 Construct the perpendicular bisector of a line segment.	**2.2.6a** The perpendicular bisector of a line segment intersects the line segment at its midpoint and forms four right angles with the line segment. **2.2.6b** The perpendicular bisector of a line segment is determined by identifying two points in a plane that are equidistant from the endpoints of the line segment and constructing a line, ray, or line segment through those two points. **2.2.6c** Every point that lies on the perpendicular bisector of a line segment is equidistant from the endpoints of the line segment.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

The town of Centerville wants to put a monument at a point that is equidistant from the post office, city hall, and the police station. There are straight roads that connect the post office to city hall, the post office to the police station, and city hall to the police station. Use what you know about perpendicular bisectors to determine where the monument should be placed. Explain how you know that the monument will be the same distance from the three buildings.



UNIT 2

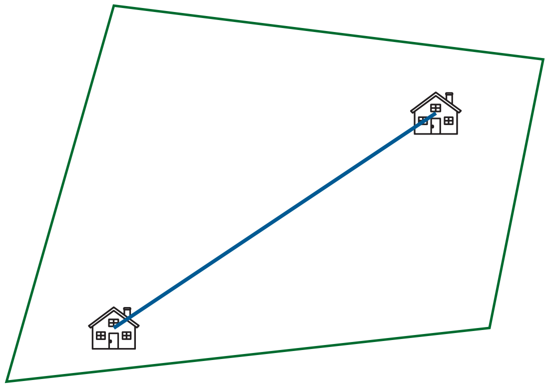
PART 1: PLACING THE BROTHERS' FENCE

This lesson introduces the concept of perpendicular bisectors to students through a scenario of two brothers who live on the same plot of land. The brothers want to build a fence that separates the land in a way they think is fair. This part of the lesson culminates in students figuring out that the points that lie on the perpendicular bisector of a line segment are equidistant from the endpoints of that segment.

- Begin by displaying for students the brothers' fence scenario described on **Handout 2.5.A: Placing the Brothers' Fence**. You could distribute both handouts if you want students to have their own copies.

Placing the Brothers' Fence

Two brothers live on the same plot of land. A straight path connects their houses. They decide to add a fence to split the property using this rule: "If the point on the property is closer to me, then it belongs to me. If it is closer to you, then it belongs to you." The brothers want the fence to include a gate that is on the path between their houses.



Where should the brothers put their fence?

Handout 2.5.A

- Provide students with time to closely observe the scenario. Then give students an opportunity to share what they notice and what they wonder about the scenario.
- Let students make some conjectures about where the fence should be built using the brothers' rule. Students may suggest all sorts of fence placements. They might also suggest fence shapes that are not linear. Make sure students understand the rule the brothers are using to split the land.
- Invite students to share their ideas. Record all ideas in a central location so everyone can see the conjectures.

Instructional Rationale

Students might wonder whether or not the rule the brothers are going to use will split the area of the property equally. This particular rule was chosen because it will lead students to discover the properties of the perpendicular bisector. Because there are many different ways the brothers could have split the land, it could be worth taking time at the end of the lesson to explore the advantages and disadvantages of different rules, as well as what the land separation would look like under different kinds of rules.

- Ask students to think about what strategies the brothers might use to determine where to put the fence. Students might suggest a guess-and-test strategy where they choose points and measure with a ruler to determine which house is closer. They might suggest using a compass to determine a radius of points equidistant from the house. They could suggest using a coordinate plane and the distance formula to find points closest to each house.
- Let students decide which method they prefer to use to do the next part of the task. Each student should find five points that are closer to the house on the right, five points that are closer to the house on the left, and five points that are the same distance from each house. **Handout 2.5.A: Placing the Brothers' Fence** is better for a purely synthetic approach, because it does not include a coordinate plane. **Handout 2.5.B: Placing the Brothers' Fence in the Coordinate Plane** presents the same scenario, except the houses are plotted on a coordinate plane.
- For students who want to use the synthetic plane, you can provide rulers and compasses. Encourage students to go beyond just guessing and testing with a ruler and to also use increasingly large circles centered at each house to see which points are closer to them. Circles that overlap will pose a challenge but are a good step toward locating the perpendicular bisector of the segment.
- For students using the coordinate plane, you can direct them to the Desmos interactive, **preap.org/brothers-fence**, which they can use to find the coordinates of the 15 points.
- Once students plot the points on their own, have them partner with another student to compare their points. Ask the pairs of students to closely

Classroom Ideas

To distinguish between the points closer to one brother or the other, students can use different colored pencils or symbols. They will need three colors or symbols to distinguish points that are closer to the brother on the left, points that are closer to the brother on the right, and points that are the same distance from each brother.

observe and analyze each other's points. Encourage students to identify both similarities and differences between the points they plotted within each of the three categories they've identified.

- After students have compared their points and made some observations, debrief with the class about what similarities and differences they noticed. Through this debrief, try to elicit from students that the points that are the same distance from each brother's house seem to lie on a line.

Guiding Student Thinking

If students need some prompting to notice the line, you could have students form small groups of four and plot all their points on the same paper. Once they have 20 points the same distance from each brother's house, they are more likely to see the shape of the line emerge.

- Before moving on to the next part of the lesson, ask students to think about where the gate should be located. Give them some time to mark on their handout where they think the gate should be located and to justify their choices with mathematical arguments.
- Have several students share their proposed gate locations and the reasons why they selected those locations. By the end of this discussion, students should recognize that if the gate must be on the path between the houses and not be closer to either brother, then the gate should be at the midpoint of the path. They should also observe that the gate would be along the line of points that are the same distance from each brother.

PART 2: CONSTRUCTING THE PERPENDICULAR BISECTOR

In this part of the lesson, students use patty paper to construct the perpendicular bisector of a line segment. They will observe that every point on the bisector is equidistant from the endpoints of the line segment being bisected.

- Begin by distributing several sheets of patty paper, and ask students to trace the line segment that represents the path between the brothers' houses. Suggest that students use a straightedge. Give students the opportunity to find the exact location of the midpoint of the line segment without any other tools.
- Some students using **Handout 2.5.B: Placing the Brothers' Fence in the Coordinate Plane** might have already found the midpoint of the segment with the midpoint formula. Ask students how they could find the midpoint of the segment if all they had was the line segment on the patty paper and did not have a coordinate

plane. Students might struggle a bit at first, but eventually some students should figure out that folding the paper in half to match up the endpoints will divide the line segment into two equal parts. Allow the students who figure this out to share their discovery with the class.

- Once students have found the midpoint of the segment by folding the patty paper, ask them to make some observations about the crease they made in the paper. Students may observe that the crease is longer than the original segment, or that the crease goes through the entire paper. Another important observation is that the crease makes a right angle with the segment, which means that the line segment and the crease are perpendicular.

Meeting Learners' Needs
Students with manual dexterity challenges may find the task of matching the endpoints of the segment particularly difficult. Encourage these students to draw a dot at each end of the segment to help them align the endpoints before folding the paper.

- If students do not suggest that the crease and the segment are perpendicular, you can prompt them to analyze the angle formed by the crease and segment. Encourage them to provide a mathematical argument for why they must be perpendicular. You can ask them questions like these to lead them to a transformation argument that justifies why the angles must be right angles:
  - How can we know that the angles are right angles?

    Maybe we could try to measure the angles with a protractor.
  - Suppose that we did not have a protractor. What do we know about right angles? Are all right angles congruent?

    We know that all right angles are congruent, because every right angle measures 90°.
  - How do we know if two angles are congruent? What is our definition of congruence?

    Two angles are congruent if one can be translated, rotated, or reflected to coincide with the other.
  - Is it possible to translate, rotate, or reflect one of the angles to coincide with the other one? Why or why not? What can we conclude about the angles?

    It is possible to reflect one of the angles to coincide with the other, because we formed the angles by folding, which could be thought of as a reflection. The angles must be congruent.

Meeting Learners' Needs
Some students will benefit from labeling the line and line segments with points. It may be easier to keep track of the angles, segments, and lines if students assign labels to them.

- What else do we know about the angles? Since the angles are adjacent to each other on a line (or segment), what do we know about their relationship?

  Because the angles are adjacent and on a line, they form a linear pair. That means the sum of their measures is 180°.

- We know that the angles are congruent and that the sum of their measures is 180°. What can we conclude about the angles? What can we conclude about the segment and the crease we made?

  Because the angles are congruent and the sum of their measures is 180°, they must each measure 90°. Since they each measure 90°, they are both right angles. This shows that the segment and the crease are perpendicular.

- It will be helpful to provide students with the proper mathematical name to the crease the students made instead of continuing to call it "the crease" or "the fold." Explain that the line along the fold they created is called the *bisector* of the segment.
- Before defining the term, ask students to think about the components of the word *bisector* and what they mean. Have them speculate on the meaning of bisector by combining the meanings of its components together.

  The prefix *bi-* means two, the stem *sect* means part, and the suffix *-or* means something that performs a function. Based on these components, a bisector may be something that forms two parts.

- Use the information that students generated about the word bisector to formally define the term: *A bisector is a geometric object that cuts another object into two equal parts.*
- Use the information that students generated about the angle formed by the fold and the segment to define the perpendicular bisector: A *perpendicular bisector is a line (or line segment) that is perpendicular to a line segment that cuts the segment into two equal parts.*
- Now, ask students to think about how they might be able to find the perpendicular bisector in situations in which they can't fold the line segment. You can relate this back to the scenario of the two brothers and the path between their houses. The

Meeting Learners' Needs
Students will benefit from deconstructing mathematical terms and connecting new words to ones they already know. For example: a *bi*cycle has two wheels, a *sect*ion of a building is a part of the building, and an accelerat*or* is the pedal that causes a car to accelerate.

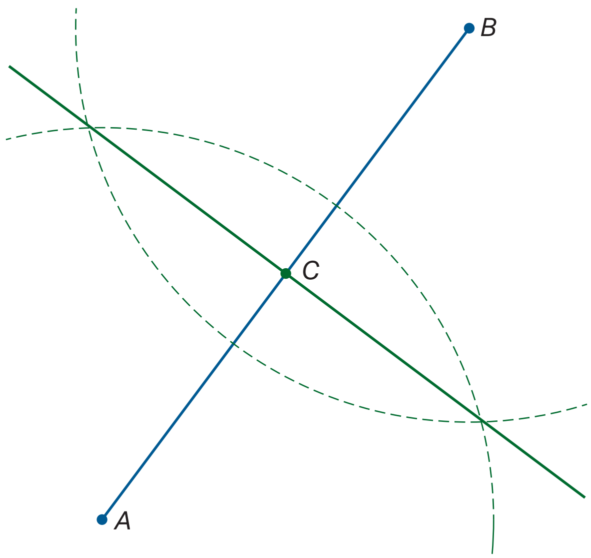
actual path cannot be folded on itself to find the midpoint. You can allow a few moments for students to make conjectures about how to find the midpoint.

- Let students know that they will use the line they found by folding the paper to develop another method for finding the perpendicular bisector of a line segment without folding. However, this method will require a compass and straightedge.
- Next, on their creased patty paper, have students carefully trace a line along the fold. Then, have students choose a point on the line they drew. They should not choose the midpoint, but any other point will work well. While students are doing that, distribute compasses to students if they do not already have them from the first part of the lesson.
- Students should open their compass so that the fixed end (without the pencil) is at an endpoint of the segment and the pencil end is on the point they chose on the bisector.

Guiding Student Thinking

Students may wonder whether it matters if they place the fixed end of the compass on the right or left endpoint. This is a good opportunity to let them explore whether the choice of right or left endpoint is important. When they realize that they can use either endpoint to begin this construction, they are ready to explore and understand the perpendicular bisector theorem.

- Ask students to use the compass to draw an arc that intersects their perpendicular bisector twice. One intersection will be at the point they chose on the bisector, and the second intersection will be on the other side of the line segment.
- Using the same compass setting, have students repeat the process from the other endpoint of the segment. The arc they draw from the other endpoint should intersect the bisector at the same points as the first arc. If their final drawing does not look like the image below, then they likely changed the compass setting before drawing the second arc.



- Ask students to make some observations about their construction and to think specifically about how they could use its steps to help them find the perpendicular bisector for any line segment.

  Every point on the arc drawn from point A is equidistant from point A, because the compass copies distances. If the same compass opening is used to draw an arc from point B, every point on the arc is equidistant from point B for the same reason. Because the same opening was used for each arc, their intersection points are the same distance from both A and B. A line that connects the arc intersection points will bisect the segment, because point C is the same distance from A as it is from B.

- Give students an opportunity to write down the steps for finding the perpendicular bisector. Then lead the class in a debrief in which the following construction steps should emerge:

  (1) Given a line segment $\overline{AB}$, place the fixed end of a compass at either endpoint (A or B) and open the compass so its opening is more than half the length of the line segment.

  (2) Use the compass to make a semicircular arc that passes through the line segment above and below where you expect its midpoint to be.

  (3) With the same compass setting, place the fixed end at the other endpoint. Use the compass to make another semicircular arc that intersects the first arc twice.

Classroom Ideas

You can use an online applet such as the one at **mathopenref.com/constbisectline.html** to show students the process multiple times. This could help them write the sequence of steps.

(4) Use a straightedge to connect the two intersection points with a line segment. This line segment is the perpendicular bisector of line segment $\overline{AB}$.

- After students have some time to write down the steps for constructing the perpendicular bisector, ask some questions like the ones below to focus students on the properties of the perpendicular bisector:
  - Is the midpoint equidistant from the endpoints of a line segment? Does the midpoint lie on the perpendicular bisector of the line segment?

    Yes, the midpoint is, by definition, the point on the line segment that is the same distance from its two endpoints. It must be on the line that cuts the segment into two equal parts.
  - What must be true about any point that is on the perpendicular bisector of a line segment? Why is this true?

    Any point on the perpendicular bisector is the same distance from the endpoints of the segment. This is true because you can use the same compass opening to draw arcs from both endpoints that intersect on the perpendicular bisector.
  - What must be true about any point that is equidistant from the two endpoints of a segment? Why is that true?

    The point must be on the perpendicular bisector. This is true because if the point is the same distance from the endpoints of the segment, then you can use the same compass opening to draw arcs from both endpoints that intersect at the point. The intersections of the arcs on either side of the line segment will form the perpendicular bisector.
- Finally, summarize the student discussion with the following theorem: *The perpendicular bisector theorem states that a point P lies on the perpendicular bisector of line segment $\overline{AB}$ if and only if it is equidistant from points A and B.*

Instructional Rationale

If students have not encountered the phrase "if and only if" before, this is a good opportunity to explore it. The phrase, sometimes abbreviated "iff," has a specific meaning in mathematics. When the phrase "if and only if" joins two sentences, it indicates that both sentences are true or that both sentences are false. The sentences are logically equivalent. A theorem that uses "if and only if" is true in both directions. These statements are called "biconditional" statements. For the perpendicular bisector theorem, this means that if a point P is on the perpendicular bisector of $\overline{AB}$, then P is equidistant from points A and B, as well as that if point P is equidistant from points A and B, then P is on the perpendicular bisector of $\overline{AB}$.

PART 3: CONNECTING THE PERPENDICULAR BISECTOR AND THE MIDPOINT FORMULA

At this point in the lesson, students know everything they need to construct the perpendicular bisector that determines the location of the brothers' fence. Because **Handout 2.5.B: Placing the Brothers' Fence in the Coordinate Plane** presents the brothers' houses as endpoints of a segment on a coordinate plane, students using that handout will verify that the perpendicular bisector intersects the segment at its midpoint. This verification will help students connect the synthetic perpendicular bisector construction with the analytic midpoint formula. This connection sets the stage for students to be able to write algebraic equations for the perpendicular bisector of a segment in the coordinate plane.

Instructional Rationale

Placing this investigation after students learn about the perpendicular bisector theorem allows them to see that this way of finding the midpoint is consistent with the midpoint formula. While the midpoint formula only works with a segment in the coordinate plane, the perpendicular bisector construction can be used both for segments that do not have coordinates and for segments in the coordinate plane. Students sometimes struggle to see the connections among topics learned in different lessons or units in a given math course. This is an opportunity to help them create more durable knowledge about geometry.

- Ask students to return to **Handout 2.5.A: Placing the Brothers' Fence**. Using what they've learned about perpendicular bisectors, have students construct the location of the fence on the property to divide it according to the brothers' rule. Students should construct the perpendicular bisector of the path between the brothers' houses using a straightedge and compass.
- Once they have constructed the perpendicular bisector, check student understanding by asking some questions like the ones below:
    - How can you be sure that the perpendicular bisector of the path between the houses is the location that the brothers want for the fence?

      Every point on the perpendicular bisector is equidistant from the endpoints of the segment. That means the fence is not closer to either house, but every other point will be closer to one brother or the other.

- What is the name of the point where the fence and the path intersect? What is the name of the point on the segment where it intersects the perpendicular bisector? How do you know? What does this mean in the context of this problem?

  In each case, it is the midpoint of the segment, because it cuts the segment into two equal parts (or is the same distance from the endpoints of the segment). In this problem, it means this is the point at which the gate should be placed.

- Ask students how they could verify that the perpendicular bisector they constructed actually intersects the path between the houses at the midpoint. If it's been some time since they used the midpoint formula, students may need some prompting to remember it. If some students used the midpoint formula in Part 1 to find the midpoint of the segment, you could have them share their method with the class.
- Have students determine the coordinates of the midpoint of the line segment using the midpoint formula and then determine if the perpendicular bisector intersects the line segment at the midpoint. If students erred slightly in plotting the coordinates or in drawing the line, they will be close but not exact. If this occurs, it is a good opportunity to have students discuss why their perpendicular bisectors are not constructed correctly. A conversation like this can help students understand that while the construction conceptually guarantees that the perpendicular bisector must intersect the segment at the midpoint, actually performing the construction might lead to some errors.

Guiding Student Thinking

At this point, you can choose to have students write the equation of the perpendicular bisector of a segment. This will help them connect the concepts of midpoints, slopes of perpendicular lines, and perpendicular bisectors.

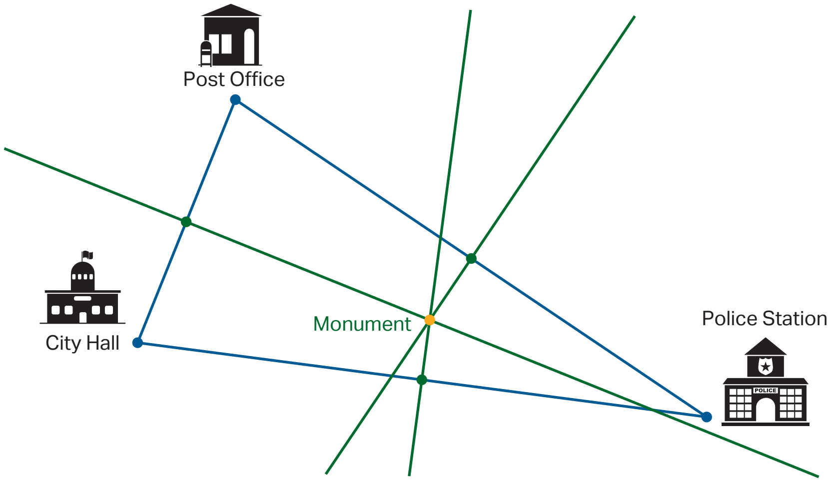
ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

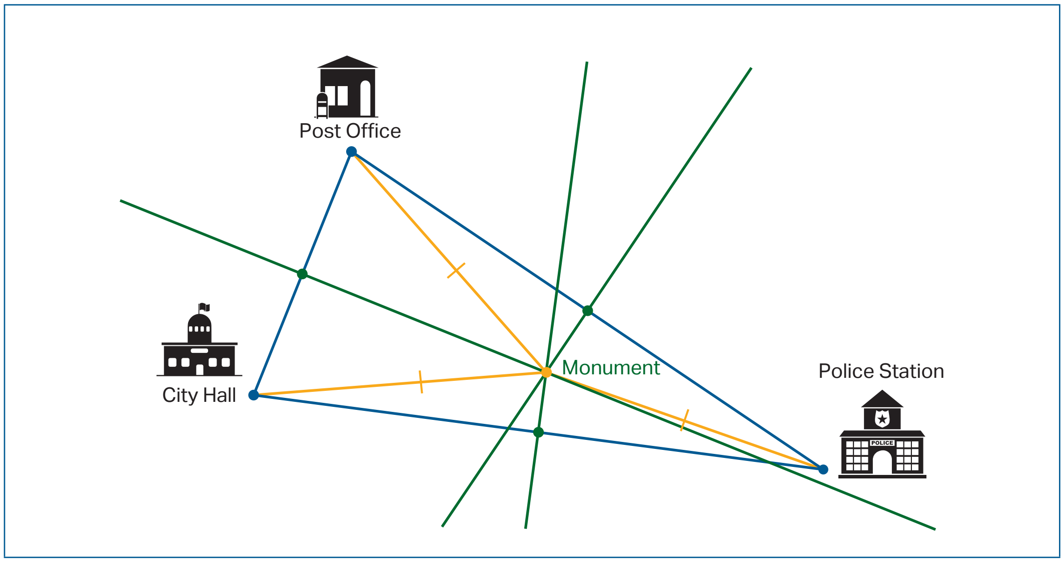
When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

The town of Centerville wants to put a monument at a point that is equidistant from the post office, city hall, and the police station. There are straight roads that connect the post office to city hall, the post office to the police station, and city hall to the police station. Use what you know about perpendicular bisectors to determine where the monument should be placed. Explain how you know that the monument will be the same distance from the three buildings.

