CollegeBoard Pre-AP®

# Pre-AP® Geometry with Statistics

## TEACHER RESOURCES

# Units 3 and 4

## ABOUT COLLEGE BOARD

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## PRE-AP EQUITY AND ACCESS POLICY

College Board believes that all students deserve engaging, relevant, and challenging grade-level coursework. Access to this type of coursework increases opportunities for all students, including groups that have been traditionally underrepresented in AP and college classrooms. Therefore, the Pre-AP program is dedicated to collaborating with educators across the country to ensure all students have the supports to succeed in appropriately challenging classroom experiences that allow students to learn and grow. It is only through a sustained commitment to equitable preparation, access, and support that true excellence can be achieved for all students, and the Pre-AP course designation requires this commitment.

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The sentence-writing strategies used in Pre-AP lessons are based upon The Writing Revolution, Inc., a national nonprofit organization that trains educators to implement The Hochman Method, an evidence-based approach to teaching writing. The strategies included in Pre-AP materials are meant to support students' writing, critical thinking, and content understanding, but they do not represent The Writing Revolution's full, comprehensive approach to teaching writing. More information can be found at **www.thewritingrevolution.org**.

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# Contents

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# Introduction to Pre-AP Geometry with Statistics

# About Pre-AP

# Introduction to Pre-AP

Every student deserves classroom opportunities to learn, grow, and succeed. College Board developed Pre-AP® to deliver on this simple premise. Pre-AP courses are designed to support all students across varying levels of readiness. They are not honors or advanced courses.

Participation in Pre-AP courses allows students to slow down and focus on the most essential and relevant concepts and skills. Students have frequent opportunities to engage deeply with texts, sources, and data as well as compelling higher-order questions and problems. Across Pre-AP courses, students experience shared instructional practices and routines that help them develop and strengthen the important critical thinking skills they will need to employ in high school, college, and life. Students and teachers can see progress and opportunities for growth through varied classroom assessments that provide clear and meaningful feedback at key checkpoints throughout each course.

## DEVELOPING THE PRE-AP COURSES

Pre-AP courses are carefully developed in partnership with experienced educators, including middle school, high school, and college faculty. Pre-AP educator committees work closely with College Board to ensure that the course resources define, illustrate, and measure grade-level-appropriate learning in a clear, accessible, and engaging way. College Board also gathers feedback from a variety of stakeholders, including Pre-AP partner schools from across the nation who have participated in multiyear pilots of select courses. Data and feedback from partner schools, educator committees, and advisory panels are carefully considered to ensure that Pre-AP courses provide all students with grade-level-appropriate learning experiences that place them on a path to college and career readiness.

## PRE-AP EDUCATOR NETWORK

Similar to the way in which teachers of Advanced Placement® (AP®) courses can become more deeply involved in the program by becoming AP Readers or workshop consultants, Pre-AP teachers also have opportunities to become active in their educator network. Each year, College Board expands and strengthens the Pre-AP National Faculty—the team of educators who facilitate Pre-AP Readiness Workshops and Pre-AP Summer Institutes. Pre-AP teachers can also become curriculum and assessment contributors by working with College Board to design, review, or pilot the course resources.

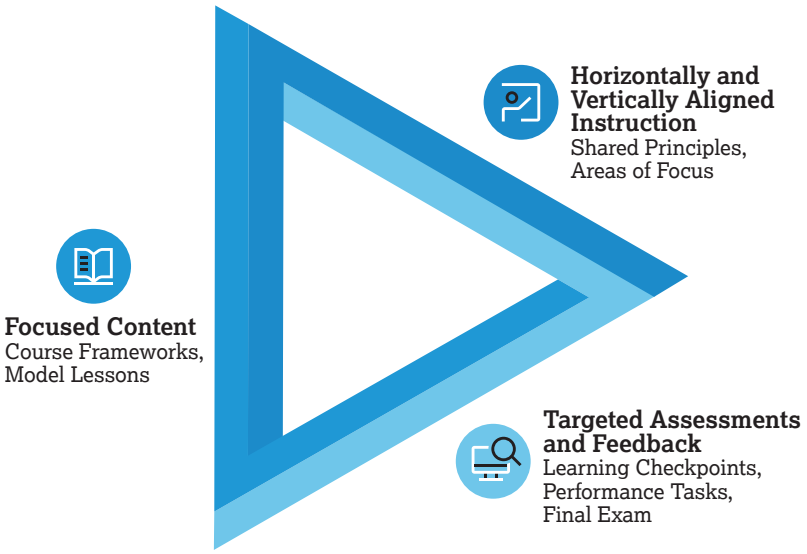
## HOW TO GET INVOLVED

Schools and districts interested in learning more about participating in Pre-AP should visit **preap.org/join** or contact us at **preap@collegeboard.org**.

Teachers interested in becoming members of Pre-AP National Faculty or participating in content development should visit **preap.org/national-faculty** or contact us at **preap@collegeboard.org**.

## Pre-AP Approach to Teaching and Learning

Pre-AP courses invite all students to learn, grow, and succeed through focused content, horizontally and vertically aligned instruction, and targeted assessments for learning. The Pre-AP approach to teaching and learning, as described below, is not overly complex, yet the combined strength results in powerful and lasting benefits for both teachers and students. This is our theory of action.




### FOCUSED CONTENT

Pre-AP courses focus deeply on a limited number of concepts and skills with the broadest relevance for high school coursework and college and career success. The course framework serves as the foundation of the course and defines these prioritized concepts and skills. Pre-AP model lessons and assessments are based directly on this focused framework. The course design provides students and teachers with intentional permission to slow down and focus.

### HORIZONTALLY AND VERTICALLY ALIGNED INSTRUCTION

Shared principles cut across all Pre-AP courses and disciplines. Each course is also aligned to discipline-specific areas of focus that prioritize the critical reasoning skills and practices central to that discipline.

## SHARED PRINCIPLES

All Pre-AP courses share the following set of research-supported instructional principles. Classrooms that regularly focus on these cross-disciplinary principles allow students to effectively extend their content knowledge while strengthening their critical thinking skills. When students are enrolled in multiple Pre-AP courses, the horizontal alignment of the shared principles provides students and teachers across disciplines with a shared language for their learning and investigation, and multiple opportunities to practice and grow. The critical reasoning and problem-solving tools students develop through these shared principles are highly valued in college coursework and in the workplace.


### Close Observation and Analysis

Students are provided time to carefully observe one data set, text image, performance piece, or problem before being asked to explain, analyze, or evaluate. This creates a safe entry point to simply express what they notice and what they wonder. It also encourages students to slow down and capture relevant details with intentionality to support more meaningful analysis, rather than rush to completion at the expense of understanding.

### Higher-Order Questioning

Students engage with questions designed to encourage thinking that is elevated beyond simple memorization and recall. Higher-order questions require students to make predictions, synthesize, evaluate, and compare. As students grapple with these questions, they learn that being inquisitive promotes extended thinking and leads to deeper understanding.

**Evidence-Based Writing**

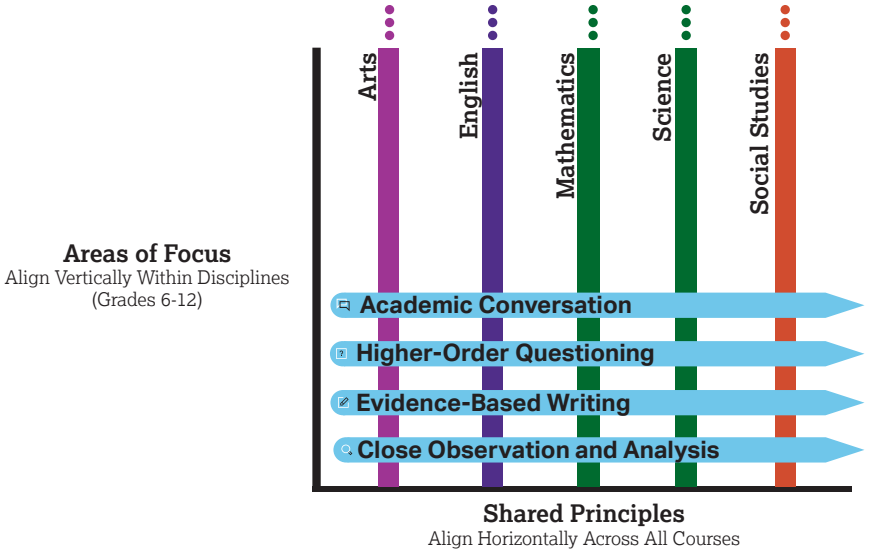
With strategic support, students frequently engage in writing coherent arguments from relevant and valid sources of evidence. Pre-AP courses embrace a purposeful and scaffolded approach to writing that begins with a focus on precise and effective sentences before progressing to longer forms of writing.