The monument should be placed at the intersection of the three perpendicular bisectors of the roads that connect each pair of buildings.



I know the monument is equidistant from all three buildings because it is on all three perpendicular bisectors. That means that the monument is the same distance from the post office as it is from city hall, the same distance from city hall as it is from the police station, and the same distance from the police station as it is from the post office. For this to be true, all of these distances must be equal.



Guiding Student Thinking

If you want to continue to support students' integration of analytic and synthetic problem solving, you can adjust the problem so the locations of the buildings are presented with coordinates. Then you can ask students to find the coordinates of the midpoints of the roads or to write the equations of the perpendicular bisectors. You could also extend the lesson by creating a scenario in which the intersection of the perpendicular bisectors of the three sides of the triangle does not lie in the interior of the triangle.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 2.5.A: Placing the Brothers' Fence

This handout provides students with a scenario in which they will explore the properties of the perpendicular bisector. See lesson for more details.

Handout 2.5.B: Placing the Brothers' Fence in the Coordinate Plane

This handout provides students with the same scenario as **Handout 2.5.A: Placing the Brothers' Fence**, but it presents the figures on a coordinate plane. See lesson for more details.

PRACTICE PERFORMANCE TASK

The Flatiron Footprint

OVERVIEW

DESCRIPTION

In this practice performance task, students use what they know about areas, intersections, altitudes, perpendicular lines, and right triangles to determine the area of a famous building's footprint.

CONTENT FOCUS

This task is designed to assess students' understanding of the distance formula, the Pythagorean theorem, and the relationship between the slopes of perpendicular lines in the plane. It is intended to be used at the end of Unit 2, Key Concept 2.1. In the task, students calculate the area of a triangle using information about its placement in the coordinate plane. This task also requires students to determine the intersection of two lines.

AREAS OF FOCUS

- Engagement in Mathematical Argumentation
- Connections Among Multiple Representations
- Greater Authenticity of Applications and Modeling

SUGGESTED TIMING

~45 minutes

HANDOUTS

Unit 2 Practice Performance Task: The Flatiron Footprint

MATERIALS

- scientific calculator or graphing utility
- graph paper (optional)

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ A formal mathematical argument establishes new truths by logically combining previously known facts. ▪ Measuring features of geometric figures is the process of assigning numeric values to attributes of the figures, which allows the attributes to be compared. ▪ Right triangles are simple geometric shapes in which we can relate the measures of acute angles to ratios of their side lengths.	
Learning Objectives	**Essential Knowledge**
2.1.8 Calculate the distance between two points.	**2.1.8a** The distance between two points in the plane is the length of the line segment connecting the points. **2.1.8b** The distance between two points in the coordinate plane can be determined by applying the Pythagorean theorem to a right triangle whose hypotenuse is a line segment formed by the two points and whose sides are parallel to each axis.
2.2.7 Construct a line, ray, or line segment perpendicular to another line, ray, or line segment.	**2.2.7a** A horizontal line, ray, or line segment in the coordinate plane is perpendicular to a vertical line, ray, or line segment if they intersect. **2.2.7b** Two perpendicular lines, rays, or line segments in the coordinate plane will intersect and have slopes that are opposite reciprocals of each other, or one will be vertical and the other will be horizontal. **2.2.7c** Applying the perpendicular bisector construction to a point on a line, ray, or line segment is sufficient to construct a line, ray, or line segment perpendicular to the given line, ray, or line segment.

UNIT 2

SUPPORTING STUDENTS

BEFORE THE TASK

In this practice performance task, students will use their knowledge of geometry in the coordinate plane to determine the area of an unusually shaped building's triangular footprint.

- Students not familiar with New York City may benefit from an introduction to the context. You can show them an image of the iconic Flatiron Building. Let students know that the building is located in New York City and is bounded by 5th Avenue, Broadway, and 22nd Street. It was not originally called the "Flatiron Building" but gained its name shortly after its construction, when people noticed that its shape is similar to a clothes iron.

Meeting Learners' Needs

In this task, students have to solve a system of equation to determine the intersection point of two lines. You could begin class with a brief warmup that helps students remember how to solve a system of linear equations.

(Alamy / Songquan Deng)

DURING THE TASK

Because this is a practice performance task, you could choose to have students engage in the task differently than a conventional assessment. You could choose an implementation strategy such as one of the following:

- Students could work in pairs to complete the task. It is not recommended for students to work in small groups. There is ample work and enough potential discussion areas for two students. However, in groups of more than two students, there may not be quite enough work for everyone to engage in the task.

- You could chunk the task into its three parts and have students complete one part at a time. Students could check their solutions with you or the scoring guidelines before moving on to the next part. During the check, spend a few moments discussing the solution with each student. Ask students to address what changes, if any, they could make to their solutions to craft more complete responses the next time they engage in a performance task.
- You could have students complete the task individually. Then, you could distribute the scoring guide to students to score their own responses or the responses of a classmate. Finally, have students reflect on their responses and their score by asking them to make recommendations about what they could do to craft a more complete response the next time they engage in a performance task.

Meeting Learners' Needs

The task requires students to calculate the area of a triangle. Because the area of a triangle is almost always taught prior to high school, students should know the correct formula. However, if any students cannot remember the formula, it would be appropriate to tell them what it is. This task is primarily assessing students' knowledge of the Pythagorean theorem, the distance formula, and the slopes of perpendicular lines in the coordinate plane. Students should not be prevented from completing the task because they do not know the area formula.

AFTER THE TASK

Whether you decide to have students score their own solutions, have students score their classmates' solutions, or score the solutions yourself, the results of the practice performance task should be used to inform instruction.

Students should understand that converting their score into a percentage does not provide a useful measure of how well they performed on the task. You can use the suggested scoring conversion guide that follows the scoring guidelines to discuss their performance.

SCORING GUIDELINES

There are 9 possible points for this performance task.

Student Stimulus and Part (a)

(a) Is the footprint of the Flatiron Building in the shape of a right triangle? Explain why or why not.

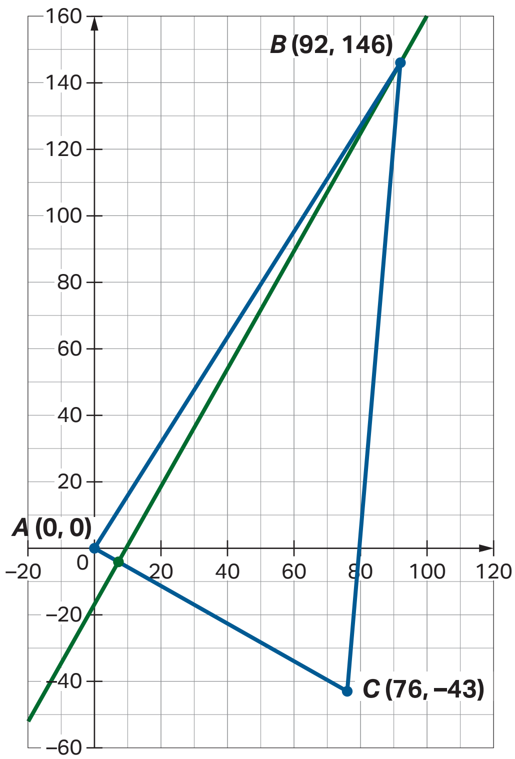
Sample Solutions	Points Possible
Using the rounded side lengths given in the problem with the Pythagorean theorem, students should explain that because $190^2 \neq 173^2 + 87^2$, the footprint of the Flatiron Building is not a right triangle. Students may calculate the lengths of the sides with the distance formula: $87.32^2 + 172.57^2 \neq 189.68^2$.	**3 points maximum** 1 point for Pythagorean theorem setup 1 point to show $190^2 \neq 173^2 + 87^2$ 1 point to conclude that the Flatiron Building is not a right triangle *Scoring note:* If students miscalculate the lengths of one or more sides with the distance formula but use the incorrect numbers to reach a consistent conclusion, they should receive 2 points.

Targeted Feedback for Student Responses
If several students make mistakes in part (a), it could mean that students need help with the distance formula or the Pythagorean theorem.

TEACHER NOTES AND REFLECTIONS

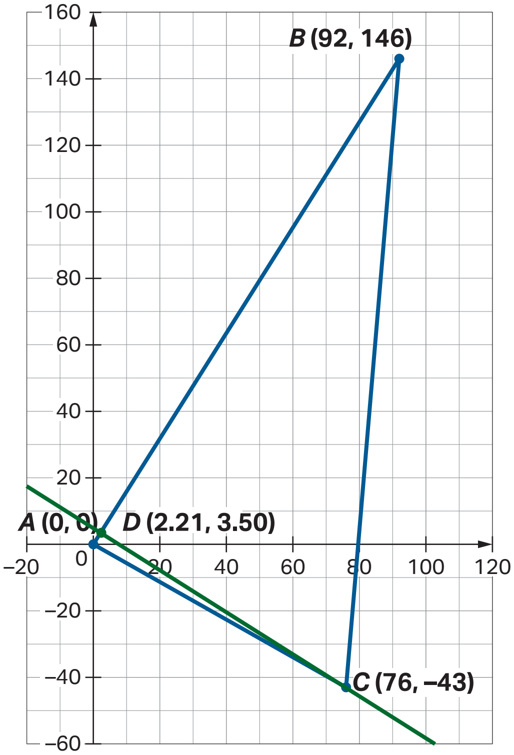
Student Stimulus and Part (b)

(b) Write an equation for a line that contains an altitude of the triangle. (Remember that an altitude is a line segment that connects a vertex of a triangle to the opposite side and is perpendicular to that side.)

Sample Solutions	Points Possible
There are three possible altitudes in the triangle and three equations that contain the altitudes. Solution 1: Using $\overline{AC}$ as the Base of the Triangle  The slope of $\overline{AC}$ is $-\frac{43}{76}$, which means the slope of the altitude must be $\frac{76}{43}$. The equation of the line that contains the altitude is $y=\frac{76}{43}(x-92)+146$, or $y=\frac{76}{43}x-\frac{714}{43}$.	**3 points maximum** 1 point for identifying the slope of the base segment 1 point for the slope of the altitude 1 point for the equation of the altitude *Scoring note:* Students should receive full credit if they have an equivalent correct linear equation and did not simplify the equation to slope-intercept form. If students incorrectly simplify the equation, they should receive a maximum of 2 points.

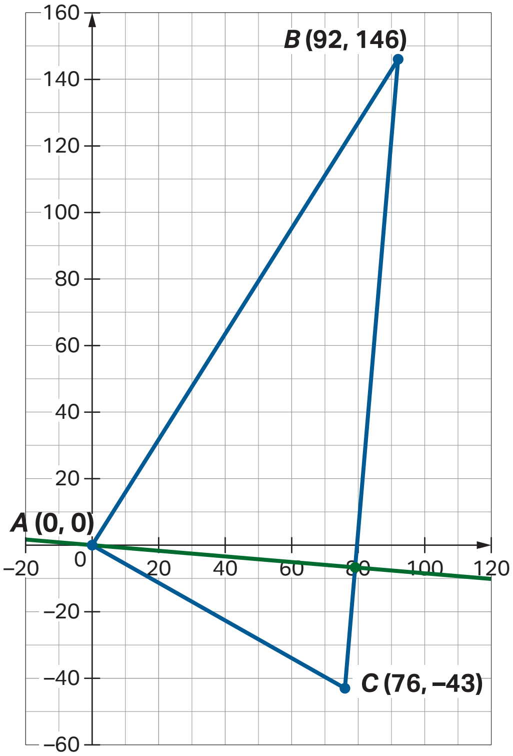
continues

Solution 2: Using $\overline{AB}$ as the Base of the Triangle



The slope of $\overline{AB}$ is $\frac{146}{92}$, which can be reduced to $\frac{73}{46}$. That means the slope of the altitude must be $-\frac{92}{146}$, or $-\frac{46}{73}$. The equation of the line that contains the altitude is $y = -\frac{46}{73}(x - 76) - 43$, or $y = -\frac{46}{73}x + \frac{357}{73}$.

continues

Solution 3: Using $\overline{BC}$ as the Base of the Triangle  The slope of $\overline{BC}$ is $\frac{189}{16}$, which means the slope of the altitude must be $-\frac{16}{189}$. The equation of the line that contains the altitude is $y = -\frac{16}{189}x$.	

Targeted Feedback for Student Responses

If several students make mistakes in part (b), it could mean that students need help with writing equations, identifying the altitude of a triangle, or understanding the relationship between the slopes of perpendicular lines in the coordinate plane.

TEACHER NOTES AND REFLECTIONS

Student Stimulus and Part (c)

(c) Determine the area of the footprint of the Flatiron Building, and round your answer to the nearest square foot. Show all your work, and explain your solution.

Sample Solutions	Points Possible
Because there are three altitudes, there are three possible ways students could determine the height of the triangle. Solution 1: Using $\overline{AC}$ as the Base of the Triangle The approximate location of the intersection point of $\overline{AC}$ and its altitude is (7.12, –4.03). The length of the altitude is about 172 feet. Using 87 feet as the length of $\overline{AC}$, the area of the triangle is approximately 7,480 square feet. Solution 2: Using $\overline{AB}$ as the Base of the Triangle The approximate location of the intersection point of $\overline{AB}$ and its altitude is (2.21, 3.50). The length of the altitude is about 87 feet. Using 173 feet as the length of $\overline{AB}$, the area of the triangle is approximately 7,526 square feet.	**3 points maximum** 1 point for the intersection point of the altitude and base 1 point for the length of the altitude 1 point for the area of the triangle

continues

Sample Solutions	Points Possible
Solution 3: Using $\overline{BC}$ as the Base of the Triangle The approximate location of the intersection point of $\overline{BC}$ and its altitude is (79.07, −6.69). The length of the altitude is about 79 feet. Using 190 feet as the length of $\overline{BC}$, the area of the triangle is approximately 7,505 square feet.	
Targeted Feedback for Student Responses	
If several students make mistakes in part (c), it could mean that students need help with the distance formula or calculating the area of a triangle.	

TEACHER NOTES AND REFLECTIONS

Points Received	Appropriate Letter Grade (If Graded)	How Students Should Interpret Their Score
8 or 9 points	A	"I know all of this geometry really well."
6 or 7 points	B	"I know all of this geometry well, but I made a few mistakes."
4 or 5 points	C	"I know some of this geometry well, but not all of it."
2 or 3 points	D	"I only know a little bit of this geometry."
0 or 1 point	F	"I don't know much of this geometry at all."

UNIT 2

LESSON 2.6

Using Right Triangles in the Coordinate Plane

OVERVIEW

LESSON DESCRIPTION

Part 1: Identifying Similar Right Triangles
Students engage in a sorting activity with similar right triangles. The activity leads to a definition of similarity based on pairs of congruent angles.

Part 2: Finding a Point along a Line Segment
Students explore how to use similar right triangles to determine the coordinates of a point along a line segment. This activity shows the usefulness of right triangles in the coordinate plane.

CONTENT FOCUS

In this lesson, students develop an informal understanding of similarity. By the end of the lesson, students should recognize that two similar right triangles have three pairs of congruent corresponding angles and proportional corresponding side lengths. The lesson focuses on right triangles because their properties allow for useful applications throughout geometry and in real-world situations. The lesson establishes the basic principles of similarity, which is necessary for understanding right triangle trigonometry.

AREAS OF FOCUS

- Engagement in Mathematical Argumentation
- Connections Among Multiple Representations

SUGGESTED TIMING

~45 minutes

HANDOUTS

Lesson

- 2.6.A: Right Triangles

Practice

- 2.6.B: Points Along a Line Segment

MATERIALS

- rulers
- protractors
- patty paper
- graph paper
- access to **Desmos.com**

COURSE FRAMEWORK CONNECTIONS

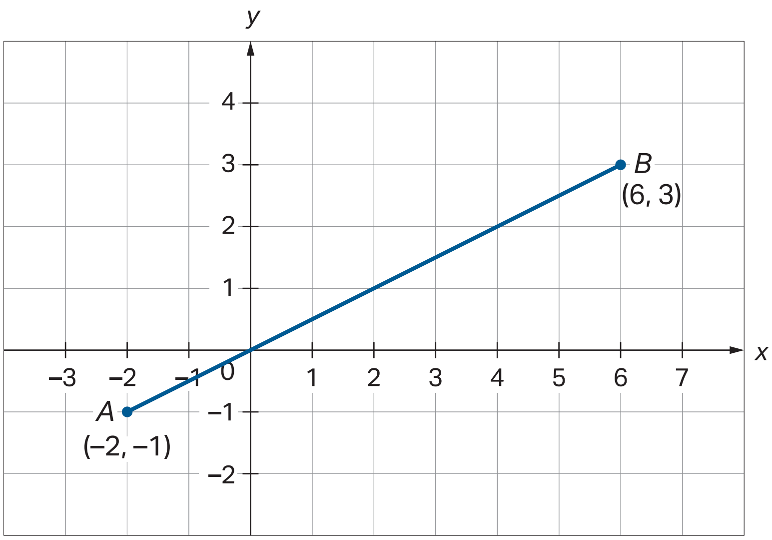
Enduring Understandings	
▪ Measuring features of geometric figures is the process of assigning numeric values to attributes of the figures, which allows the attributes to be compared. ▪ Right triangles are simple geometric shapes in which we can relate the measures of acute angles to ratios of their side lengths.	
Learning Objectives	**Essential Knowledge**
2.3.1 Prove whether two right triangles are similar using informal similarity transformations.	**2.3.1a** Two right triangles are similar if and only if one triangle can be translated, reflected, and/or rotated so it coincides with the other after dilating one triangle by a scale factor. **2.3.1b** Two right triangles are similar if and only if their corresponding angles have equal measures. **2.3.1c** Two right triangles are similar if and only if their corresponding side lengths are in proportion.
2.3.2 Determine the coordinates of a point on a line segment.	**2.3.2a** The coordinates of a point along a line segment in the coordinate plane that divides the line segment into a given ratio can be determined using similar triangles.

UNIT 2

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

Point C is located on line segment $\overline{AB}$. Point A is at $(-2, -1)$, and point B is at $(6, 3)$.



(a) Use similar right triangles to find the coordinates of point C if it is located $\frac{1}{4}$ the distance from A to B. Show or explain your work.

(b) Use similar right triangles to find the coordinates of point D if it is located $\frac{1}{6}$ of the distance from B to A. Show or explain your work.

(c) Use similar right triangles to find the coordinates of point E if it is located in such a way that the ratio of $m\overline{AD}$ to $m\overline{DB}$ is 2:3. Show or explain your work.

PART 1: IDENTIFYING SIMILAR RIGHT TRIANGLES

This lesson opens with a sorting activity involving similar right triangles. Students group right triangles based on some of their measurable attributes. The activity leads to a definition of similarity based on pairs of congruent angles.

- Begin by asking students a question to motivate the activity:
  - What makes two objects similar? What is an example of two objects that are similar but not exactly the same?
- Allow students a minute or two to think about these questions individually before asking them to share their answers with a partner. After students have talked with a partner for a few minutes, ask several pairs of students to share with the class. Listen for statements that explain that some but not all attributes of the objects would be the same. For example, two shirts could be considered similar if they both have short sleeves, even if they have different designs; two pencils could be considered similar if one is mechanical and one is not, because they both write in graphite.
- Use the examples that students provide to develop the idea that two objects are similar when some, but not all, of their attributes match. Therefore, the two objects will have some differences as well. You do not have to formally define similarity at this point. Students will know enough to engage in the sorting activity. The mathematical definition of similarity will make more sense after the activity.