**Academic Conversation**

Through peer-to-peer dialogue, students' ideas are explored, challenged, and refined. As students engage in academic conversation, they come to see the value in being open to new ideas and modifying their own ideas based on new information. Students grow as they frequently practice this type of respectful dialogue and critique and learn to recognize that all voices, including their own, deserve to be heard.

## AREAS OF FOCUS

The areas of focus are discipline-specific reasoning skills that students develop and leverage as they engage with content. Whereas the shared principles promote horizontal alignment across disciplines, the areas of focus provide vertical alignment within a discipline, giving students the opportunity to strengthen and deepen their work with these skills in subsequent courses in the same discipline.




For information about the Pre-AP mathematics areas of focus, see page 15.

## TARGETED ASSESSMENTS FOR LEARNING

Pre-AP courses include strategically designed classroom assessments that serve as tools for understanding progress and identifying areas that need more support. The assessments provide frequent and meaningful feedback for both teachers and students across each unit of the course and for the course as a whole. For more information about assessments in Pre-AP Geometry with Statistics, see page 60.

## Pre-AP Professional Learning

Pre-AP teachers are required to engage in two professional learning opportunities. The first requirement is designed to help prepare them to teach their specific course. There are two options to meet the first requirement: the Pre-AP Summer Institute (Pre-APSI) and the Online Foundational Module Series. Both options provide continuing education units to educators who complete them.

- The Pre-AP Summer Institute is a four-day collaborative experience that empowers participants to prepare and plan for their Pre-AP course. While attending, teachers engage with Pre-AP course frameworks, shared principles, areas of focus, and sample model lessons. Participants are given supportive planning time where they work with peers to begin to build their Pre-AP course plan.
- The Online Foundational Module Series is available to all teachers of Pre-AP courses. This 12- to 20-hour course supports teachers in preparing for their Pre-AP course. Teachers explore course materials and experience model lessons from the student's point of view. They also begin to plan and build their own course so they are ready on day one of instruction.

The second professional learning requirement is to complete at least one of the Online Performance Task Scoring Modules, which offer guidance and practice applying Pre-AP scoring guidelines to student work.

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# About the Course

# Introduction to Pre-AP Geometry with Statistics

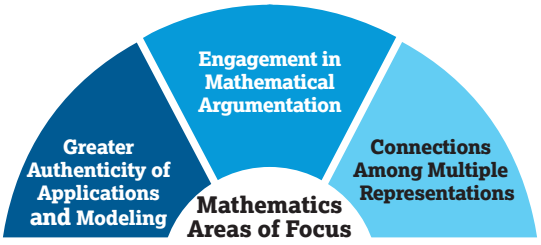
Pre-AP Geometry with Statistics is designed to provide students with a meaningful conceptual bridge between algebra and geometry to deepen their understanding of mathematics. Students often struggle to see the connections among their mathematics courses. In this course, students are expected to use the mathematical knowledge and skills they have developed previously to problem solve across the domains of algebra, geometry, and statistics.

Rather than seeking to cover all topics traditionally included in a standard geometry or introductory statistics textbook, this course focuses on the foundational geometric and statistical knowledge and skills that matter most for college and career readiness. The Pre-AP Geometry with Statistics Course Framework highlights how to guide students to connect core ideas within and across the units of the course, promoting a coherent understanding of measurement.

The components of this course have been crafted to prepare not only the next generation of mathematicians, scientists, programmers, statisticians, and engineers, but also a broader base of mathematically informed citizens who are well equipped to respond to the array of mathematics-related issues that impact our lives at the personal, local, and global levels.

## PRE-AP MATHEMATICS AREAS OF FOCUS

The Pre-AP mathematics areas of focus, shown below, are mathematical practices that students develop and leverage as they engage with content. They were identified through educator feedback and research about where students and teachers need the most curriculum support. These areas of focus are vertically aligned to the mathematical practices embedded in other mathematics courses in high school, including AP, and in college, giving students multiple opportunities to strengthen and deepen their work with these skills throughout their educational career. They also support and align to the AP Calculus Mathematical Practices, the AP Statistics Course Skills, and the mathematical practices listed in various state standards.



### Greater Authenticity of Applications and Modeling

***Students create and use mathematical models to understand and explain authentic scenarios.***

Mathematical modeling is a process that helps people analyze and explain the world. In Pre-AP Geometry with Statistics, students explore real-world contexts where mathematics can be used to make sense of a situation. They engage in the modeling process by making choices about what aspects of the situation to model, assessing how well the model represents the available data, drawing conclusions from their model, justifying decisions they make through the process, and identifying what the model helps clarify and what it does not.

In addition to mathematical modeling, Pre-AP Geometry with Statistics students engage in mathematics through authentic applications. Applications are similar to modeling problems in that they are drawn from real-world phenomena, but they differ because the applications dictate the appropriate mathematics to use to solve the problem. Pre-AP Geometry with Statistics balances these two types of real-world tasks.

### Engagement in Mathematical Argumentation

***Students use evidence to craft mathematical conjectures and prove or disprove them.***

Reasoning and proof lie at the heart of the discipline of mathematics. Mathematics is both a way of thinking and a set of tools for solving problems. Pre-AP Geometry with Statistics students gain proficiency in deductively reasoning with axioms and theorems to reach logical conclusions. Students also develop skills in using statistical and probabilistic reasoning to make sense of data and craft assertions using data as evidence and support. Students learn how to quantify chance and make inferences about populations. Through these two different types of mathematical argumentation, students learn how to be critical of their own reasoning and the reasoning of others.

### Connections Among Multiple Representations

***Students represent mathematical concepts in a variety of forms and move fluently among the forms.***

Pre-AP Geometry with Statistics students explore how to weave together multiple representations of geometric and statistics concepts. Every mathematical representation illuminates certain characteristics of a concept while also obscuring other aspects. Often, geometric reasoning is used to make sense of algebraic calculations. Likewise, algebraic techniques can be used to solve problems involving geometry. Patterns in data can emerge by depicting the data visually. Statistical calculations are important and valuable, but they make more sense to students when they are conceptually grounded in and related to graphical representations of data. With experience that continues to develop in Pre-AP Geometry with Statistics, students become equipped with a nuanced understanding of which representations best serve a particular purpose.

## PRE-AP GEOMETRY WITH STATISTICS AND CAREER READINESS

The Pre-AP Geometry with Statistics course resources are designed to expose students to a wide range of career opportunities that depend on geometry and statistics knowledge and skills. Examples include not only field-specific specialty careers such as mathematicians or statisticians, but also other endeavors where geometry and statistics knowledge is relevant, such as architects, carpenters, engineers, mechanics, actuaries, and programmers.

Career clusters that involve geometry and statistics, along with examples of careers in mathematics or related to mathematics, are provided below and on the following page. Teachers should consider discussing these with students throughout the year to promote motivation and engagement.

| Career Clusters Involving Mathematics |
| --- |
| agriculture, food, and natural resources<br>architecture and construction<br>arts, A/V technology, and communications<br>business management and administration<br>finance<br>government and public administration<br>health science<br>information technology<br>manufacturing<br>marketing<br>STEM (science, technology, engineering, and math)<br>transportation, distribution, and logistics |

| Examples of Geometry Related Careers | Examples of Statistics Related Careers |
|---|---|
| animator | drafter |
| architect | economist |
| cartographer | financial analyst |
| drafter | mathematics teacher |
| mathematician | meteorologist |
| mathematics teacher | professor |
| professor | programmer |
| programmer | research analyst |
| surveyor | statistician |

*Source for Career Clusters*: "Advanced Placement and Career and Technical Education: Working Together." Advance CTE and the College Board. October 2018. **https://careertech.org/resource/ap-cte-working-together**.

For more information about careers that involve mathematics, teachers and students can visit and explore the College Board's Big Future resources: **https://bigfuture.collegeboard.org/majors/math-statistics-mathematics**.

## SUMMARY OF RESOURCES AND SUPPORTS

Teachers are strongly encouraged to take advantage of the full set of resources and supports for Pre-AP Geometry with Statistics, which is summarized below. Some of these resources must be used for a course to receive the Pre-AP Course Designation. To learn more about the requirements for course designation, see details below and on page 68.