Meeting Learners' Needs

If students have difficulty naming qualities that make two objects similar, you can ask them what qualities make two objects the same. Some examples familiar to students might be copies of handouts a teacher makes for a class or uniforms that a sports team might wear. Then you can prompt them to think about when two of those things might be nearly, but not exactly, the same.

- Distribute **Handout 2.6.A: Right Triangles** to each pair of students. Ask students to sort the triangles into groups that are similar to each other and to record each group as a list. Students can use rulers, protractors, and patty paper to support their choices. As you circulate around the room, encourage students to present geometric explanations for why the triangles in each group are similar.

  There are four groups of similar triangles: (1) ΔA, ΔD, ΔG; (2) ΔB, ΔF, ΔL; (3) ΔE, ΔI, ΔJ; (4) ΔC, ΔH, ΔK

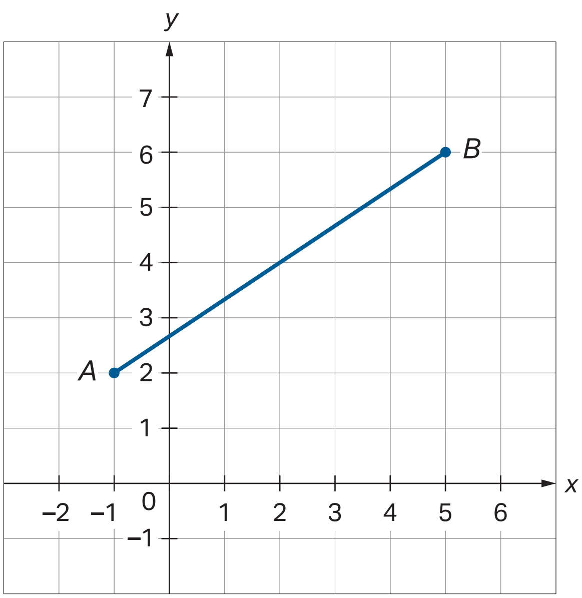
- Once students have made their groupings, have two pairs of students join together to present their groupings and compare their criteria for making these groups. Then engage in a class debrief. As you solicit responses from students, listen for explanations that reference the angle measures and those that reference the side lengths.

- It is possible that students will share that they compared the overall shape of the triangles to group them. You can probe deeper into their thinking by asking them questions like:
  - What specifically about the triangles did you compare? What parts of the triangle were the same and which parts were different?

    We noticed that the angle measures were the same, but that the corresponding sides had different lengths. Because the angle measures were the same, the triangles had the same shape but were different sizes.
- Once students understand that for triangles to be similar, some of their attributes (the angle measures) have to be the same but other attributes (the side lengths) do not have to be the same, you can share this definition for similarity: *Two right triangles are similar if they have three pairs of corresponding congruent angles.*
- It is valuable to introduce the notation for similarity here. You can use this notation with triangles from **Handout 2.6.A: Right Triangles** or with a new set of triangles you or the students create. For example, from **Handout 2.6.A: Right Triangles**, we can say that $\Delta A \sim \Delta D \sim \Delta G$.

PART 2: FINDING A POINT ALONG A LINE SEGMENT

This part of the lesson focuses students' attention on the usefulness of right triangles in the coordinate plane. Students use similar right triangles and their understanding of ratios and proportions to determine the coordinates of a point along a line segment.

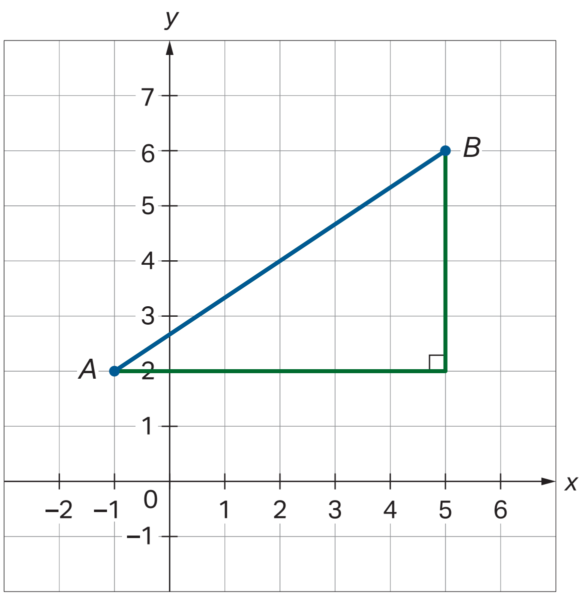


- To motivate this part of the lesson, we return to the scenario of the two brothers who have houses on a plot of land and a straight path between the houses (Lesson 2.5). Distribute graph paper to students, or you can use **Handout 2.5.B: Placing the Brothers' Fence in the Coordinate Plane** if you choose. Project or display the following scenario so all students can see it:

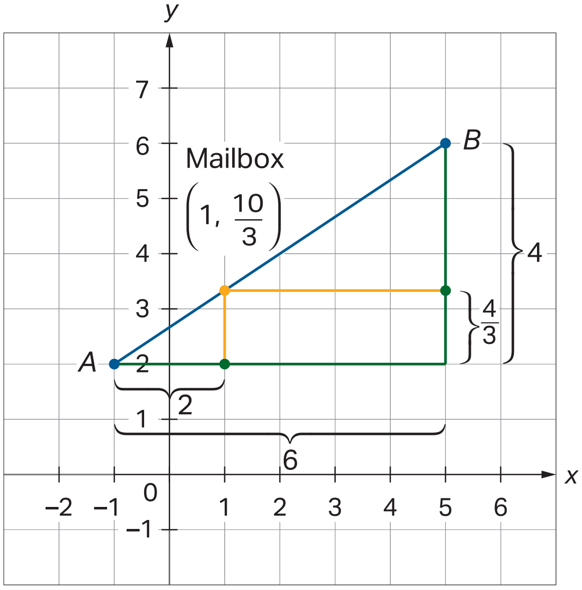
Suppose that the brother in the house in the lower left, which we'll call House A, decided to place a mailbox at a location along the path that is $\frac{1}{3}$ of the way from his house to the other house, which we'll call House B. House A is located at $(-1, 2)$, and House B is located at $(5, 6)$. What would be the coordinates of the mailbox?

- Give students a little time to closely observe the task and the graph and to ask clarifying questions. Ask students to share some initial thoughts they have about the problem with their partner. Have one or two students share their thoughts.
- It is possible that some students will suggest using the distance formula to find the distance between the houses and then dividing the result by 3 to find the distance from House A to the mailbox. This is a good suggestion, but it ultimately is not helpful if students do not know how to use the distance to the mailbox to find its coordinates. If students suggest that approach, probe their thinking with some higher-order questions such as the following:
  - Why do you think the distance formula will be helpful? How will you use the distance you find to determine the coordinates?

    We can use the distance formula to calculate the distance between the brothers' houses. Then we can divide the total distance by 3 to figure out the distance from House A to the mailbox. The distance we get tells us how far to move along the path from House A. But that might not help us get the coordinates of the mailbox.
- If no students suggest adding two auxiliary segments to form a right triangle, then you could prompt them by drawing vertical and horizontal legs to form a right triangle whose hypotenuse is $\overline{AB}$, as shown in the image.



- Some students might remark that this looks like a slope triangle. This is an important insight, because all the slope triangles that can be drawn from a given segment will be similar. In this lesson, students will use similar right triangles to find the coordinates of the mailbox.
- Have students work in pairs to create a right triangle whose dimensions are $\frac{1}{3}$ of the corresponding dimensions of the triangle you drew for them and that shares point A with the larger triangle. Then ask students to use this new triangle to find the coordinates of the mailbox.
- As you circulate around the room, provide students with help if they need it. Also listen for student explanations that can be shared during the debrief about the problem.
- Once most pairs of students seem to have figured out where the mailbox should be located, have one student share their solution and their diagram with the class. It will look something like this diagram and explanation:



The horizontal leg of the large triangle is 6 units. So, the horizontal leg of the small triangle is $\frac{1}{3}$ of 6, or 2 units. Because the x-coordinate of House A is –1, we can move 2 units to the right to see where the smaller horizontal leg ends, at $x=-1+2=1$. The vertical leg of the large triangle is 4 units. So, the vertical leg of the smaller triangle is $\frac{1}{3}$ of 4, or $\frac{4}{3}$ units. Because the y-coordinate of House A is 2, we can move $\frac{4}{3}$ units up from 2 to see where the smaller vertical leg ends, at $y=2+\frac{4}{3}=\frac{10}{3}$. The coordinates of the mailbox are $\frac{1}{3}$ the distance along the path from House A to House B. The mailbox is located at $\left(-1+2, 2+\frac{4}{3}\right)=\left(1, \frac{10}{3}\right)$. I found this by figuring out $\frac{1}{3}$ of the horizontal distance and $\frac{1}{3}$ of the vertical distance between the houses.

- Ask students to consider how they would find the location of a mailbox for House B if the brother who lived there wanted it $\frac{1}{3}$ of the way along the path from House B to House A.

  Students should reason that they should subtract $\frac{1}{3}$ of 6, or 2, from the x-coordinate of House B and subtract $\frac{1}{3}$ of 4, or $\frac{4}{3}$, from the y-coordinate of House B. So the location of the mailbox for House B is $\left(5-2, 6-\frac{4}{3}\right)=\left(3, \frac{14}{3}\right)$.
- You can use the Desmos interactive, **preap.org/brothers-mailbox**, to show students how the triangles change when the mailbox is $\frac{1}{3}$ of the distance from House B instead of $\frac{1}{3}$ of the distance from House A. In the interactive, you can also adjust the fraction of the distance between the houses.
- To check for understanding, ask students some culminating questions for this part of the lesson. Questions like the following are good examples of higher-order questions that move students' thinking beyond simple calculations:
  - What is the relationship between the point $\frac{1}{3}$ of the distance from point B to point A and the point $\frac{2}{3}$ of the distance from point A to point B?

    The point $\frac{1}{3}$ of the distance from point B to point A is the same as the point $\frac{2}{3}$ of the distance from point A to point B.
  - Use what you know to locate the point halfway between House A and House B.

    The horizontal leg of the large triangle has a length of 6 units. So, the horizontal leg of the small triangle has a length of $\frac{1}{2}$ of 6, or 3 units. We can add 3 to the x-coordinate of House A, or subtract 3 from the x-coordinate of House B. The vertical leg of the large triangle has a length of 4 units. So, the length of the vertical leg of the smaller triangle is $\frac{1}{2}$ of 4, or 2 units. We can add 2 to the y-coordinate of House A, or subtract 2 from the y-coordinate of House B. The point halfway along the path is $(2,4)$.

- Do you know another way to find the point halfway along a line segment? Do both methods produce the same coordinates? Why or why not?

  You can find the point halfway along a line segment using the midpoint formula. Both formulas have to produce the same coordinates, because there can be only one point at the middle of the segment.

- Students may find it difficult to reason algebraically, but the right triangle method of adding half the horizontal and vertical distances to the x- and y-coordinates respectively will result in the midpoint formula:

$$\left(x_1+\frac{x_2-x_1}{2}, y_1+\frac{y_2-y_1}{2}\right)$$
$$\left(\frac{2x_1+x_2-x_1}{2}, \frac{2y_1+y_2-y_1}{2}\right)$$
$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

- At this point, you can provide students with an opportunity to practice what they've learned about using right triangles in the coordinate plane. **Handout 2.6.B: Points Along a Line Segment** has several problems intended to help students deepen their understanding of the lesson content.

Meeting Learners' Needs

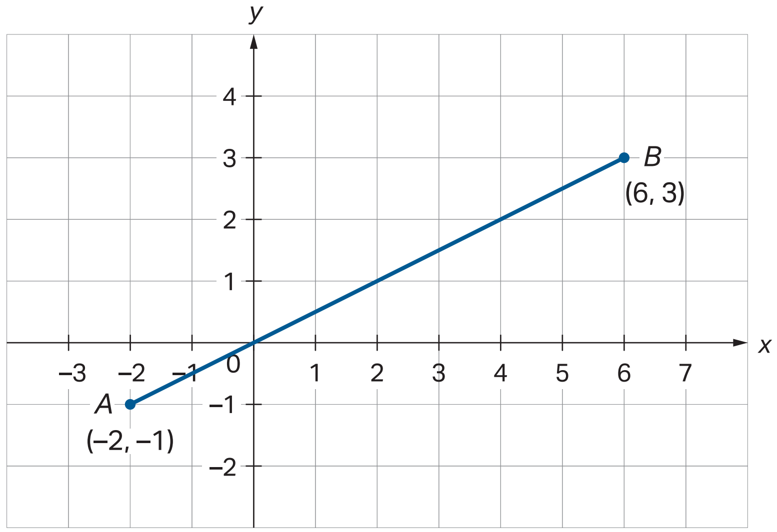
Some students might wonder if they can use a method like the midpoint formula to find the coordinates of a point that is a given fraction of the distance between the endpoints of a segment. You can explore with those students the general approach of adding the desired fraction of the distance between the points to one of the endpoints. Algebraically, the coordinates can be calculated with the formula, $\left(x_1+\frac{x_2-x_1}{n}, y_1+\frac{y_2-y_1}{n}\right)$, where $\frac{1}{n}$ is the desired fraction.

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

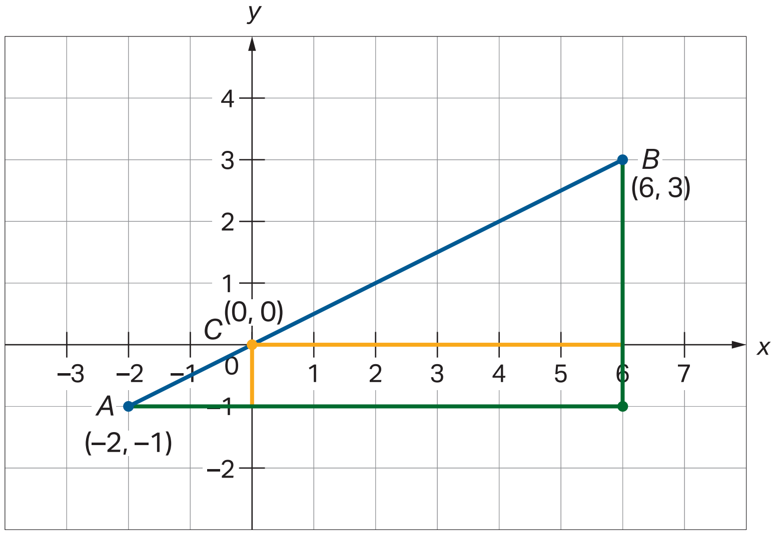
When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

Point C is located on line segment $\overline{AB}$. Point A is at $(-2, -1)$, and point B is at $(6, 3)$.



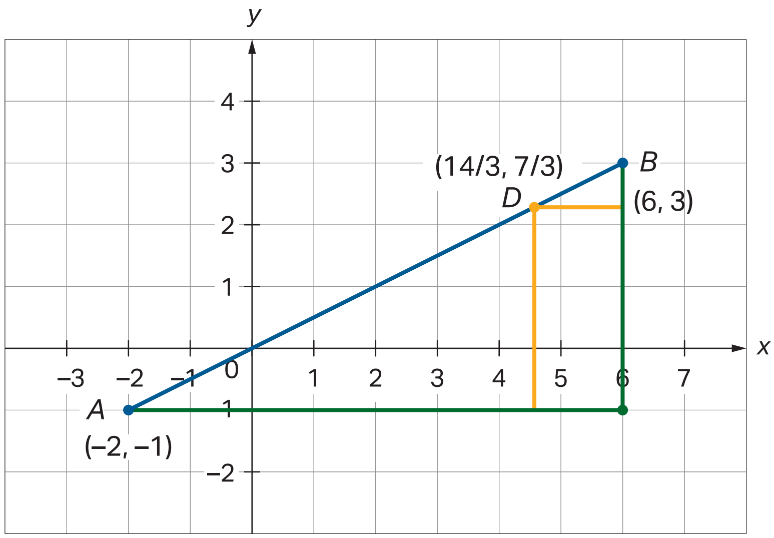
(a) Use similar right triangles to find the coordinates of point C if it is located $\frac{1}{4}$ of the distance from A to B. Show or explain your work.

The horizontal distance from A to B is 8 units, so $\frac{1}{4}$ of the horizontal distance is 2 units. The vertical distance from A to B is 4 units, so $\frac{1}{4}$ of the vertical distance is 1 unit. Point C is located at $(-2+2, -1+1) = (0, 0)$.



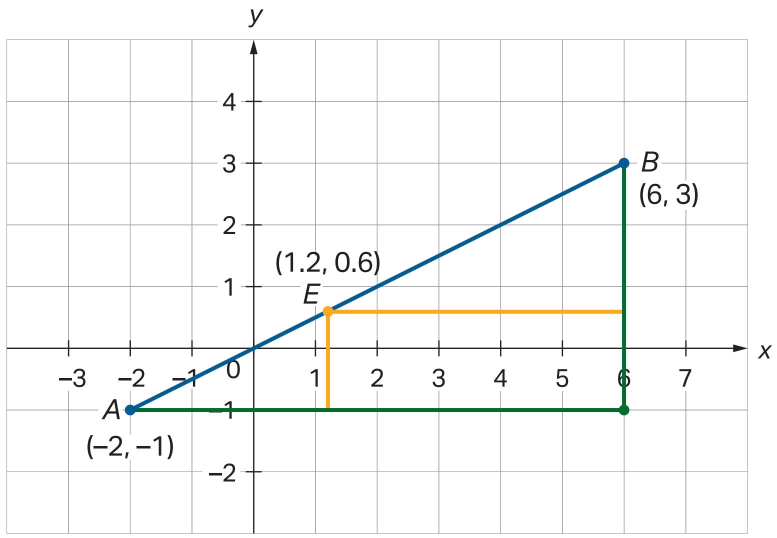
(b) Use similar right triangles to find the coordinates of point D if it is located $\frac{1}{6}$ of the distance from B to A. Show or explain your work.

The horizontal distance from B to A is 8 units, so $\frac{1}{6}$ of the horizontal distance is $\frac{4}{3}$ units. The vertical distance from B to A is 4 units, so $\frac{1}{6}$ of the vertical distance is $\frac{2}{3}$ unit. Point D is located at $\left(6-\frac{4}{3}, 3-\frac{2}{3}\right)=\left(\frac{14}{3}, \frac{7}{3}\right)$.



(c) Use similar right triangles to find the coordinates of point E if it is located in such a way that the ratio of $m\overline{AD}$ to $m\overline{DB}$ is 2:3. Show or explain your work.

If the ratio of the parts of the segment is 2:3, then there are 5 parts in total. Point E is located $\frac{2}{5}$ of the distance from A to B, (or, equivalently, $\frac{3}{5}$ of the distance from B to A). The horizontal distance is 8 units, so $\frac{2}{5}$ of the horizontal distance is $\frac{16}{5}$ units. The vertical distance from A to B is 4 units, so $\frac{2}{5}$ of the vertical distance is $\frac{8}{5}$ units. Point E is located at $\left(-2+\frac{16}{5},-1+\frac{8}{5}\right)=\left(\frac{6}{5},\frac{3}{5}\right)$.



Guiding Student Thinking

Students may be able to find the proportional side lengths of the smaller similar triangle but forget to use the coordinates of an endpoint to locate the point along the segment. Students might not take into account the direction they should move, so they might have added when they should have subtracted. Encourage students to think about which endpoint they are starting at and whether or not they should add or subtract to find the coordinate they need.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

UNIT 2

Handout 2.6.A: Right Triangles

This handout presents a sorting activity for similar right triangles. See lesson for answers.

Handout 2.6.B: Points Along a Line Segment

This structured problem set will help students practice the skill of finding a point along a line segment and will also deepen students' understanding of similar right triangles. No images of right triangles are shown with the solutions below, but students should be encouraged to draw right triangles to aid their conceptual understanding and their calculations.

1. The horizontal distance between A and B is 5 units; the vertical distance between A and B is 9 units.

   (a) Point C is located at $\left(2+\frac{1}{4}\bullet 5, 7-\frac{1}{4}\bullet 9\right)=\left(\frac{13}{4},\frac{19}{4}\right)$.

   (b) Point D is located at $\left(2+\frac{2}{5}\bullet 5, 7-\frac{2}{5}\bullet 9\right)=\left(4,\frac{17}{5}\right)$.

   (c) Point E is located at $\left(7-\frac{3}{4}\bullet 5, -2+\frac{3}{4}\bullet 9\right)=\left(\frac{13}{4},\frac{19}{4}\right)$; point E is in the same location as point C from part (a).