### COURSE FRAMEWORK

The framework defines what students should know and be able to do by the end of the course. It serves as an anchor for model lessons and assessments, and it is the primary document teachers can use to align instruction to course content. **Use of the course framework is required**. *For more details see page 24.*

### MODEL LESSONS

Teacher resources, available in print and online, include a robust set of model lessons that demonstrate how to translate the course framework, shared principles, and areas of focus into daily instruction. **Use of the model lessons is encouraged but not required**. *For more details see page 58.*

### LEARNING CHECKPOINTS

Accessed through Pre-AP Classroom (the Pre-AP digital platform), these short formative assessments provide insight into student progress. They are automatically scored and include multiple-choice and technology-enhanced items with rationales that explain correct and incorrect answers. **Use of one learning checkpoint per unit is required**. *For more details see page 60.*

### PERFORMANCE TASKS

Available in the printed teacher resources as well as on Pre-AP Classroom, performance tasks allow students to demonstrate their learning through extended problem-solving, writing, analysis, and/or reasoning tasks. Scoring guidelines are provided to inform teacher scoring, with additional practice and feedback suggestions available in online modules on Pre-AP Classroom. **Use of each unit's performance task is required**. *For more details see page 61.*

### PRACTICE PERFORMANCE TASKS

Available in the student resources, with supporting materials in the teacher resources, these tasks provide an opportunity for students to practice applying skills and knowledge as they would in a performance task, but in a more scaffolded environment. **Use of the practice performance tasks is encouraged but not required**. *For more details see page 62.*

### FINAL EXAM

Accessed through Pre-AP Classroom, the final exam serves as a classroom-based, summative assessment designed to measure students' success in learning and applying the knowledge and skills articulated in the course framework. **Administration of the final exam is encouraged but not required**. *For more details see page 63.*

### PROFESSIONAL LEARNING

Both the four-day Pre-AP Summer Institute (Pre-APSI) and the Online Foundational Module Series support teachers in preparing and planning to teach their Pre-AP course. **All Pre-AP teachers are required to either attend the Pre-AP Summer Institute or complete the module series. In addition, teachers are required to complete at least one Online Performance Task Scoring module**. *For more details see page 11.*

# Course Map

## PLAN

The course map shows how components are positioned throughout the course. As the map indicates, the course is designed to be taught over 140 class periods (based on 45-minute class periods), for a total of 28 weeks.

Model lessons are included for approximately 50% of the total instructional time, with the percentage varying by unit. Each unit is divided into key concepts.

## TEACH

The model lessons demonstrate how the Pre-AP shared principles and mathematics areas of focus come to life in the classroom.

*Shared Principles*
Close observation and analysis
Higher-order questioning
Evidence-based writing
Academic conversation

*Areas of Focus*
Greater authenticity of applications and modeling
Engagement in mathematical argumentation
Connections among multiple representations

## ASSESS AND REFLECT

Each unit includes two learning checkpoints and a performance task. These formative assessments are designed to provide meaningful feedback for both teachers and students.