   (d) Point F is located at $\left(7-\frac{2}{5}\bullet 5, -2+\frac{2}{5}\bullet 9\right)=\left(5,\frac{8}{5}\right)$.

2. The horizontal distance between A and B is 8 units. The vertical distance between A and B is 12 units.

   (a) Point C is located $\frac{1}{8}$ of the distance from A to B. The horizontal distance from A to C is 1 unit. This means that the fractional horizontal distance is $\frac{1}{8}$ unit. The vertical distance from A to C is $\frac{3}{2}$ units. This means that the fractional vertical distance is $\frac{3/2}{12}=\frac{1}{8}$ unit. The ratio of $m\overline{AC}$ to $m\overline{AB}$ is 1:8.

   (b) If point C is $\frac{1}{8}$ of the distance from A to B, then it must be located $\frac{7}{8}$ of the distance from B to A. This is because point C splits the segment into two parts, and the fractions of the two parts must add to 1, which is the total distance from A to B (or from B to A).

   (c) The ratio of $m\overline{AC}$ to $m\overline{CB}$ is 1:7.

   (d) The part:part ratio can be used to determine the part:whole ratio because the sum of the two parts has to equal the whole.

3. The horizontal distance between M and N is 10 units. The vertical distance between M and N is 10 units.
   (a) If the ratio of $m\overline{MQ}$ to $m\overline{QN}$ is 1:3, then the ratio of $m\overline{MQ}$ to $m\overline{MN}$ is 1:4. This is because the sum of the two parts must be equal to the whole, and 1 + 3 = 4.
   (b) If point R is located on $\overline{MN}$ so that $m\overline{MR} : m\overline{RN}$ is 3:4, then R is located $\frac{3}{7}$ of the distance from M to N. Point R is located at $\left(-2+\frac{3}{7}\bullet 10, 5-\frac{3}{7}\bullet 10\right)=\left(\frac{16}{7},\frac{5}{7}\right)$.
4. If point W is located on $\overline{ST}$ so that $m\overline{SW} : m\overline{WT}$ is 1:5, that means that point W is $\frac{1}{6}$ of the distance from S to T. Because point V is $\frac{1}{5}$ of the distance from S to T, point W is closer to point S.

UNIT 2

LESSON 2.7

Similarity and the Pythagorean Theorem

OVERVIEW

LESSON DESCRIPTION

Part 1: Exploring Pythagorean Triples
Students explore common Pythagorean triples in the geometric context of similar triangles. Students observe that multiples of Pythagorean triples are also Pythagorean triples.

Part 2: Using Similarity to Prove the Pythagorean Theorem
Students develop a proof of the Pythagorean theorem using the similarity of right triangles. This part of the lesson actively engages students in rigorous reasoning and sense making with a particular goal.

Part 3: Using the Pythagorean Theorem in Context
Students have an opportunity to build skill with the Pythagorean theorem through contextual scenarios involving the side lengths of right triangles.

AREAS OF FOCUS

- Engagement in Mathematical Argumentation
- Greater Authenticity of Applications and Modeling

SUGGESTED TIMING

~90 minutes

HANDOUTS

Practice

- 2.7: Using the Pythagorean Theorem in the Real World

MATERIALS

- calculator

CONTENT FOCUS

This lesson expands on students' previous work with the Pythagorean theorem. Students explore sets of values to identify Pythagorean triples in arithmetic and geometric contexts. Finally, students develop a proof of the Pythagorean theorem using similar triangles.

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ A formal mathematical argument establishes new truths by logically combining previously known facts. ▪ Measuring features of geometric figures is the process of assigning numeric values to attributes of the figures, which allows the attributes to be compared. ▪ Right triangles are simple geometric shapes in which we can relate the measures of acute angles to ratios of their side lengths.	
Learning Objectives	**Essential Knowledge**
2.3.2 Determine the coordinates of a point on a line segment.	**2.3.2a** The coordinates of a point along a line segment in the coordinate plane that divides the line segment into a given ratio can be determined using similar triangles.
2.3.3 Prove the Pythagorean theorem using similar right triangles.	**2.3.3a** An altitude drawn from the right angle of a right triangle to the hypotenuse creates similar right triangles. **2.3.3b** When an altitude is constructed from the right angle to the hypotenuse of a right triangle, the proportions of the side lengths of the similar right triangles formed can be used to prove the Pythagorean theorem.

UNIT 2

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

A man has three children. They are not triplets. By a remarkable coincidence, however, they were all born on the same day in different years. By another remarkable coincidence, their ages right now form a Pythagorean triple. The oldest child is nine years older than the youngest, and the middle child is two years younger than the oldest.

(a) How old are each of the three children? Show your work.

(b) The man knows that a multiple of a Pythagorean triple is also a Pythagorean triple. He reasons that in the year when the ages of his three children have doubled, their ages will again form a Pythagorean triple. Is he correct? Why or why not?

PART 1: EXPLORING PYTHAGOREAN TRIPLES

In this part of the lesson, students explore similarity as it relates to Pythagorean triples. Students learn the definition of a Pythagorean triple and the fact that multiples of Pythagorean triples are also Pythagorean triples.

Instructional Rationale

Pythagorean triples—sets of three integers that satisfy the Pythagorean relationship—have fascinated humans for a very long time. A clay tablet from Babylonia, written in about 1800 BCE, shows tables of integers that solve the equation $a^2 + b^2 = c^2$. Integer solutions to the Pythagorean equation are studied in the branch of mathematics known as number theory. The purpose of Part 1 of the lesson is to allow students to see that integer solutions to the Pythagorean equation are rare and that restricting the solutions to positive integers is simply a preference, not a necessity. The intended focus for students is on the similarity argument that scaling a Pythagorean triple will create another Pythagorean triple. Pythagorean triples have limited applications, so it is not necessary for students to memorize sets of Pythagorean triples.

- Begin by presenting this task to students:

  Which of these sets of three values could be the side lengths of a right triangle?

  $(3,4,5)$; $(1,1,\sqrt{2})$; $(5,7,9)$; $(2,5,\sqrt{29})$; $(7,24,25)$; $(1,1,3)$; $(6,11,\sqrt{17})$; $(33,56,65)$
- Provide students with time to test each set. Some students may benefit from partnering with another student if they have trouble getting started.
- Once students have selected which sets could be the lengths of the sides of a right triangle, have them share their choices and explain their reasoning.

  The sets $(3,4,5)$; $(1,1,\sqrt{2})$; $(2,5,\sqrt{29})$; $(7,24,25)$; and $(33,56,65)$ could be the sides of a right triangle, because they are solutions to the equation given by the Pythagorean theorem.
- Draw students' attention to the sets that only have integer values. You could ask them the following:
  - Notice that some of the sets of values only include integers. Do you think there are other sets of integers that will satisfy the Pythagorean theorem?
- Allow some time for students to think about the question and to make some conjectures about the answer. Then ask students to see if they can find two more sets of three integers that will satisfy the Pythagorean theorem.

Instructional Rationale

It is valuable for students to search for some Pythagorean triples using a calculator. This exercise gives them a chance to practice their calculator skills in a low-risk situation. Because most students will require multiple attempts to find a Pythagorean triple using trial and error, there should be little stigma attached to being incorrect.

- Give students a little time to find one or two Pythagorean triples. It is likely that a student will stumble upon $(6,8,10)$, $(9,12,15)$, $(5,12,13)$, or $(8,15,17)$. If no students find a triple, you can turn their attention to the sets from the first task: $(3,4,5)$, $(7,24,25)$, and $(33,56,65)$. At this point, you can define a Pythagorean triple as "a set of three positive integers that satisfy the equation $a^2+b^2=c^2$."
- Ask students to choose any Pythagorean triple, one from the first task or one that they found. Then ask the following:
  - Suppose that you doubled all the integers of your Pythagorean triple. Do you think the new values would also be a Pythagorean triple? Try it and see what happens.
  - Suppose that you tripled all the integers of your Pythagorean triple. Do you think the new values would also be a Pythagorean triple?
- Allow time for students to double and then triple their chosen Pythagorean triples and determine if each new set satisfies the equation $a^2+b^2=c^2$.

  My doubled and tripled sets are also Pythagorean triples.
- Now that students have explored Pythagorean triples, it's important to have them synthesize their understanding. You could ask some questions like the following:
  - Do you think you can multiply your Pythagorean triple by any positive integer and get another Pythagorean triple? Try a few values and see what happens.

    Based on my work, I think any new set formed by multiplying each value of a Pythagorean triple by a positive integer will also be a Pythagorean triple.
- Now ask students to make a conjecture about whether or not a positive integer multiple of any Pythagorean triple will also be a Pythagorean triple.

  It looks like any new set formed by multiplying each value of a Pythagorean triple by the same positive integer will also be a Pythagorean triple.

Guiding Student Thinking

It is likely that students will be convinced that any positive integer multiple of a Pythagorean triple is also a Pythagorean triple even though they only explored a few examples. It is important to point out that they don't know for sure that this conjecture is true, but they have established evidence to support that it may be true. In mathematics, we need to prove the statement is true for all multiples to be certain that the conjecture is true.

- Guide students through the algebraic steps to prove their conjecture. You might need to remind students that the goal of a proof is to reason logically from known information to reach a desired conclusion. You can use a table format like the one shown below. While working through the steps, ask students for suggestions for a possible next step and to give the explanations that lead from one statement to the next.

	Statements	Reasons
1	Suppose we know that (a,b,c) is a Pythagorean triple, and we want to know if a multiple of (a,b,c) is also a Pythagorean triple.	State the given information and the goal of the proof.
2	Because (a,b,c) is a Pythagorean triple, we know that it satisfies $a^2+b^2=c^2$.	This is the Pythagorean theorem.
3	We can choose any positive integer to multiply each value in the Pythagorean triple by. We can represent this integer with k.	This returns to the hypothesis and reminds us of the goal.
4	If we multiply (a,b,c) by k, we will get a new set, (ka,kb,kc).	Multiplication by the integer we chose
5	Now we have to see if the values in the set (ka,kb,kc) satisfy the Pythagorean theorem.	Testing the Pythagorean theorem
6	$(ka)^2+(kb)^2\stackrel{?}{=}(kc)^2$ $k^2a^2+k^2b^2\stackrel{?}{=}k^2c^2$	Square each term.

	Statements	Reasons
7	$k^2\left(a^2+b^2\right)\stackrel{?}{=}k^2c^2$	Use the distributive property to rewrite the left side by factoring out the constant k^2.
8	$k^2\left(a^2+b^2\right)=k^2c^2$	This is a true statement because we know that $a^2+b^2=c^2$ is a true statement, and multiplying both sides of an equation by the same value, k^2, does not change the truth of the equation.
9	Therefore, if (a,b,c) is a Pythagorean triple, then any positive integer multiple of it, (ka,kb,kc), will also be a Pythagorean triple.	

Guiding Student Thinking

It is tempting to divide both sides of the statement in line 7 by k^2 because it looks like an equation, but we don't know if the right side is equal to the left side. Therefore, we should not divide both sides of the equation by the same value. However, since we already know that the Pythagorean equation is true, we could have multiplied both sides by k^2 to yield another true statement.

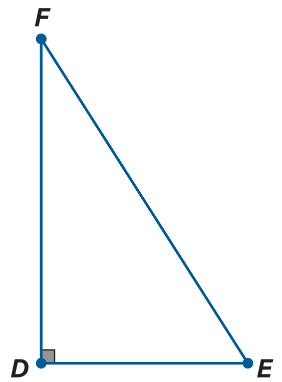
PART 2: USING SIMILARITY TO PROVE THE PYTHAGOREAN THEOREM

In this part of the lesson, students work collaboratively to craft a proof of the Pythagorean theorem using similar right triangles. This may be the most advanced proof students have developed at this point in the course, and it may take some time to for them to understand the process. While the reasoning used here is somewhat more advanced than what students have already experienced, the mathematical concepts necessary to complete the proof are within students' knowledge base. Engaging in a real mathematical problem can sometimes lead down wrong paths, but remind students that this is part of the process. Through this structured activity, students explore several features of mathematical proof writing.

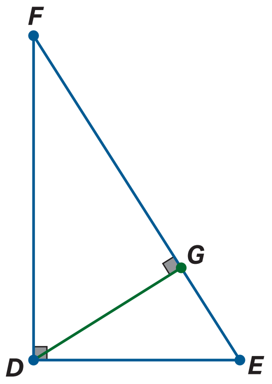
Instructional Rationale

There are many proofs of the Pythagorean theorem that are usually included in a geometry course. Proofs of the Pythagorean theorem have been written by humans for millennia. The one shown here is perhaps different from the more conventional proofs that use areas of squares and triangles. The proofs that use area should be reserved for later in the course. This proof relies on similarity and reasoning with proportions, so it connects to concepts in this section of the course. The proof developed in this lesson is not very different from the one Euclid included in the *Elements*.

- Let students know that they will be figuring out how to prove the Pythagorean theorem using what they know about similar right triangles.
- Have each student draw a large right triangle with vertices D, E, and F and a right angle at D, and display one so everyone can see it. Alternatively, you could provide students with a right triangle.



- Ask students to draw a line segment from point D perpendicular to side $\overline{FE}$. Label the point of intersection G. Then ask students to closely observe the resulting figure. Have students write down what they notice and what they wonder about the resulting figure. Allow some individual think time, and then encourage students to share their observations with a partner.



- Have students share some of their observations and record them for the entire class to see. At this point, everything students noticed and wondered should be shared and written down for the class.
- Through the discussion, listen for students to notice that the resulting figure has more right triangles and to wonder if the new triangles have any relationship to each other or to the original triangle. Probe these observations by asking questions such as these:
  - How many triangles do you see? Are they all right triangles? How do you know?

    There are three triangles: ΔFDE, ΔDGE, and ΔDGF. The original triangle was given as a right triangle, and the two new triangles are right triangles because we drew $\overline{DG}$ perpendicular to $\overline{FE}$.
  - Can you tell if the triangles are similar? What information would you need to show that the triangles are similar?

    To show that two triangles are similar, we would need to know that the corresponding angles are congruent or that the corresponding sides are in proportion.

Guiding Student Thinking

The next step is for students to justify that the three triangles are similar by looking at their angle measures. Some students will need to use numerical examples of angle measures to explain which angles are congruent, while other students will be able to reason abstractly about the measures of the angles.

- You may need to prompt students to focus on the angle measures. Once they realize that comparing angles is necessary for the next step in the proof, you can proceed. Because there was no angle measure given for $\angle F$, it could reasonably be any acute angle. Let pairs of students choose different values for $m\angle F$ and use those values to determine the measurements of the rest of the angles in the figure.
- Once students determine all the angle measures in the figure, have them identify which angles are congruent, using the appropriate notation.

Classroom Ideas

If students draw their triangles on small whiteboards, you can have several students share their congruent angles at the same time.

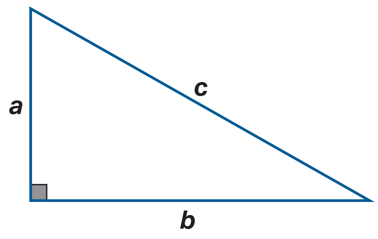
- Have students share their congruent angles. Encourage them to explain why angles would have to be congruent regardless of the measure of $\angle F$.

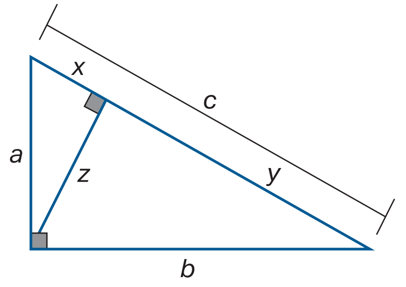
  $\angle FDE \cong \angle FGD \cong \angle DGE$, because they are right angles and all right angles are congruent. $\angle DFG$ and $\angle DEG$ are complementary because they are the acute angles of ΔDEF. $\angle DFG$ and $\angle FDG$ are complementary because they are the acute angles of ΔDGF. This means that $\angle DEG \cong \angle FDG$. Likewise, $\angle GDE$ and $\angle DEG$ are complementary because they are the acute angles of ΔDGE. This means that $\angle GDE \cong \angle DFG$.

- Once students have identified the congruent angles, have them indicate which triangles are similar using the conventional notation. They should also be able to explain the order of the vertices in their notation. Then ask them why the triangles are similar. They will need this information to prove the Pythagorean theorem using similarity.

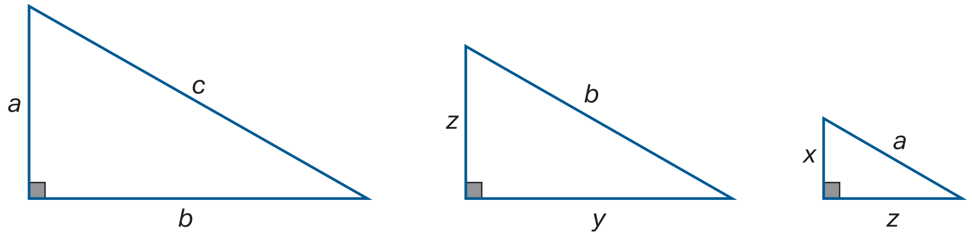
  "Big" triangle ΔFDE is similar to "medium" ΔFGD is similar to "little" ΔDGE. In correct notation, $\Delta FDE \sim \Delta FGD \sim \Delta DGE$. We list the vertices of the triangles in these orders to show which angles are congruent.

- The next part of the proof relies on writing proportions using the side lengths of the triangles. To make this process a little easier for students, suggest labeling each side of the triangles with a different variable. This will help focus students' attention on the lengths of the sides instead of the angles.





- Some students will need to see the triangles drawn separately to write the similarity statement. Have students draw the three triangles in the same orientation, like in this image:



- After students have their three similar triangles set up, have students write as many true proportions as possible. At this point in the proof, it is probably unclear to students what the next step will be, so writing down what relationships they know from the similar triangles is a good idea.
- Allow some time for students to write down the sets of proportions from the triangles.

  There are three sets of proportions: (1) relating the legs of the triangles $\frac{a}{b}=\frac{z}{y}=\frac{x}{z}$; (2) relating the shorter leg to the hypotenuse $\frac{a}{c}=\frac{z}{b}=\frac{x}{a}$; and (3) relating the longer leg to the hypotenuse $\frac{b}{c}=\frac{y}{b}=\frac{z}{a}$.
- Students might write down the proportions as an equation of two ratios, so there could be as many as nine proportions (or equivalent forms) from students. Record all the proportions so everyone in the class can see them.

Instructional Rationale

Before moving to the next step, it is important to take a step back to remind students about what they're trying to do: prove the Pythagorean theorem. Students sometimes get the impression that crafting a proof is a straightforward process of logical deductions, but that is not always true. Often, there are many potential next steps, and most of these will be false starts. Students should understand that this is part of doing mathematics. A helpful strategy is to stop and analyze all available information with an eye toward the eventual goal. In this case, the goal is to manipulate these proportions into $a^2 + b^2 = c^2$.

- Take a moment with students to reflect on the goal of the task. You can ask some questions like these:
  - What is the goal of this task?

    The goal is to prove the Pythagorean theorem.
  - What equation should we get at the end of the proof?

    We should get $a^2 + b^2 = c^2$.
  - Can we use any of the proportions we listed to express a^2, b^2, or c^2?
- Let students have some time to puzzle through the proportions to see which ones will lead them to a^2 and b^2. It is not possible to express c^2 from any of the proportions they wrote down.

  We can rewrite $\frac{a}{c} = \frac{x}{a}$ to show that $a^2 = cx$ and we can rewrite $\frac{b}{c} = \frac{y}{b}$ to show that $b^2 = cy$.
- Once students have rewritten the proportions, have them take a moment to reflect, again, on the goal and what they are trying to achieve. Ask students some questions like the following:
  - Is there a way to make a new equation that looks almost like the Pythagorean equation from these two equations?

    We can add the equations together to get a new equation with $a^2 + b^2$ on the left and $cx + cy$ on the right.

Meeting Learners' Needs

Many students will remember the procedure of cross-multiplying as a way to express proportions as equivalent equations. To reinforce why the procedure works, you can review the properties of equality that underlie the technique. Remind students that the multiplicative property of equality states that multiplying both sides of an equation by a nonzero quantity results in a true equation. Choosing a particular number to multiply so that it "undoes" the division in the proportion is an application of the inverse property of multiplication. Show students that multiplying both sides of $\frac{a}{c} = \frac{x}{a}$ by c produces $c \bullet \frac{a}{c} = c \bullet \frac{x}{a}$, which simplifies to $a = \frac{cx}{a}$. Next, multiplying

UNIT 2

- It might take some time for students to recognize that adding the equations is the next step. Give them some time to make suggestions for how to manipulate the two equations. As time permits, follow as many suggestions as possible, even those that are false starts.

both sides by a produces $a \bullet a = a \bullet \frac{cx}{a}$, or $a^2 = cx$. In the same manner, $\frac{b}{c} = \frac{y}{b}$ can be rewritten as $b^2 = cy$. The cross-multiplication technique is a shortcut in which both multiplications are performed simultaneously.

Guiding Student Thinking

Often, students will not take the risk of making a suggestion, for fear of being wrong. You can reduce anxiety by assuring students that all suggestions are welcome. To show students that you take all suggestions seriously, follow suggestions down paths that are not ultimately productive. Remind students that they cannot know whether the idea they follow is right until they are well on their way down the path. Occasionally, a suggestion will lead to an unanticipated but useful understanding or result. Let students know that doing mathematics often involves productive struggle.

- Display the equation $a^2 + b^2 = cx + cy$ for everyone to see, and ask students for ideas about how the equation could be manipulated or rewritten at this point. It may take some prompting for students to remember how to factor the right side of the equation.
- Display the rewritten equation $a^2 + b^2 = c(x + y)$ for students. Draw their attention to the quantity $(x + y)$. Ask them what they know about the value of $(x + y)$. You may have to direct them back to the triangle diagrams they drew.

  The segment addition postulate tells us that $x + y = c$. We can substitute c for $(x + y)$ in the equation.

- Display the equation after making the substitution: $a^2 + b^2 = c \bullet c$. Ask students what they observe.

  Because $c \bullet c = c^2$, we have shown that $a^2 + b^2 = c^2$, which is the Pythagorean equation.

- Because it has taken a considerable amount of time and reasoning to get to this point, give students time to retrace their steps so they can explain the entirety of their proof. This is a good opportunity to have students engage in evidence-based writing to develop their skills in written communication.

PART 3: USING THE PYTHAGOREAN THEOREM IN CONTEXT

The Pythagorean theorem can be extremely helpful for solving problems in real life. Students will develop a deeper conceptual understanding of, and procedural fluency with, the Pythagorean theorem through some carefully constructed problems that involve right triangles.

- Following the exploration of similarity and the Pythagorean theorem, provide students with an opportunity to build their skill with the Pythagorean theorem through contextualized scenarios on **Handout 2.7: Using the Pythagorean Theorem in the Real World**. The problems on the handout are designed to promote problem-solving skills and support productive struggle in learning mathematics.
- To complete these problems, students should collaborate with one or two classmates. The problems steadily increase in conceptual difficulty. The problems involving TV sizes and the Pythagorean spiral are more difficult than typical Pythagorean theorem problems, but they are also significantly more interesting.
- Students are likely to use a variety of different solution techniques for these problems. It can be worthwhile for students to share their techniques with the class. You could have pairs of students create posters of their solutions and have other students respond to the solutions.
- This handout can also be used to encourage students' evidence-based writing. Because the problems involve a step or two beyond calculating the length of a side of a right triangle, students should explain how they arrived at their answers.
- Solutions for the problems are included in the Assess and Reflect on the Lesson section below.

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

A man has three children. They are not triplets. By a remarkable coincidence, however, they were all born on the same day in different years. By another remarkable coincidence, their ages right now form a Pythagorean triple. The oldest child is 9 years older than the youngest, and the middle child is 2 years younger than the oldest.

(a) How old are each of the three children? Show your work.

The children are 8, 15, and 17 years old. I let the age of the youngest child be represented by x. The oldest child is 9 years older, so the oldest child's age can be represented by $(x+9)$. Because the middle child is 2 years younger than the oldest child, the middle child's age can be represented by $(x+9-2)$, which can be simplified to $(x+7)$. Using the Pythagorean theorem:

$$x^2+(x+7)^2=(x+9)^2$$
$$x^2+x^2+14x+49=x^2+18x+81$$
$$x^2-4x-32=0$$
$$(x-8)(x+4)=0$$

This means that the youngest child is either 8 or –4 years old. Since negative numbers do not make sense in this context, the youngest child is 8, so the other two children are 15 and 17 years old.

(b) The man knows that a multiple of a Pythagorean triple is also a Pythagorean triple. He reasons that in the year when the ages of his three children have doubled, their ages will again form a Pythagorean triple. Is he correct? Why or why not?

The man is not correct. While it is true that if all the ages were doubled, the ages would form Pythagorean triple, this situation would not happen in the same year. You can observe that this is not true by taking each child's age as its own case:

Case 1—doubling 8: This would happen in 8 years, making the children's ages 16, 23, and 25. Because $16^2+23^2\neq 25^2$, the ages are not a Pythagorean triple.

Case 2—doubling 15: This would happen in 15 years, making the children's ages 23, 30, and 32. Because $23^2+30^2\neq 32^2$, the ages are not a Pythagorean triple.

Case 3—doubling 17: This would happen in 17 years, making the children's ages 25, 32, and 34. Because $25^2+32^2\neq 34^2$, the ages are not a Pythagorean triple.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

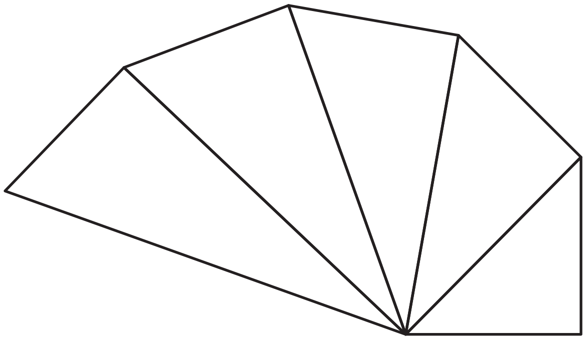
Handout 2.7: Using the Pythagorean Theorem in the Real World

1. Kayla's height above the boat can be found using the Pythagorean theorem: $181^2 + x^2 = 240^2$. Kayla is about 157.6 meters above the boat.
2. Using the Pythagorean theorem, the side along the edge of the improperly irrigated part of the field is calculated to be 80 m. The area of the improperly irrigated part of the field is $\frac{1}{2}(80)(150) = 6{,}000$ square meters. You can convert this area to acres by dividing by 4,000, because 1 acre ≈ 4,000 square meters. Therefore, the improperly irrigated part of the field is 1.5 acres. If soybeans sold for \$12 a bushel and each acre could have produced 54 bushels, then the amount of money lost was $1.5(54)(12) =$ \$972.
3. The distance from A to B in feet is $1.4(5{,}280) = 7{,}392$ feet. The distance from A to C in feet is $\sqrt{7{,}392^2 + 158^2} \approx 7{,}393.69$. Therefore, the total distance tunneled by the company, from A to C and from C to B, to the nearest hundredth of a foot is $7{,}390.31 + 158 \approx 7{,}551.69$ feet. Converting this distance to miles, the company dug 1.43 miles instead of 1.4 miles. This means they dug about 0.03 miles more than necessary, at an additional cost of \$3.6 million.
4. (a) The heights and widths of the TVs have to satisfy both relationships. If we represent width as w, height as h, and TV size as t, we can write the equations $w = \frac{4}{3}h$ and $h^2 + w^2 = t^2$. The first equation can be substituted into the second, leaving one variable to solve for. A 15" TV would have a height of 9" and a width of 12". A 42" TV would have a height of 25.2" and a width of 33.6".

   (b) The heights and widths of the TVs have to satisfy both relationships: $w = \frac{16}{9}h$ and $h^2 + w^2 = t^2$. Again, the first equation can be substituted into the second, leaving one variable to solve for. A 42" TV would have a height of about 20.6" and a width of about 36.6". A 60" TV would have a height of about 29.4" and a width of about 52.3".

   (c) If two TVs have the same aspect ratio and the same diagonal length, it is not possible for them to have different screen lengths and widths. The relationship between the width and height is fixed by the aspect ratio, so there is only one pair of solutions to the equations $w = \frac{16}{9}h$ and $h^2 + w^2 = t^2$.

5. (a) The spiral with the fourth and fifth triangles would look like this:



(b) The lengths of the hypotenuses are $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$, and $\sqrt{6}$.

(c) The pattern seems to be that the value under the square root increases by 1 with every additional triangle. This suggests that the length of the hypotenuse of the nth triangle is $\sqrt{n+1}$.

(d) If the pattern holds, then the length of the hypotenuse of the 10th triangle will be $\sqrt{11}$.

LESSON 2.8

Introducing the Tangent Ratio

OVERVIEW

LESSON DESCRIPTION

Part 1: Surveying for a Bridge
Students explore a scenario in which indirect measurement is necessary to determine the distance across a creek. The task creates a need for trigonometry. Students return to the task to solve it in the third part of the lesson.

Part 2: Connecting Angle Measures and Side Length Ratios in Similar Right Triangles
Students investigate the ratios of the side lengths of similar right triangles to discover that these ratios are constant for a given angle measure. Trigonometric ratios are defined as functions whose input is the angle measure and whose output is the ratio of the side lengths in similar right triangles.

Part 3: Surveying for a Bridge with the Tangent Function
Students return to the land surveying scenario to apply what they know about the tangent function to solve the problem.

CONTENT FOCUS

Through this lesson, students should begin to understand that trigonometric ratios naturally emerge from the similarity of right triangles. This lesson begins by having students think about the relationship between the measure of an angle in a triangle and the ratio of certain sides of that triangle. Through this lesson and the next, students see that the sine, cosine, and tangent ratios depend only on the angle used as a

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~90 minutes

LESSON SEQUENCE

This lesson is part of a lesson sequence (~225 minutes total) with Lesson 2.9.

HANDOUTS

Lesson

- 2.8.A: Surveying for a Bridge
- 2.8.B: Measuring Similar Right Triangles

Practice

- 2.8.C: Practice Using the Tangent Ratio

MATERIALS

- graph paper
- protractors
- straightedges and rulers
- compasses
- calculators

reference and not on a specific triangle used to compute the ratio. This understanding is crucial for thinking of the trigonometric ratios as functions whose input is an angle measure and whose output is a ratio of side lengths of all similar right triangles that include that angle measure.

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings

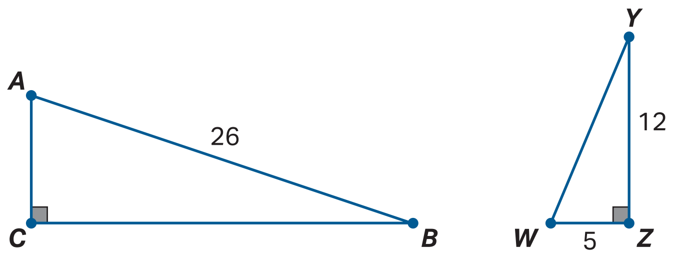
- Measuring features of geometric figures is the process of assigning numeric values to attributes of the figures, which allows the attributes to be compared.
- Right triangles are simple geometric shapes in which we can relate the measures of acute angles to ratios of their side lengths.

Learning Objectives	Essential Knowledge
2.3.4 Associate the measures of an acute angle, $\angle A$, in a right triangle to ratios of the side lengths.	**2.3.4c** The tangent of the measure of $\angle A$ is the ratio of the length of the side opposite the angle and the length of the side adjacent to the angle.
2.3.5 Explain why a trigonometric ratio depends only on an angle measure of a right triangle and not on the side lengths.	**2.3.5a** Trigonometric ratios are functions whose input is an acute angle measure and whose output is a ratio of two side lengths in a right triangle. **2.3.5b** The ratio of the lengths of two sides of a right triangle will equal the ratio of the lengths of the corresponding sides of a similar right triangle. Therefore, the ratios of the sides depend only on the angle measure.
2.3.7 Model contextual scenarios using right triangles.	**2.3.7a** Contextual scenarios that involve nonvertical and nonhorizontal segments or the distance between two points that do not lie on a vertical or horizontal line can be modeled by right triangles. **2.3.7b** Trigonometric ratios can be used to solve problems or model scenarios involving angles of elevation or depression.

UNIT 2

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.



In the diagram above, $\Delta ABC \sim \Delta WYZ$.

(a) What is the ratio that defines $\tan(Y)$?

(b) What is the ratio that defines $\tan(B)$?

(c) Which value is larger: $\tan(W)$ or $\tan(A)$? Justify your answer.

(d) Which value is larger: $\tan(W)$ or $\tan(B)$? Justify your answer.

PART 1: SURVEYING FOR A BRIDGE

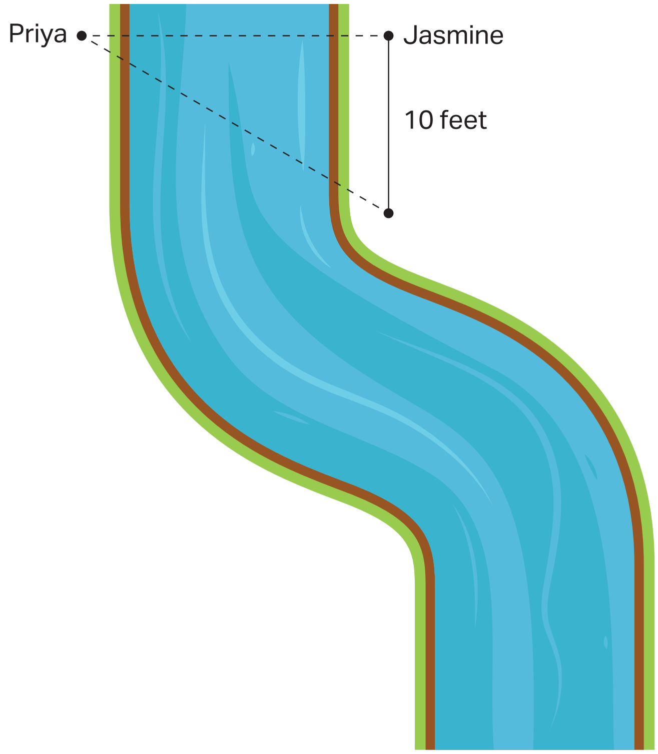
This lesson begins with a contextual scenario involving land surveying. Students cannot solve the problem given what they know at this point. The purpose of this scenario is twofold: first, to create a mathematical need for trigonometry and, second, to show how the relationships of side lengths in right triangles are useful for indirect measurement. In the third part of the lesson, students return to solve this problem using the tangent ratio.

- Share the scenario from **Handout 2.8.A: Surveying for a Bridge** with your class. You can choose to display the scenario for everyone to see, or you could distribute the handout. A snapshot of this handout can be seen on the following page.
- Ask students to take a few moments to closely observe the problem. Have students share what they notice and what they wonder about the problem. Students should observe that a right triangle is formed, as stated in the problem, and that they know the measures of the angles and the length of one leg of the triangle.
- Ask students if it is possible to determine the length of the other leg of the triangle, which represents where the bridge should be located. Expect students to suggest that if they knew the length of the hypotenuse, they would be able to figure out the length of the leg. Some students might suggest creating similar right triangles somehow. They should conclude that it is not possible to solve the problem with the information given and the tools they have.
- You can let students know that there is a way to relate the angles and the sides of a right triangle and that they will carry out an investigation to figure out the relationship.

Surveying for a Bridge

Priya and Jasmine are engineers who have been asked by the local government to design a pedestrian bridge across a river in the town park. They need to figure out how long to make the bridge. Engineers use a device called a *theodolite* to measure the angle between two points from a fixed position. They use this information to figure out the distance between locations that are too far away from each other to measure directly.

Priya sets up the theodolite on one side of the river. Jasmine sets up two spots on the opposite side of the river that are 10 feet apart.



Suppose that Priya and Jasmine are directly across the river from each other in the location they want to put the bridge. There is a right triangle formed by Priya's theodolite and the two spots Jasmine set up. Priya determines that the angle between the two spots Jasmine set up measures 20°. What is the distance across the river?

Handout 2.8.A

PART 2: CONNECTING ANGLE MEASURES AND SIDE LENGTH RATIOS IN SIMILAR RIGHT TRIANGLES

In this part of the lesson, students measure the lengths of the legs of similar right triangles and calculate the ratios of these lengths. Students notice that the ratios of the lengths of corresponding legs of similar right triangles are equal, which implies that the ratio of the legs depends only the angle measure.

Instructional Rationale

This activity is best done in groups of four students, because each student will create and measure the leg lengths of a right triangle and then compare the ratios of the lengths they measured as a group. If completed in pairs, each student will create and measure two right triangles. It is not recommended that a group of more than four students engage in the activity, because there are not enough tasks for more than four students to complete.

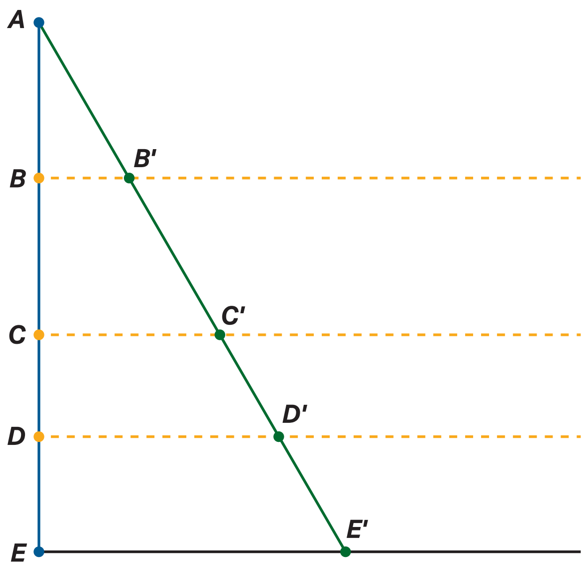
- To begin, arrange your students into groups of four. Distribute one copy of **Handout 2.8.B: Measuring Similar Right Triangles** to each group. The handout has a line segment, $\overline{AE}$, and a ray perpendicular to $\overline{AE}$ with endpoint E.

	Horizontal Leg Length	Vertical Leg Length	
$\Delta ABB'$			
$\Delta ACC'$			
$\Delta ADD'$			
$\Delta AEE'$			

Handout 2.8.B

- Ask one student in each group to choose a point that is located somewhere on the ray emanating from point E and label that point E'. This selection will set the angle measure for $\angle A$ for the similar right triangles the group will create. You do not have to tell students why they are choosing E', because they will figure it out later in the lesson. The student should measure $\overline{AE}$ and $\overline{EE'}$ to the nearest millimeter and record the lengths in the table on **Handout 2.8.B: Measuring Similar Right Triangles**.

- Have each successive student choose a point along $\overline{AE}$, labeled B, C, and D, respectively. Then they should use a straightedge and compass to construct line segments perpendicular to $\overline{AE}$ that intersect $\overline{AE'}$ at points B', C', and D'. They should measure the vertical and horizontal lengths they created and record them in the table on the handout. The resulting diagram should look something like the following:



- This process will have created four (similar) right triangles: $\Delta ABB'$, $\Delta ACC'$, $\Delta ADD'$, and $\Delta AEE'$.
- Now ask students to think about whether or not the triangles they created are similar and to justify their reasoning.

  The triangles have $\angle A$ in common, and the angles $\angle ABB'$, $\angle ACC'$, $\angle ADD'$, and $\angle AEE'$ are congruent because they are right angles, which means that $\angle AB'B$, $\angle AC'C$, $\angle AD'D$, and $\angle AE'E$ must also be congruent. Because the corresponding angles are congruent, the triangles must be similar.

Classroom Ideas

You could modify this task slightly by choosing points B, C, and D and constructing the rays perpendicular to $\overline{AE}$. In that case, students should still choose point E' and measure the lengths of the similar triangles.

- Students should use the triangle sum theorem or parallel lines intersected by a transversal as a justification for why the third angles in the triangles are congruent.

- Ask students to pass around the handout again. This time, each student should calculate the ratio of the horizontal leg length to the vertical leg length for the triangle they formed and record it in the last column of the table on the handout. Have the first student label the column as "Ratio of horizontal and vertical leg lengths." Give students some time to closely observe the values of the ratios, and then ask students to share their observations.

  The ratio of the horizontal leg length to the vertical leg length is the same (or nearly the same) for each pair.

- It's important at this point in the lesson to have students synthesize their thoughts. To check for understanding, pose questions like these to the students:
  - Suppose you have a right triangle that is similar to the ones in your group and has a vertical leg length of 1,000 mm. How long do you think its horizontal leg will be? Explain your answer.