**Note:** The final exam, offered during a six-week window in the spring, is not represented on the map.

**UNIT 1** Measurement in Data

~35 Class Periods

**Pre-AP model lessons provided for 100% of instructional time in this unit**

**KEY CONCEPT 1.1**
The Shape of Data

Learning Checkpoint 1

**KEY CONCEPT 1.2**
Chance Events

**KEY CONCEPT 1.3**
Inferences from Data

Learning Checkpoint 2

Performance Task for Unit 1

## UNIT 2 Tools and Techniques of Geometric Measurement

~35 Class Periods

**Pre-AP model lessons provided for approximately 50% of instructional time in this unit**

**KEY CONCEPT 2.1**

Measurement in Geometry

Learning Checkpoint 1

**KEY CONCEPT 2.2**

Parallel and Perpendicular Lines

**KEY CONCEPT 2.3**

Measurement in Right Triangles

Learning Checkpoint 2

Performance Task for Unit 2

## UNIT 3 Measurement in Congruent and Similar Figures

~35 Class Periods

**Pre-AP model lessons provided for approximately 30% of instructional time in this unit**

**KEY CONCEPT 3.1**

Transformations of Points in a Plane

**KEY CONCEPT 3.2**

Congruent and Similar Polygons

Learning Checkpoint 1

**KEY CONCEPT 3.3**

Measurement of Lengths and Angles in Circles

Learning Checkpoint 2

Performance Task for Unit 3

## UNIT 4 Measurement in Two and Three Dimensions

~35 Class Periods

**Pre-AP model lessons provided for approximately 10% of instructional time in this unit**

**KEY CONCEPT 4.1**

Area as a Two-Dimensional Measurement

**KEY CONCEPT 4.2**

**Learning Objectives 4.2.1–4.2.4**

Volume as a Three-Dimensional Measurement

Learning Checkpoint 1

**KEY CONCEPT 4.2 (*continued*)**

**Learning Objectives 4.2.5–4.2.7**

Volume as a Three-Dimensional Measurement

**KEY CONCEPT 4.3**

Measurements of Spheres

Learning Checkpoint 2

Performance Task for Unit 4

# Pre-AP Geometry with Statistics Course Framework

## INTRODUCTION

Based on the Understanding by Design® (Wiggins and McTighe) model, the Pre-AP Geometry with Statistics Course Framework is back mapped from AP expectations and aligned to essential grade-level expectations. The course framework serves as a teacher's blueprint for the Pre-AP Geometry with Statistics instructional resources and assessments.

The course framework was designed to meet the following criteria:

- **Focused:** The framework provides a deep focus on a limited number of concepts and skills that have the broadest relevance for later high school, college, and career success.
- **Measurable:** The framework's learning objectives are observable and measurable statements about the knowledge and skills students should develop in the course.
- **Manageable:** The framework is manageable for a full year of instruction, fosters the ability to explore concepts in depth, and enables room for additional local or state standards to be addressed where appropriate.
- **Accessible:** The framework's learning objectives are designed to provide all students, across varying levels of readiness, with opportunities to learn, grow, and succeed.

## COURSE FRAMEWORK COMPONENTS

The Pre-AP Geometry with Statistics Course Framework includes the following components:

### Big Ideas

The big ideas are recurring themes that allow students to create meaningful connections between course concepts. Revisiting the big ideas throughout the course and applying them in a variety of contexts allows students to develop deeper conceptual understandings.

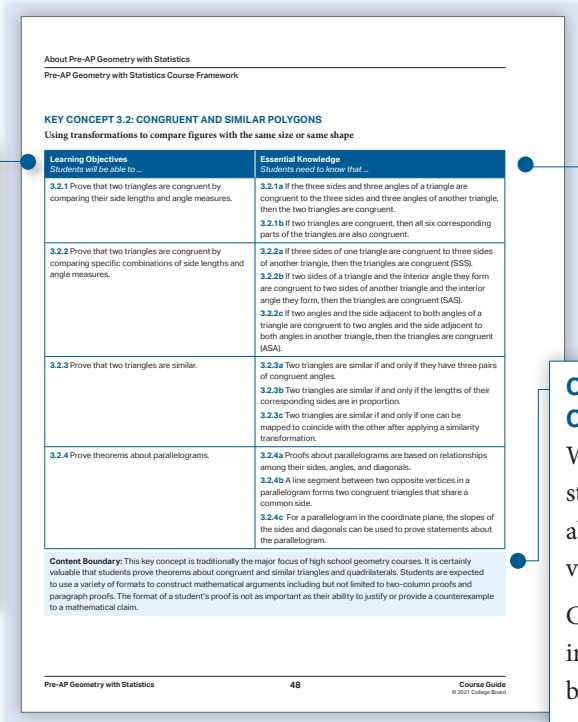
### Enduring Understandings

Each unit focuses on a small set of enduring understandings. These are the long-term takeaways related to the big ideas that leave a lasting impression on students. Students build and earn these understandings over time by exploring and applying course content throughout the year.

### Key Concepts

To support teacher planning and instruction, each unit is organized by key concepts. Each key concept includes relevant **learning objectives** and **essential knowledge statements** and may also include **content boundary and cross connection statements**. These are illustrated and defined below.

**Learning Objectives:**
These objectives define what a student needs to be able to do with essential knowledge to progress toward the enduring understandings. The learning objectives serve as actionable targets for instruction and assessment.