    Answers will depend on the measure of $\angle A$ formed in each student group, but students should use the constant ratio of the horizontal leg length to the vertical leg length from their triangle to determine the length of the horizontal leg in this similar right triangle.

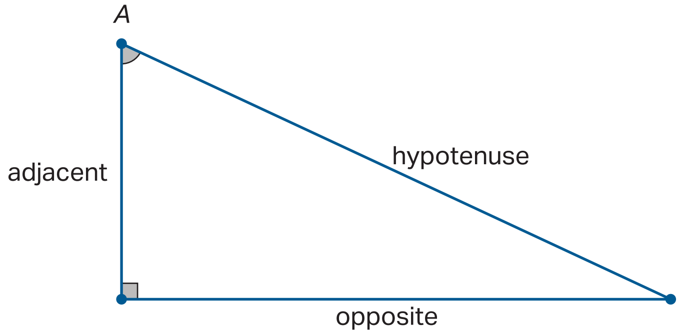
  - Suppose you have a right triangle that is similar to yours and has a horizontal leg length of 1,000 mm. How long do you think its vertical leg will be? Explain your answer.

    Answers will depend on the measure of $\angle A$ formed in each student group, but students should use the constant ratio of the horizontal leg length to the vertical leg length from their triangle to determine the length of the vertical leg in this similar right triangle.

  - What would be the angle measures of a similar right triangle with a vertical leg length of 1,000 mm? What would be the angle measures of a similar right triangle with a horizontal leg length of 1,000 mm? How do you know?

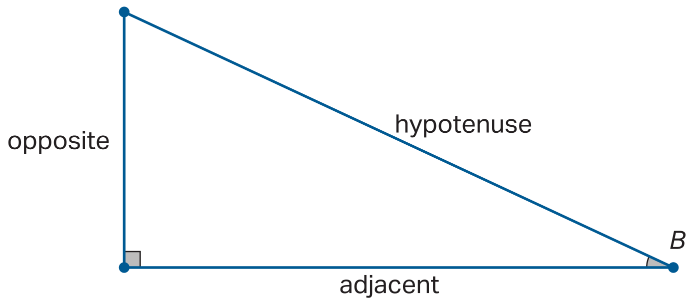
    The angle measures would be the same as those in $\Delta AEE'$ (or any of the other student-drawn triangles), because similar right triangles have congruent corresponding angles.

- This is an appropriate time to introduce, or reinforce, some terminology for students. You can use the student-drawn triangles as a reference diagram, or you can draw a new triangle, as shown below:



The longest side of the right triangle is called the hypotenuse. The leg adjacent to $\angle A$ *is the side of* $\angle A$ *that is not the hypotenuse. The leg opposite to* $\angle A$ *is the leg of the triangle that is not a side of* $\angle A$.

- It can be helpful for students to see another version of the same triangle to make the connection that the sides identified as “opposite” and “adjacent” depend on which of the acute angles is being referenced. For example, the designations for $\angle B$ are shown below:

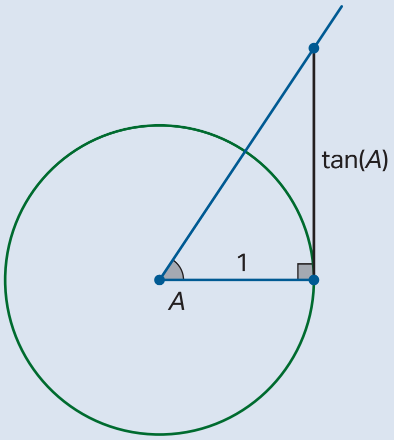


- Ask students to refer back to their triangles from **Handout 2.8.B: Measuring Similar Right Triangles** to identify which legs are the opposite and adjacent legs of $\angle A$. Ask a few students to share their thoughts to check their understanding.
- Let students know that the ratio of the lengths of the opposite leg and the adjacent leg has a particular name. Define for them:

The tangent of an angle A, denoted by $\tan(A)$, *is the ratio of the opposite leg length to the adjacent leg length. Therefore,* $\tan(A) = \dfrac{\text{opposite leg length}}{\text{adjacent leg length}}$.

Guiding Student Thinking

Some students may wonder why this is called the “tangent” ratio. They might know the word “tangent” from the phrase “go off on a tangent,” which refers to pursuing a new and only slightly related line of conversation. It may be premature to tell students where the word comes from at this point. You could wait to discuss it until Unit 3, when they explore circles in detail. The word *tangent* is derived from the Latin word *tangere*, which means “to touch.” The tangent ratio is closely related to tangents of a circle, which are lines or segments that intersect a circle at exactly one point and are perpendicular to the radius. In a unit circle, the length of the “tangent segment” is the tangent ratio of the central angle formed by the radius of the circle through the point of tangency and a ray whose endpoint is the center of the circle and that intersects the endpoint of the line segment tangent to the circle, as shown below:



- Ask students to refer back to their triangles in **Handout 2.8.B: Measuring Similar Right Triangles** and to calculate the tangent of $\angle A$, denoted as $\tan(A)$. Ask a few students to share their tangent values to check their understanding. All students within each group—who are working from similar triangles—should have the same value for $\tan(A)$.
- To help students summarize the main points of the lesson, you may ask questions such as the following:
  - Suppose you constructed a really small right triangle with an angle measure of 10° and a really big right triangle with an angle measure of 10°. Is $\tan(10°)$ the same or different in the two triangles?

    The value of $\tan(10°)$ will be the same for both triangles. Because the triangles are similar, the ratio of the opposite leg length and the adjacent leg length will be equal.

- Suppose you have two right triangles, ΔLMN and ΔSTR, where the right angles are $\angle M$ and $\angle T$. If $\tan(L) = \tan(S)$, is it true that $\angle L \cong \angle S$?

  Yes. If two acute angles have the same tangent value, then the ratios of the opposite leg length and the adjacent leg length will be equal. If those ratios are equal, then the angles have to be congruent.

- The tangent of an acute angle in a triangle relates three pieces of information. What are these three pieces of information?

  The three pieces of information that the tangent ratio relates are the measure of an acute angle in the triangle and the lengths of the sides opposite and adjacent to the angle.

Instructional Rationale

Some texts will not use the parentheses in "$\tan(A)$" especially when A is a numerical value. Pre-AP will use parentheses throughout the materials to reinforce the function qualities of all trigonometric ratios.

PART 3: SURVEYING FOR A BRIDGE WITH THE TANGENT FUNCTION

Now that students have started to explore the relationship between the leg lengths and angle measures of right triangles, they are ready to return to the land surveying problem from Part 1 of the lesson. They use what they know about the tangent ratio to determine the distance across the river.

- Direct students' attention back to the scenario on **Handout 2.8.A: Surveying for a Bridge**. If you did not distribute the handout at the beginning of the lesson, you should distribute it now.
- Let students work collaboratively in pairs to see if they can figure out how to use the tangent ratio to find the distance across the river.
- As you circulate around the room, provide some hints to students who are struggling to relate the information in the problem.

Guiding Student Thinking

You may need to further explain to some students that the tangent of an angle is a relationship between the measure of the angle and the ratio of the sides opposite and adjacent to the angle. Some students will need help setting up this relationship in the bridge length problem because, unlike the similar triangles activity from Part 2, there is information missing in the bridge scenario.

UNIT 2

- Ask students to share the relationship they wrote to relate the information in the scenario. They should get one of two equations:

  $\tan(20°) = \dfrac{10}{\text{distance across river}}$ or $\tan(70°) = \dfrac{\text{distance across river}}{10}$

- If students only suggest one equation, challenge them to write a second equation using a different angle. If students express some skepticism that both equations will result in the same answer, that should be honored. It is not obvious yet that the equations will give the same solution. As students become more fluent with the tangent function, they will see how the tangent ratios of complementary angles are reciprocals.

- This is a good opportunity to reinforce students' skills in algebraic manipulations by having them rewrite the equations to isolate the unknown value. This also allows you to explain to students how to use their calculators to determine tangent ratios of a given angle.

- Every calculator model has distinct nuances in how it handles trigonometric functions. Many calculators use radians as the default unit for angle measures. If students do not have the correct setting for the angle measurement unit, they will almost always get an incorrect value.

  $\text{distance across river} = \dfrac{10}{\tan(20°)}$

  $\text{distance across river} \approx 27.475 \text{ feet}$

  or

  $\text{distance across river} = 10 \bullet \tan(70°)$

  $\text{distance across river} \approx 27.475 \text{ feet}$

- Conclude the lesson by making sure that students relate their solution back to the problem context and use correct units.

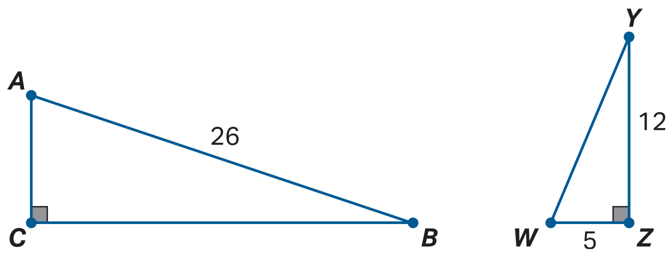
  The bridge needs to be at least 27.475 feet long to span the river.

There is a short practice set on **Handout 2.8.C: Practice Using the Tangent Ratio**. In it, students will have an opportunity to develop some procedural fluency with the tangent ratio and to explore some applications of the tangent function. Included with the problems is an interesting connection between the slope of a line and the tangent of the angle the line makes with the *x*-axis. This could be assigned to be completed in class or out of class.

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.



In the diagram above, $\Delta ABC \sim \Delta WYZ$.

(a) What is the ratio that defines $\tan(Y)$?

$\tan(Y) = \frac{m\overline{WZ}}{m\overline{YZ}} = \frac{5}{12}$

(b) What is the ratio that defines $\tan(B)$?

Because the triangles are similar and $\angle B \cong \angle Y$, that means $\tan(B) = \tan(Y)$, so $\tan(B) = \frac{5}{12}$. Alternatively, students may use the Pythagorean theorem to find $m\overline{WY} = 13$ and then determine that the scale factor between the similar triangles is $\frac{m\overline{AB}}{m\overline{WY}} = \frac{26}{13} = 2$. Using the scale factor with corresponding sides of the similar triangles, $m\overline{AC} = 10$ and $m\overline{BC} = 24$, leads to $\tan(B) = \frac{10}{24} = \frac{5}{12}$.

(c) Which value is larger: $\tan(W)$ or $\tan(A)$? Justify your answer.

Because the triangles are similar, $\angle W \cong \angle A$. Therefore, $\tan(W) = \tan(A)$.

(d) Which value is larger: $\tan(W)$ or $\tan(B)$? Justify your answer.

$\tan(W) = \frac{m\overline{YZ}}{m\overline{WZ}} = \frac{12}{5}$. Because $\tan(B) = \frac{5}{12}$, we can determine that $\tan(W) > \tan(B)$.

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 2.8.A: Surveying for a Bridge

See lesson for solution.

UNIT 2

Handout 2.8.B: Measuring Similar Right Triangles

See lesson for solution.

Handout 2.8.C: Practice Using the Tangent Ratio

1. (a) $\tan(M) = \dfrac{45}{28}$

   (b) $\tan(N) = \dfrac{28}{45}$

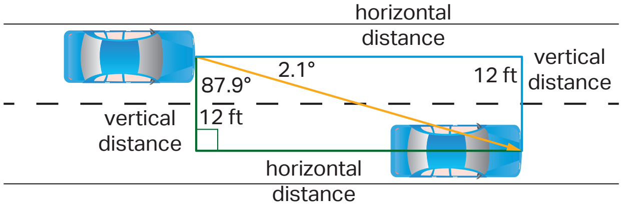
2. (a) The values are equal. Because $m\angle Q = m\angle L = 65°$, it can be concluded that $\tan(Q) = \tan(L) = \tan(65°)$.

   (b) The values are equal. By definition, $\tan(J) = \dfrac{m\overline{KL}}{m\overline{JK}}$. Because the triangles are similar, $\dfrac{m\overline{QR}}{m\overline{SR}} = \dfrac{m\overline{KL}}{m\overline{JK}}$, which means that $\tan(J) = \dfrac{m\overline{QR}}{m\overline{SR}}$.

   (c) Because $m\angle J = m\angle S = 25°$, it can be concluded that $\tan(S) = \tan(25°) \approx 0.4663$.

3. The bee was about 8.75 meters from the daffodil. This is found by setting up and solving the relationship $\tan(5°) = \dfrac{\text{distance away from daffodil}}{100}$.

4. A car will travel about 327.5 feet to complete a lane change. The car turns 2.1° from the direction of the starting lane. The vertical distance that the car travels is 12 feet, assuming that the car moves from the center of one lane to the center of the next lane. The horizontal distance the car travels can be found by setting up and solving $\tan(2.1°) = \dfrac{12}{\text{horizontal distance}}$. So, the horizontal distance is about 327.3 feet. Using the Pythagorean theorem to find the length of the hypotenuse of the triangle, the actual distance traveled is about 327.5 feet.

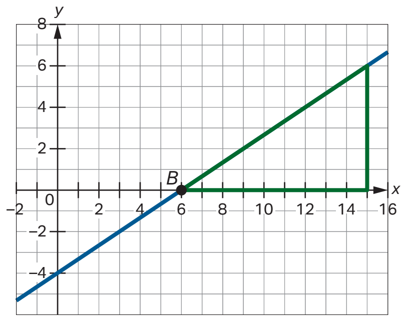


Guiding Student Thinking

Problem 4 requires students to use the tangent ratio to find the horizontal distance traveled and then to use the Pythagorean theorem to find the actual distance the car travels to change lanes. This problem can be directly solved using the cosine ratio but is included here to help students connect all their available tools for working with right triangles. They will revisit this problem in the next lesson.

5. (a) The slope of the line is $\frac{2}{3}$, because line has a vertical change of 2 for each horizontal change of 3. (Students may reference the coefficient of x, $\frac{2}{3}$, from the algebraic representation of the line.)

(b) Answers will vary. One possible slope triangle is shown here:



The length of the horizontal leg is 9, and the length of the vertical leg is 6.

(c) The tangent of B is $\frac{6}{9}$, which can be reduced to $\frac{2}{3}$.

(d) Conjecture: The slope of the line and the tangent of the acute angle formed by the line and the x-axis are equal.

(e) Yes, the conjecture will be true for all nonright angles formed by the x-axis and a line in the coordinate plane. This is because it is always possible to draw a slope triangle using the acute angle formed by the line and the x-axis. The tangent of an angle in a triangle is the ratio of the opposite leg length and the adjacent leg length. Because the opposite leg length is the vertical change of the line and the adjacent leg length is the horizontal change of the line, the tangent ratio can be rewritten as

$$\tan(\text{angle with } x\text{-axis}) = \frac{\text{opposite leg length}}{\text{adjacent leg length}} = \frac{\text{vertical change of line}}{\text{horizontal change of line}}.$$

Guiding Student Thinking

Problem 5 has students investigate an important connection between linear functions and geometry: the slope of a line is equal to the tangent of the angle that the line forms with the x-axis. To ensure that students understand this relationship, you could explore this problem during class time. This problem also provides an excellent opportunity to encourage connections among multiple representations as well as engagement in mathematical argumentation.

UNIT 2

LESSON 2.9

The Sine and Cosine Ratios

OVERVIEW

LESSON DESCRIPTION

Part 1: Finding the Distance Between Stars
Students explore a scenario in which indirect measurement is necessary to determine the distance between two distant stars. The task creates a need for a new trigonometric ratio beyond the tangent ratio. Students solve the problem in the second part of the lesson.

Part 2: Defining the Sine and Cosine Ratios
Students use what they learned about the tangent ratio to develop the sine and cosine ratios. They identify that these trigonometric ratios are functions whose input is an angle measure and whose output is a ratio of side lengths.

Part 3: Finding the Sine, Cosine, and Tangent of Common Angles
Students use the Pythagorean theorem to explore the relationship among the side lengths of 45-45-90 and 30-60-90 triangles. They use the side length relationships to exactly compute the sine, cosine, and tangent ratios of 30°, 45°, and 60°.

CONTENT FOCUS

In the previous lesson, students explored the idea that angles and side lengths are only two of the useful measures in a right triangle. The ratios of pairs of legs are also helpful and informative. Students worked with the tangent ratio, defined as a function whose input is an angle and whose output is a ratio of the lengths of the opposite and adjacent sides. In this lesson they use

AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~135 minutes

LESSON SEQUENCE

This lesson is part of a lesson sequence (~225 minutes total) with Lesson 2.8.

HANDOUTS

Lesson

- 2.9.A: Finding Distances Between Stars
- 2.9.B: Measuring Similar Right Triangles

Practice

- 2.9.C: Practice with Trigonometric Ratios

MATERIALS

- straightedge
- protractors
- calculators
- compass (optional)

a similar approach to develop and work with the sine and cosine ratios. Students also explore the trigonometric ratios of special right triangles.

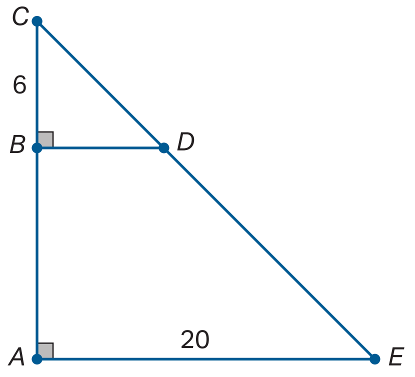
COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ Measuring features of geometric figures is the process of assigning numeric values to attributes of the figures, which allows the attributes to be compared. ▪ Right triangles are simple geometric shapes in which we can relate the measures of acute angles to ratios of their side lengths.	
Learning Objectives	**Essential Knowledge**
2.3.4 Associate the measures of an acute angle, $\angle A$, in a right triangle to ratios of the side lengths.	**2.3.4a** The sine of the measure of $\angle A$ is the ratio of the length of the side opposite the angle and the length of the hypotenuse. **2.3.4b** The cosine of the measure of $\angle A$ is the ratio of the length of the side adjacent to the angle and the length of the hypotenuse.
2.3.5 Explain why a trigonometric ratio depends only on an angle measure of a right triangle and not on the side lengths.	**2.3.5a** Trigonometric ratios are functions whose input is an acute angle measure and whose output is a ratio of two side lengths in a right triangle. **2.3.5b** The ratio of the lengths of two sides of a right triangle will equal the ratio of the lengths of the corresponding sides of a similar right triangle. Therefore, the ratios of the sides depend only on the angle measure.
2.3.7 Model contextual scenarios using right triangles.	**2.3.7a** Contextual scenarios that involve nonvertical and nonhorizontal segments or the distance between two points that do not lie on a vertical or horizontal line can be modeled by right triangles. **2.3.7b** Trigonometric ratios can be used to solve problems or model scenarios involving angles of elevation or depression.

FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

In the triangle below, $m\overline{CB} = 6$, $m\overline{AE} = 20$, and $\cos(C) = \frac{3}{5}$.



(a) What is $\sin(C)$? Explain your answer.

(b) What is $\tan(E)$? Explain your answer.

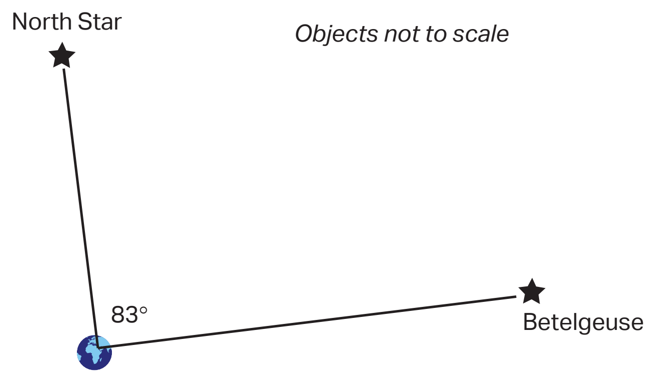
(c) What is $m\overline{CE}$? Explain your answer.

PART 1: DEFINING THE SINE AND COSINE RATIOS

This lesson begins with a contextual scenario involving the relationship between distant stars as seen from Earth. Like the land surveying problem from Lesson 2.8, this problem cannot be solved with tools the students have at this point. Also like the land surveying problem, the purpose of this task is do two things: first, to reinforce how right triangles are useful for indirect measurements in many scenarios, and second, to create a need for another trigonometric ratio when the tangent does not suffice.

- Share the scenario from **Handout 2.9.A: Finding Distances Between Stars** with your class. You can choose to display the scenario for everyone to see, or you could distribute the handout.