About Pre-AP Geometry with Statistics

Pre-AP Geometry with Statistics Course Framework

**KEY CONCEPT 3.2: CONGRUENT AND SIMILAR POLYGONS**

**Using transformations to compare figures with the same size or same shape**

| Learning Objectives<br>*Students will be able to ...* | Essential Knowledge<br>*Students need to know that ...* |
|---|---|
| **3.2.1** Prove that two triangles are congruent by comparing their side lengths and angle measures. | **3.2.1a** If the three sides and three angles of a triangle are congruent to the three sides and three angles of another triangle, then the two triangles are congruent.<br>**3.2.1b** If two triangles are congruent, then all six corresponding parts of the triangles are also congruent. |
| **3.2.2** Prove that two triangles are congruent by comparing specific combinations of side lengths and angle measures. | **3.2.2a** If three sides of one triangle are congruent to three sides of another triangle, then the triangles are congruent (SSS).<br>**3.2.2b** If two sides of a triangle and the interior angle they form are congruent to two sides of another triangle and the interior angle they form, then the triangles are congruent (SAS).<br>**3.2.2c** If two angles and the side adjacent to both angles of a triangle are congruent to two angles and the side adjacent to both angles in another triangle, then the triangles are congruent (ASA). |
| **3.2.3** Prove that two triangles are similar. | **3.2.3a** Two triangles are similar if and only if they have three pairs of congruent angles.<br>**3.2.3b** Two triangles are similar if and only if the lengths of their corresponding sides are in proportion.<br>**3.2.3c** Two triangles are similar if and only if one can be mapped to coincide with the other after applying a similarity transformation. |
| **3.2.4** Prove theorems about parallelograms. | **3.2.4a** Proofs about parallelograms are based on relationships among their sides, angles, and diagonals.<br>**3.2.4b** A line segment between two opposite vertices in a parallelogram forms two congruent triangles that share a common side.<br>**3.2.4c** For a parallelogram in the coordinate plane, the slopes of the sides and diagonals can be used to prove statements about the parallelogram. |

**Content Boundary:** This key concept is traditionally the major focus of high school geometry courses. It is certainly valuable that students prove theorems about congruent and similar triangles and quadrilaterals. Students are expected to use a variety of formats to construct mathematical arguments including but not limited to two-column proofs and paragraph proofs. The format of a student's proof is not as important as their ability to justify or provide a counterexample to a mathematical claim.

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**Essential Knowledge Statements:**
Each essential knowledge statement is linked to a learning objective. One or more essential knowledge statements describe the knowledge required to perform each learning objective.

**Content Boundary and Cross Connection Statements:**
When needed, content boundary statements provide additional clarity about the content and skills that lie within versus outside of the scope of this course.

Cross connection statements highlight important connections that should be made between key concepts within and across the units.

## BIG IDEAS IN PRE-AP GEOMETRY WITH STATISTICS

While the Pre-AP Geometry with Statistics framework is organized into four core units of study, the content is grounded in three big ideas, which are cross-cutting concepts that build conceptual understanding and spiral throughout the course. Since these ideas cut across units, they serve as the underlying foundation for the enduring understandings, key concepts, and learning objectives that make up the focus of each unit. A deep and productive understanding in Pre-AP Geometry with Statistics relies on these three big ideas:

- **Measurement:** Measurement is the quantification of features of an object or a phenomenon. In geometry, measuring objects allows us to draw meaningful conclusions about those objects. Measurement provides relatable real-world applications in one, two, and three dimensions.
- **Transformation:** A transformation is a function, which means that it associates one set of objects with another. When a mathematical object is transformed, some of its measurements change while other measurements do not change. Congruence and similarity are defined through transformations, which puts the focus on measurements that are affected by transformations and those that are not. An understanding how data distributions are affected by transformations enhances the connections between probability and statistics.
- **Comparison and Composition:** Throughout mathematics, new and more complex concepts are understood in terms of simpler, previously explored concepts. In geometry, this mode of thinking allows for the deconstruction of two- and three-dimensional shapes for further investigation. This interpretation relies on the recognition that complex objects are composed of, and can be compared to, simpler objects. For statistics, this means using measures of center and spread to characterize complex data distributions.

## OVERVIEW OF PRE-AP GEOMETRY WITH STATISTICS UNITS AND ENDURING UNDERSTANDINGS

| Unit 1: Measurement in Data | Unit 2: Tools and Techniques of Geometric Measurement |
|---|---|
| ▪ Statistics are numbers that summarize large data sets by reducing their complexity to a few key values that model their center and spread.<br>▪ Distributions are functions whose displays are used to analyze data sets.<br>▪ Probabilistic reasoning allows us to anticipate patterns in data.<br>▪ The method by which data are collected influences what can be said about the population from which the data were drawn, and how certain those statements are. | ▪ A formal mathematical argument establishes new truths by logically combining previously known facts.<br>▪ Measuring features of geometric figures is the process of assigning numeric values to attributes of the figures, which allows the attributes to be compared.<br>▪ Pairs of lines in a plane that never intersect or that intersect at right angles have special geometric and algebraic properties.<br>▪ Right triangles are simple geometric shapes in which we can relate the measures of acute angles to ratios of their side lengths. |
| **Unit 3: Measurement in Congruent and Similar Figures** | **Unit 4: Measurement in Two and Three Dimensions** |
| ▪ Transformations are functions that can affect the measurements of a geometric figure.<br>▪ Congruent figures have equal corresponding angle measures and equal distances between corresponding pairs of points.<br>▪ Similar figures have equal corresponding angle measurements, and the distances between corresponding pairs of points are proportional.<br>▪ The geometry of a circle is completely determined by its radius. | ▪ The area of a figure depends on its height and its cross-sectional widths.<br>▪ The volume of a solid depends on its height and its cross-sectional areas.<br>▪ The geometry of a sphere is completely determined by its radius. |