Finding Distances Between Stars

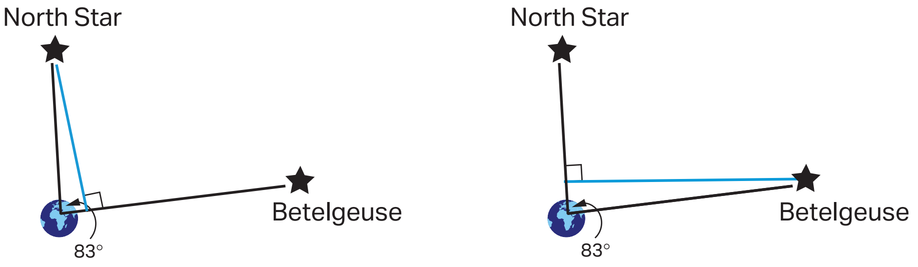


Viewed from Earth, the North Star and Betelgeuse are separated by an angle of about 83°. If the North Star is about 430 light years from Earth, and Betelgeuse is about 640 light years from Earth, then about how far apart in light years are the North Star and Betelgeuse?

Handout 2.9.A

- Ask students to take a few moments to closely observe the problem. Have students share what they notice and what they wonder about the problem. Students will likely say that a triangle is formed by the two stars and the Earth but that it is not a right triangle.
- If students have limited experience creating auxiliary lines in diagrams, you may need to show them how to do the next step. Ask students to figure out where they could place a line segment through a point representing one of the stars so that it is perpendicular to the line segment that includes the other star. This means their

auxiliary segment either would include the North Star and be perpendicular to the segment from Earth to Betelgeuse (often pronounced "beetle-juice") or would include Betelgeuse and be perpendicular to the segment from Earth to the North Star. This technique is often referred to as "dropping a perpendicular." It is a helpful method for imposing a right triangle onto a diagram where one is not given. The resulting diagrams should look something like the following. Note that these diagrams are not to scale.



- Give students some time to work individually or collaboratively to complete their diagrams with other relevant information from the problem. Encourage students to think about ways that the right triangle and the tangent ratio might help them solve the problem. Have some students share their conclusions.

  It is not possible to solve the problem, because we only have information about the hypotenuse of the triangle and the angles in the triangle. Because the tangent relates the angle to the ratio of the lengths of the opposite and adjacent sides, we cannot use the tangent in this problem.

- Once students recognize that the tangent ratio, even coupled with the Pythagorean theorem, is not enough to solve the problem, you can share with them that they will investigate ratios of other combinations of sides next.

PART 2: DEFINING THE SINE AND COSINE RATIOS

This portion of the lesson should proceed similarly to Part 2 of Lesson 2.8. **Handout 2.9.B: Measuring Similar Right Triangles** has a similar setup to **Handout 2.8.B: Measuring Similar Right Triangles**. The table under the diagram includes a column for the length of the hypotenuse and two blank columns where students will eventually record the ratio of the lengths of the horizontal leg and the hypotenuse, as well as the ratio of the lengths of the vertical leg and the hypotenuse.

Classroom Ideas

You could have students use **Handout 2.8.B: Measuring Similar Right Triangles** again so they do not have to recreate the right triangles. Then they would only have to add three additional columns to the table on that handout.

- Provide students with time to pass the diagram around the group, create the right triangles, and measure the lengths of the sides.
- As groups finish recording the measurements, ask students to pass around the handout again. This time, each student should calculate the ratio of the lengths of the horizontal leg and the hypotenuse for the triangle they formed and record it in the fourth column in the table. Have the first student label this column as "Ratio of horizontal leg and hypotenuse lengths."
- Give students some time to closely observe the values of the ratios, and then ask students to share their observations.

  The ratio of the length of the horizontal leg to the length of the hypotenuse is the same (or almost the same) for each triangle.

- To check for understanding, pose some questions like these to the students:
  - What would be the angle measures of any right triangle similar to yours? How do you know?

    The angle measures would be the same as those in $\Delta AEE'$ (or any of the other student-drawn triangles), because similar right triangles have congruent corresponding angles.

  - Suppose you have a right triangle that is similar to those in your group and that the length of its horizontal leg is 1,000 mm. How long do you think its hypotenuse leg is? Explain your answer.

    Answers will depend on the measure of $\angle A$ formed in each student group, but students should use the constant ratio of the lengths of the horizontal leg and the hypotenuse from their triangle to determine the length of the hypotenuse in this similar right triangle.

  - Suppose you have a right triangle that is similar to those in your group and that the length of its hypotenuse is 1,000 mm. How long do you think its horizontal leg is? Explain your answer.

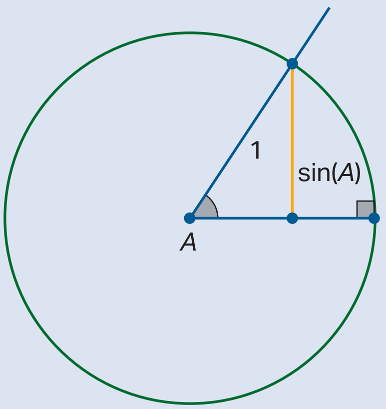
    Answers will depend on the measure of $\angle A$ formed in each student group, but students should use the constant ratio of the lengths of the horizontal leg and the hypotenuse from their triangle to determine the length of the horizontal leg in this similar right triangle.

- You may have to remind students that the horizontal leg of their similar triangles is the leg opposite to $\angle A$. Let students know that the ratio of the lengths of the opposite leg and the hypotenuse has a particular name. Define for them:

  The sine of an angle A, denoted by $\sin(A)$, *is the ratio of the length of the opposite leg to the length of the hypotenuse. Therefore,* $\sin(A) = \dfrac{\text{length of the opposite leg}}{\text{length of the hypotenuse}}$.

Guiding Student Thinking

Some students may wonder why this is called the "sine" ratio. It is a little complicated, but may be worth explaining because it shows how mathematical terminology is developed by humans and, in this case, by human error. Like the tangent ratio, the word for the sine ratio is related to segments in a circle. The Sanskrit word for a half-chord was *jiva* and was translated into Arabic as *jiba*. The word was written down in ancient Arabic texts as *jb*. The diagram of a unit circle below illustrates that the length of the vertical segment labeled "sin(A)" is half of a chord of the circle, and its length is the sine of $\angle A$.



The correct meaning of the word *jb* in the Arabic texts was misunderstood. Europeans thought that *jb* was the word *jayb*, which means fold or pocket. It was translated into Latin as *sinus*, which is where the modern word *sine* comes from.

- Ask students to pass around the handout again. This time, each student should calculate the ratio of the lengths of the vertical leg and the hypotenuse for the triangle they formed and record it in the fifth column in the table. Have the first student label this column as "Ratio of vertical leg and hypotenuse lengths."

- Give students some time to closely observe the values of the ratios, and then ask students to share their observations.

  The ratio of the length of the vertical leg to the length of the hypotenuse is the same (or nearly the same) for each triangle.

- To check for understanding, pose some questions like these to the students:
  - Suppose you have a right triangle that is similar to the triangles in your group and that the length of its vertical leg is 1,000 mm. What is the length of its hypotenuse? Explain your answer.

    Answers will depend on the measure of $\angle A$ formed in each student group, but students should use the constant ratio of the lengths of the vertical leg and the hypotenuse to determine the length of the hypotenuse in this similar right triangle.

  - Suppose you have a right triangle that is similar to the triangles in your group and that the length of its hypotenuse is 1,000 mm. What is the length of its vertical leg? Explain your answer.

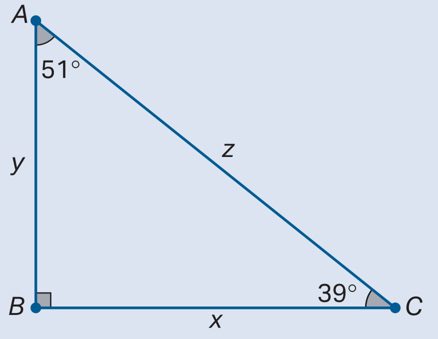
    Answers will depend on the measure of $\angle A$ formed in each student group, but students should use the constant ratio of the lengths of the vertical leg and the hypotenuse to determine the length of the vertical leg in this similar right triangle.

- You may have to remind students that the vertical leg of their similar triangles is the leg adjacent to $\angle A$. Let students know that the ratio of the lengths of the opposite leg and the hypotenuse has a particular name. Define for them:

  The cosine of an angle A, denoted by $\cos(A)$, *is the ratio of the length of the adjacent leg to the length of the hypotenuse. Therefore,* $\cos(A) = \dfrac{\text{length of the adjacent leg}}{\text{length of the hypotenuse}}$.

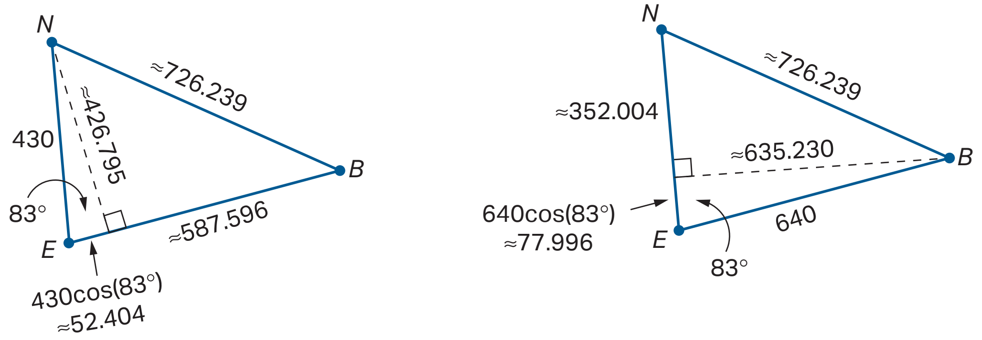
Guiding Student Thinking

Students may wonder why this ratio is called the "cosine." It is because the *cosine* ratio is the *sine* of the *co*mplementary angle. You can illustrate this for students by pointing out the relationships in the diagram below:



$\cos(A) = \frac{y}{z}$, and $\sin(C) = \frac{y}{z}$. Angles A and C are complementary, and $\cos(A) = \sin(C)$.

- Students can now solve the problem from the beginning of the lesson and determine the approximate distance between the North Star and Betelgeuse. Using the sine or cosine ratio and the Pythagorean theorem, students should determine that the stars are about 726 light years apart.



- If you haven't introduced the term trigonometry to students yet, this is an appropriate time to do so. *Trigonometry* is the study of the relationship between the angles and side lengths of a triangle. You can break down the word into its parts to help students remember what it means: *tri-* means three; *gon* comes from the Greek word *gonia*, which means angle; and *-metry* comes from the Greek work *metron*, which means measurement. So *trigonometric* translates to three-angle measurement.

- If you have a word wall, take some time to have students create definitions and examples for the words tangent, sine, cosine, and trigonometry.

Meeting Learners' Needs
Many students will benefit from using a Frayer model to organize the definition, characteristics, examples, and nonexamples of each term. You could have students include a diagram with the definitions. You can use **Handout 3.1.C: Vocabulary Graphic Organizer** as a template. Or, you can have students write the term in the center of a standard piece of paper and then fold it in half lengthwise and widthwise to create four areas for the definition, characteristics, examples, and nonexamples.

PART 3: FINDING THE SINE, COSINE, AND TANGENT OF COMMON ANGLES

In this part of the lesson, students continue to think about each trigonometric ratio as a function whose input is an angle measure and whose output is a ratio of certain sides of a right triangle. They explore the relationships between the sides of two special right triangles and determine the exact values of the trigonometric ratios of 30°, 45°, and 60°.

- To begin this part of the lesson, ask students questions like the ones below to check for understanding. You could use these questions as a warm-up if this part of the lesson coincides with the beginning of a new class period.
    - Suppose that we have three right triangles, all of different sizes, and each one has a 32° angle. What can we conclude about the three triangles?

      All three triangles have angles measuring 32°, 58°, and 90°. Therefore, the triangles are similar.
    - Because the triangles are similar, what can we conclude about the sides of the triangles?

      The corresponding sides will be in proportion.
    - Think about the sides opposite and adjacent to the 32° angle in the largest right triangle and in the smallest right triangle. Will the value of $\tan(32°)$ be different if you use the side lengths of the largest triangle or the smallest triangle? Why or why not?

      The value of $\tan(32°)$ is the same no matter which triangle you use to calculate it. The ratio of the side opposite 32° and the side adjacent to 32° will be the same in every triangle with a 32° angle.

- Will the value of $\sin(32°)$ be different if you use the side lengths of the largest triangle or the smallest triangle? Will the value of $\cos(32°)$ be different if you use the side lengths of the largest triangle or the smallest triangle? Why or why not?

  The values of $\sin(32°)$ and $\cos(32°)$ do not change no matter which triangle you use to calculate them. The ratio of the side opposite 32° and the hypotenuse or the ratio of the side adjacent to 32° and the hypotenuse will be the same in every right triangle with a 32° angle.

- Could there be another acute angle besides 32° that has the same sine, cosine, or tangent as 32°? Why or why not?

- The last question in the sequence above is important for students to consider. Students will need some time to figure out whether or not two different acute angle measures could have the same sine, cosine, or tangent ratio. If students need support to reach the conclusion that no two acute angles will have equal sines, cosines, or tangents, suggest that they try to draw congruent right triangles that have different angles but equal side lengths. Students should state that it is impossible to do that because the angle measures must be equal in congruent triangles, by definition. Likewise, if the triangles are different sizes and have different angle measures, they cannot be similar, by definition, so the ratios of the sides will not be equal.

Instructional Rationale

In Algebra 2, students will discover that it is possible for two different angle measures to have the same sine, cosine, or tangent but that the angles are not both acute. The relationship between angle measure and the ratio of two sides of a right triangle is still a function, because each input (angle measure) only leads to one output (ratio of sides).

- Ask students what they remember about functions from Algebra 1. Expect them to provide some terms like domain and range, input and output, independent and dependent variables, or "$f(x)$". Ask them what makes a relationship between two quantities (the input and the output) a function. It may take some prompting for them to recall that a function is a relation in which each input is associated with only one output.
- Ask students to think about the three trigonometric ratios and if it is possible to think about each as a function. Some example questions that you could use are provided here:

- If you think about the sine (or the cosine or the tangent) as a relation, what would be the input, and what would be the output?

  The input is the angle measure, and the output is a ratio of the lengths of two sides of the right triangle.

- Is the sine (or the cosine or the tangent) a function? Does each input have only one output?

  Each input has only one output. A specific angle always has the same sine (or cosine or tangent).

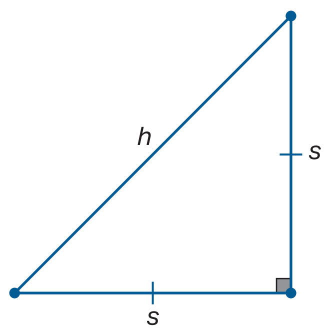
Instructional Rationale

In Algebra 2, students will learn that the domain of the trigonometric functions can be extended to all real numbers, but that is beyond the scope of this course. At this point, it is probably a good idea to avoid asking about the range of the sine, cosine, and tangent, because students likely have not had sufficient experience to figure that out. If there is time when your students explore circles in the coordinate plane, you can revisit right triangle trigonometry and explore the range of the trigonometric functions at that time.

- Let students know that they are going to spend a little time focusing on the relationships between the side lengths of some common right triangles.
- Begin by having students work individually or in pairs to determine the length of the hypotenuse of an isosceles right triangle that you give them. Vary the leg lengths of these triangles among the students so that the relationship between the sides emerges when they share their solutions.

Classroom Ideas

You could have students do their work on individual whiteboards throughout this section so all students can quickly share their results.



- Once students work through their isosceles right triangles, have them share the length of the hypotenuse. Record the leg and hypotenuse lengths of each triangle where everyone can see them. In order for students to observe a pattern in the lengths of the hypotenuses, make sure that they write their answers in an exact form by simplifying the square root expression.

Meeting Learners' Needs
Some students will need a just-in-time reminder about how to simplify a square root. This is a good opportunity to build their procedural fluency with that process.

- Ask students to make a conjecture about the relationship between the lengths of a leg and the hypotenuse of an isosceles right triangle.

  It looks like the hypotenuse is the length of a leg times the square root of 2.

- Remind students that having a lot of evidence that something is true is not the same as proving it in mathematics. Ask them if it is possible to prove that the relationship between the lengths of the sides of isosceles right triangles is true. After students have made several suggestions about what to do, if no one suggested drawing a generic isosceles right triangle, you can show them one.
- Ask students to work through the Pythagorean theorem to show that the length of the hypotenuse (h) is equal to the length of the legs (s) times the square root of 2:

$$\begin{aligned} s^2 + s^2 &= h^2 \\ 2s^2 &= h^2 \\ \sqrt{2s^2} &= \sqrt{h^2} \\ s\sqrt{2} &= h \end{aligned}$$

- Ask some clarifying questions to check for understanding:
  - Suppose you have an isosceles right triangle with a leg length of 250 feet. What would be the length of the hypotenuse?

    The hypotenuse would be $250\sqrt{2}$ feet.

  - Suppose you have an isosceles right triangle with a hypotenuse whose length is 250 feet. What would be the length of each of its legs?

    The length of each leg would be $\frac{250}{\sqrt{2}}$ feet.

Guiding Student Thinking

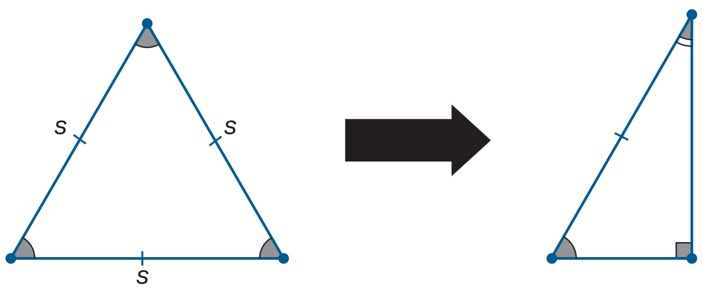
It might take some time for students to work through the second question. Instead of having them memorize two separate formulas, you can build on their understanding of how to solve equations by providing an opportunity to solve $s\sqrt{2}=h$ for s. Because $h=250$, they can solve the equation $s\sqrt{2}=250$ by dividing both sides by $\sqrt{2}$. It is not necessary for students to rationalize the denominator at this point.

- Ask students to think about the angle measures of an isosceles right triangle. Students should be able to work out that the other two angles are each 45°. Let students know that these kinds of triangles are often called "45-45-90" triangles.
- Ask the students to label the two angles measuring 45° and then to use their triangles to determine $\sin(45°)$, $\cos(45°)$, and $\tan(45°)$.
- Have students share their solutions. Students should find that each triangle yields the same values of $\sin(45°)=\frac{1}{\sqrt{2}}$, $\cos(45°)=\frac{1}{\sqrt{2}}$, and $\tan(45°)=1$, because the value of each ratio depends only on the angle measure, not on the specific triangle used.

Guiding Student Thinking

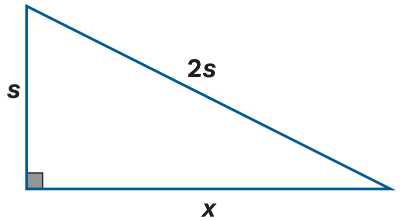
If your local standards require that students rationalize the denominator, take some time to help students rewrite their fractions. Most importantly, students should understand that $\frac{1}{\sqrt{2}}$ and $\frac{\sqrt{2}}{2}$ are equivalent. Students should able to use these expressions flexibly. In some problem-solving situations, a square root expression in the denominator of a fraction is preferable to a rationalized denominator. The problem specifics will determine which form students use.

- Now you can turn students' attention to an equilateral triangle. Ask students to think about dividing an equilateral triangle in half through one of the vertices. Ask students what they notice and what they wonder about the new triangle.



- Students should notice that the new triangle is a right triangle. They may deduce that the shortest leg is half the length of the hypotenuse of the triangle. They may wonder about the length of the side perpendicular to the shortest side.
- Have students work individually or in pairs to determine the length of the missing side of the triangle that you give them. Vary the lengths of the sides of these 30-60-90 triangles among the students so the relationship between the side lengths emerges when students share their solutions. In order for students to observe a pattern in the lengths of the hypotenuses, make sure they write their answers in an exact form by simplifying the square root expression.
- Once students work through their examples of these triangles, have them share the side lengths. Record all the side lengths where everyone can see them. Ask students to make a conjecture about the relationships between the side lengths of this triangle.

  It looks like the hypotenuse is the twice the length of the shortest leg and the other leg is the shortest leg times the square root of 3.
- Like before, remind students that having a lot of evidence that something is true is not the same as proving it. Ask them how they can prove that this relationship between the lengths of this type of triangle would always be true. Students should suggest using a generic right triangle obtained by cutting an equilateral triangle in half:



- In this case, students should be able to write the length of the hypotenuse in terms of the length of the shortest side. Their next task is to write the length of the other leg in terms of the length of the shortest side—that is, as a multiple of *s*. Ask students to work through the Pythagorean theorem to show that the length of the other leg, *x*, is equal to the length of shortest leg, *s*, times the square root of 3:

$$\begin{aligned} s^2 + x^2 &= (2s)^2 \\ s^2 + x^2 &= 4s^2 \\ x^2 &= 3s^2 \\ \sqrt{x^2} &= \sqrt{3s^2} \\ x &= s\sqrt{3} \end{aligned}$$

- Ask students to think about the angle measures of this kind of right triangle. Students should be able to work out that, because each angle in an equilateral triangle measures 60°, cutting one in half creates a 30° angle, and the perpendicular legs meet at 90°. Let students know that these kinds of triangles are sometimes called "30-60-90" triangles.
- Ask some clarifying questions to check for understanding:
  - Suppose you have a 30-60-90 right triangle whose shortest leg is 250 feet. What are the lengths of the hypotenuse and the longer leg?

    The hypotenuse would be 500 feet, and the longer leg would be $250\sqrt{3}$ feet.
  - Suppose you have a 30-60-90 right triangle whose hypotenuse is 250 feet. What are the lengths of the shortest leg and the longer leg?

    The shortest leg would be 125 feet, and the longer leg would be $125\sqrt{3}$ feet.
  - Suppose you have a 30-60-90 right triangle whose longer leg is 250 feet. What are the lengths of the shortest leg and the hypotenuse?

    The shortest leg would be $\frac{250}{\sqrt{3}}$ feet, and the hypotenuse would be $\frac{500}{\sqrt{3}}$ feet.
- Ask the students to label the angles measuring 30°, 60°, and 90°.
- Have them use their triangles to determine $\sin(30°)$, $\cos(30°)$, and $\tan(30°)$.
- Have students share their solutions. Students should find that everyone has the same values of $\sin(30°)=\frac{1}{2}$, $\cos(30°)=\frac{\sqrt{3}}{2}$, and $\tan(30°)=\frac{1}{\sqrt{3}}$, because the value of each ratio depends only on the angle measure, not on the specific triangle used.
- Now ask the students to use their triangles to determine $\sin(60°)$, $\cos(60°)$, and $\tan(60°)$.
- Have students share their solutions. Students should find that everyone has the same values of $\sin(60°)=\frac{\sqrt{3}}{2}$, $\cos(60°)=\frac{1}{2}$, and $\tan(60°)=\sqrt{3}$, because the value of each ratio depends only on the angle measure, not the on specific triangle used.
- You can highlight the fact that $\cos(60°)$ and $\sin(30°)$ are the same because the cosine of a 60° angle is equal to the sine of the complementary angle, 30°. Students can refer back to their triangles to confirm that the adjacent side of one acute angle is the opposite side of the other, and vice versa.
- If you have a word wall or a location for important formulas, you can spend some time having students add the 45-45-90 and 30-60-90 triangle relationships, as well as the sine, cosine, and tangent of 45°, 30°, and 60°.

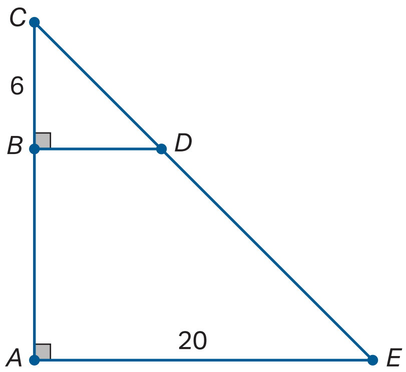
- The problems included on **Handout 2.9.C: Practice with Trigonometric Ratios** can be used to deepen students' understanding of how to use trigonometric ratios with similar triangles and to broaden students' understanding of how right triangles can be used to model real-world phenomena. These problems are sufficiently challenging that students could work collaboratively to solve them.

ASSESS AND REFLECT ON THE LESSON

FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

In the triangle below, $m\overline{CB} = 6$, $m\overline{AE} = 20$, and $\cos(C) = \frac{3}{5}$.



(a) What is $\sin(C)$? Explain your answer.

Because $\cos(C) = \frac{3}{5} = \frac{m\overline{BC}}{m\overline{CD}}$ and $m\overline{BC} = 6$, then $m\overline{CD} = 10$. By the Pythagorean theorem, $m\overline{BD} = 8$. Therefore, $\sin(C) = \frac{8}{10} = \frac{4}{5}$.

(b) What is $\tan(E)$? Explain your answer.

We know that $\angle E \cong \angle D$ because $\Delta BCD \sim \Delta ACE$, so we also know that $\tan(E) = \tan(D)$. Therefore, $\tan(E) = \frac{6}{8} = \frac{3}{4}$.

(c) What is $m\overline{CE}$? Explain your answer.

Because $\sin(C) = \frac{4}{5} = \frac{m\overline{AE}}{m\overline{CE}}$ and $m\overline{AE} = 20$, then $m\overline{CE} = 25$.

Guiding Student Thinking

Be sure students set up the trigonometric ratios correctly in parts (a) and (b). You may need to discuss the similarity of the triangles in the diagram before students can feel confident enough to take on part (c).

HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

Handout 2.9.A: Finding Distances Between Stars

See lesson for solutions.

Handout 2.9.B: Measuring Similar Right Triangles

See lesson for solutions.

Handout 2.9.C: Practice with Trigonometric Ratios

1. (a) $\sin(Y) = \frac{20}{29}$

   (b) $\cos(Y) = \frac{21}{29}$

   (c) $\tan(Y) = \frac{20}{21}$

   (d) $\sin(Z) = \frac{21}{29}$

   (e) $\cos(Z) = \frac{20}{29}$

   (f) $\tan(Z) = \frac{21}{20}$

2. (a) $m\overline{CB} = 144$; $m\overline{WZ} = \frac{65}{72}$; $m\overline{WY} = \frac{97}{72}$

   (b) $\tan(A) = \tan(W) = \frac{72}{65}$

   (c) No angles in the diagram have a cosine of $\frac{97}{65}$.

Guiding Student Thinking

If students answer that $\angle A$ and $\angle W$ have cosines of $\frac{97}{65}$, they are likely inverting the side lengths in the ratio. It is true that $\cos(A) = \cos(W) = \frac{65}{97}$.

3. Because $m\overline{AB} = 13$ by the Pythagorean theorem, we are trying to determine which expression has a value of 13.

(a) $12\sin(A) = 12 \bullet \frac{12}{13} \neq 13$

(b) $\frac{12}{\cos(A)} = \frac{12}{5/13} \neq 13$

(c) $\frac{12}{\sin(A)} = \frac{12}{12/13} = 13$

(d) $\frac{12}{\cos(B)} = \frac{12}{12/13} = 13$

(e) $5\sin(B) = 5 \bullet \frac{5}{13} \neq 13$

(f) $\frac{5}{\cos(A)} = \frac{5}{5/13} = 13$

Guiding Student Thinking

Students may recognize problem 4 from **Handout 2.8.C: Practice with Trigonometric Ratios**. The problem can be solved using the tangent ratio, but it is more direct to use the sine ratio. See answers in Lesson 2.8 for diagram.

4. A car travels about 327.478 feet to complete a lane change. This is found by solving the equation $\sin(2.1°) = \frac{12}{\text{distance traveled by car}}$.

5. (a) Boat 1, on the left side of the diagram, will travel an approximate distance of 292.380 meters, which is found by solving the equation $\sin(20°) = \frac{100}{\text{Boat 1 distance}}$. Boat 2, on the right side of the diagram, will travel an approximate distance of 293.641 meters. This is found by solving the equation $\sin(22°) = \frac{110}{\text{Boat 2 distance}}$. If both boats are moving at the same speed, Boat 1 will cross the finish line first.

(b) If each boat is moving at 10 m/s, then it will take Boat 1 $\frac{292.380}{10} = 29.238$ seconds, and Boat 2 $\frac{293.641}{10} \approx 29.364$ seconds, to cross the finish line. This means that Boat 2 crosses the finish line about 0.126 seconds after Boat 1.

6. The cosine ratio relates the Earth-Moon distance and the Earth-Sun distance: $\cos(89.85°) = \frac{\text{E-M distance}}{\text{E-S distance}}$. The value of $\cos(89.85°)$ is about 0.0026. This means that the Earth-Moon distance is 0.0026 times the Earth-Sun distance. Calculating the reciprocal will show the Earth-Sun distance as a multiple of the Earth-Moon distance. $\frac{1}{\cos(89.85°)} = \frac{\text{E-S distance}}{\text{E-M distance}} \approx 382$. Therefore, the Sun is about 382 times the distance from the Earth that the Moon is.

Unit 2

Performance Task

PERFORMANCE TASK

UNIT 2

Prove Me Wrong

OVERVIEW

DESCRIPTION
In this performance task, students determine the validity of several arguments by engaging in an extended error analysis that synthesizes what they know about midpoints, angle bisectors, perpendicular lines, and right triangle trigonometry.

CONTENT FOCUS
This task is designed to assess students' understanding of the relationships between angle bisectors, midpoints, and right triangle trigonometry. It is intended to be used at the end of Unit 2: Tools and Techniques of Geometric Measurement, because it draws on concepts from across the unit.

AREA OF FOCUS

- Engagement in Mathematical Argumentation

SUGGESTED TIMING

~45 minutes

HANDOUTS

Unit 2 Performance Task: Prove Me Wrong

MATERIALS

- scientific calculator

UNIT 2

COURSE FRAMEWORK CONNECTIONS

Enduring Understandings	
▪ A formal mathematical argument establishes new truths by logically combining previously known facts. ▪ Measuring features of geometric figures is the process of assigning numeric values to attributes of the figures, which allows the attributes to be compared. ▪ Right triangles are simple geometric shapes in which we can relate the measures of acute angles to ratios of their side lengths.	
Learning Objectives	**Essential Knowledge**
2.1.10 Solve problems involving a segment bisector or an angle bisector.	**2.1.10a** The midpoint of a line segment is the point located on the line segment equidistant from the endpoints. **2.1.10b** In the coordinate plane, the x- and y-coordinates of the midpoint of a line segment are the arithmetic means of the corresponding coordinates of the endpoints. **2.1.10c** A bisector of an angle is a line, ray, or line segment that contains the vertex of the angle and divides the angle into two congruent adjacent angles.
2.3.4 Associate the measures of an acute angle, $\angle A$, in a right triangle to ratios of the side lengths.	**2.3.4a** The sine of the measure of $\angle A$ is the ratio of the length of the side opposite the angle and the length of the hypotenuse. **2.3.4b** The cosine of the measure of $\angle A$ is the ratio of the length of the side adjacent to the angle and the length of the hypotenuse. **2.3.4c** The tangent of the measure of $\angle A$ is the ratio of the length of the side opposite the angle and the length of the side adjacent to the angle.

Learning Objectives	Essential Knowledge
2.3.6 Determine an acute angle measure in a right triangle, given a ratio of its side lengths, using an understanding of inverses.	**2.3.6a** For acute angles in a right triangle, the angle measure and the ratio of the lengths of any two specific sides have a one-to-one correspondence. **2.3.6b** Given a ratio of any two side lengths in a right triangle, it is possible to determine the acute angle measures of the right triangle.

UNIT 2

SCORING GUIDELINES

There are 9 possible points for this performance task.

Student Stimulus and Part (a)

(a) Convince your friend that they are wrong. Use what you know about trigonometry to explain why $\angle CAD$ and $\angle DAB$ are not congruent.

Sample Solutions	Points Possible
Because $m\angle CAB = \tan^{-1}\left(\frac{4}{3}\right) \approx 53.130°$, $m\angle DAB = \tan^{-1}\left(\frac{2}{3}\right) \approx 33.690°$, and $33.690°$ is more than half of $53.130°$, you can conclude that $\overrightarrow{AD}$ does not bisect $\angle CAB$. Note: Students do not have to use the tangent ratio to find $m\angle CAB$, as the sine and cosine are just as straightforward to use. Students could find the length of the hypotenuse of ΔDAB and then use the sine or cosine ratios.	**3 points maximum** 1 point for the angle measure of $\angle CAB$ 1 point for angle measure of $\angle DAB$ 1 point for a correct conclusion that $\overrightarrow{AD}$ does not bisect $\angle CAB$, with reasoning

Targeted Feedback for Student Responses
If several students make mistakes in part (a), it could mean that students are unsure how to determine and compare the angle measures.

TEACHER NOTES AND REFLECTIONS

Student Stimulus and Part (b)

(b) Locate the point E on $\overline{BC}$ that lies on the angle bisector of $\angle CAB$. How far is point E from point B? Show all your work.

Sample Solutions	Points Possible
Because $m\angle CAB \approx 53.130°$, the measure of $\angle EAB$ will measure approximately 26.565°. Let x be the length of side $\overline{EB}$. The value of x can be found by solving this equation: $\tan(26.565°) = \frac{x}{3}$ The value of x is approximately 1.499. The angle bisector of $\angle CAB$ will pass through a point about 1.499 units from point B. The value of x is exactly 1.5, but the trigonometry required to find this value is beyond the scope of the course. Students should only be expected to find a decimal approximation of the value of x.	**3 points maximum** 1 point for half the angle measure of $\angle CAB$, or the measure of $\angle EAB$ 1 point for correctly setting up the equation 1 point for correctly solving the equation for x *Scoring note*: If students make an arithmetic mistake in determining $m\angle CAB$ or $m\angle EAB$ but use their incorrect values in the correct manner to try to find the length of $\overline{EB}$, they should receive 2 out of 3 points.

Targeted Feedback for Student Responses
If several students make mistakes in part (b), it could mean that students do not understand that the angle bisector cuts the angle in half or how to solve for the measure of half the angle.

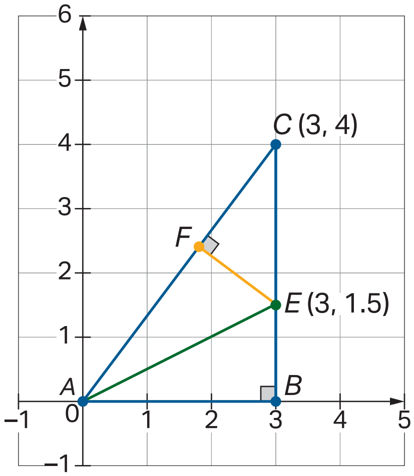
TEACHER NOTES AND REFLECTIONS

Student Stimulus and Part (c)

(c) After you convince your friend that the bisector of $\angle CAB$ doesn't go through the midpoint of $\overline{BC}$, they say to you, "Now I remember! Every point on an angle bisector is the same distance from the sides of the angle." They draw a line segment from point E that is perpendicular to $\overline{AC}$ at point F and say, "That means that $\overline{EB}$ and $\overline{EF}$ are congruent." Is your friend correct? You can use algebra or trigonometry to confirm or disprove their statement.

Sample Solutions	Points Possible
There are at least two possible methods to determine if the friend is correct about this assertion. Method 1: Using Trigonometry We know that $\angle FAE \cong \angle BAE$, because they are formed by the angle bisector of $\angle CAB$. We know that $\sin(\angle BAE) = \frac{m\overline{EB}}{m\overline{AE}}$ and $\sin(\angle FAE) = \frac{m\overline{FE}}{m\overline{AE}}$. Because $\angle FAE \cong \angle BAE$, it must be true that $\sin(\angle FAE) \cong \sin(\angle BAE)$. This means $\frac{m\overline{FE}}{m\overline{AE}} = \frac{m\overline{EB}}{m\overline{AE}}$, and therefore $m\overline{FE} = m\overline{EB}$. This proves that points on the angle bisector are equidistant from the sides of the angle. Therefore, the friend is correct.	**3 points maximum** Method 1 1 point for writing the sine ratios of $\angle FAE$ and $\angle BAE$ 1 point for equating the sine ratios of $\angle FAE$ and $\angle BAE$ 1 point for concluding that $m\overline{FE} = m\overline{EB}$ and that the points along the angle bisector are equidistant from the sides of the angle

continues

Method 2: Using Algebra  From part (b) we know that point E is about 1.5 units above point B. Using point $A(0, 0)$, the equation of side containing C is $y=\frac{4}{3}x$. The equation of the line perpendicular to $\overline{AC}$ that passes through point $E(3, 1.5)$ is $y=-\frac{3}{4}(x-3)+1.5$, or $y=-\frac{3}{4}x+\frac{15}{4}$. The point of intersection is point F. The intersection of $\overline{AC}$ and $\overline{EF}$ is the intersection of their equations, that is, the point (x, y) that satisfies both equations. The point of intersection can be found by substitution, so if $y=-\frac{3}{4}x+\frac{15}{4}$ and $y=\frac{4}{3}x$, then $\frac{4}{3}x=-\frac{3}{4}x+\frac{15}{4}$. Solving for x, we find $x=\frac{9}{5}$. Substituting this into either equation and solving for y, we find $y=\frac{12}{5}$, so F has coordinates $\left(\frac{9}{5},\frac{12}{5}\right)$. Using the Pythagorean theorem or the distance formula, the distance between point E and the intersection point F is 1.5, the same distance point E is from side $\overline{AB}$.	Method 2 1 point for the equation of the lines containing $\overline{AC}$ and $\overline{EF}$ 1 point for the coordinates of point F 1 point for showing that the distance from E to F is equal to the distance from E to B

Targeted Feedback for Student Responses

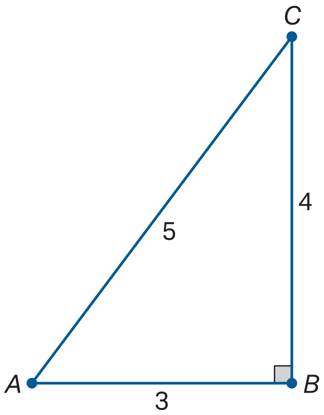
If several students make mistakes in method 1 of part (c), it could mean that students do not understand how to relate the side lengths to the angle measures. If several students make mistakes in method 2 of part (c), it could mean that students do not know how to write the equation of a line or do not understand the relationship between the slopes of perpendicular lines.

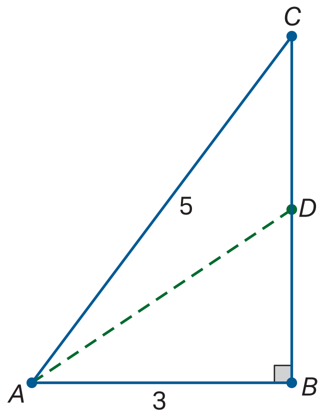
TEACHER NOTES AND REFLECTIONS

Points Received	Appropriate Letter Grade (If Graded)	How Students Should Interpret Their Score
8 or 9 points	A	"I know all of this geometry really well."
6 or 7 points	B	"I know all of this geometry well, but I made a few mistakes."
4 or 5 points	C	"I know some of this geometry well, but not all of it."
2 or 3 points	D	"I only know a little bit of this geometry."
0 or 1 point	F	"I don't know much of this geometry at all."

Prove Me Wrong

PERFORMANCE TASK





Your friend gives you the right triangle above and to the left and says, “Bisecting an angle is really easy. Take triangle ABC. If I wanted to bisect $\angle CAB$, all I need to do is find the midpoint of $\overline{BC}$. The ray that goes from A through the midpoint of $\overline{BC}$ will cut $\angle CAB$ in half, because it cuts the side $\overline{BC}$ in half.” Your friend then sketches the triangle above and to the right.

(a) Convince your friend that they are wrong. Use what you know about trigonometry to explain why $\angle CAD$ and $\angle DAB$ are not congruent.

(b) Locate the point E on $\overline{BC}$ that lies on the angle bisector of $\angle CAB$. How far is point E from point B? Show all your work.

(c) After you convince your friend that the bisector of $\angle CAB$ doesn’t go through the midpoint of $\overline{BC}$, they say to you, “Now I remember! Every point on an angle bisector is the same distance from the sides of the angle.” They draw a line segment from point E that is perpendicular to $\overline{AC}$ at point F and say, “That means that $\overline{EB}$ and $\overline{EF}$ are congruent.” Is your friend correct? You can use algebra or trigonometry to confirm or disprove their statement.