## Unit 1: Measurement in Data

**Suggested Timing: Approximately 7 weeks**

This unit offers a sustained and focused examination of statistics and probability to support the development of students' quantitative literacy. Statistics and probability help us perform essential real-world tasks such as making informed choices, deciding between different policies, and weighing competing knowledge claims. While topics of statistics and probability are commonplace in high school geometry courses, students often have limited opportunities to engage in statistical and probabilistic reasoning and sense-making. To move students toward a sophisticated understanding of data, students are expected to think about data sets as distributions which are functions that associate data values with their frequency or their probability. This encourages students to connect their knowledge of functions to concepts of statistics and probability, creating a more complete understanding of mathematics. Throughout the unit, students generate their own data through surveys, experiments, and simulations that investigate some aspect of the real world. They engage in statistical calculations and probabilistic reasoning as methods of analysis to make sense of data and draw inferences about populations. Incorporating statistics and probability in the same course as geometry allows students to experience two distinct forms of argumentation: geometrical reasoning as drawing conclusions with certainty about an ideal mathematical world, and probabilistic reasoning as drawing less-than-certain conclusions about the real world. The conclusions of a probability argument are presented as ranges that have varying degrees of certainty.

### ENDURING UNDERSTANDINGS

*Students will understand that ...*

- Statistics are numbers that summarize large data sets by reducing their complexity to a few key values that model their center and spread.
- Distributions are functions whose displays are used to analyze data sets.
- Probabilistic reasoning allows us to anticipate patterns in data.
- The method by which data are collected influences what can be said about the population from which the data were drawn, and how certain those statements are.

## KEY CONCEPTS

- **1.1: The shape of data –** Identifying measures of center and spread to summarize and characterize a data distribution
- **1.2: Chance events –** Exploring patterns in random events to anticipate the likelihood of outcomes
- **1.3: Inferences from data –** Using probability and statistics to make claims about a population

## KEY CONCEPT 1.1: THE SHAPE OF DATA

**Identifying measures of center and spread to summarize and characterize a data distribution**

| **Learning Objectives**<br>*Students will be able to ...* | **Essential Knowledge**<br>*Students need to know that ...* |
|---|---|
| **1.1.1** Determine appropriate summary statistics for a quantitative data distribution. | **1.1.1a** A data distribution is a function whose input is each value in a data set and whose output is the corresponding frequency of that value.<br>**1.1.1b** Summary statistics describe the important features of data distributions including identifying a typical value, also called the center of the data, and describing the clustering of the data around the typical value, also called the spread of the data.<br>**1.1.1c** The mean and the median summarize a data distribution by identifying a typical value, or center, of the distribution. The mean and median have the same units as the values in the data distribution.<br>**1.1.1d** The standard deviation, interquartile range, and range summarize a data distribution by quantifying the variability, or spread, of the data set. The standard deviation, interquartile range, and range have the same units as the values in the data distribution. |
| **1.1.2** Create a graphical representation of a quantitative data set. | **1.1.2a** A boxplot summarizes a quantitative data set by partitioning its values into four groups, each consisting of the same number of data values. Boxplots are used to depict the spread of a distribution.<br>**1.1.2b** A histogram summarizes a quantitative data set by partitioning its values into equal-width intervals and displaying bars whose heights indicate the frequency of values contained in each interval. Histograms are used to depict the shape of a distribution. |
| **1.1.3** Analyze data distributions with respect to their centers. | **1.1.3a** The mean is the only point in the domain of a distribution where the sum of the deviations, or differences, between the mean and each point in the distribution is zero.<br>**1.1.3b** The mean can be thought of as the center of mass of the data set. It is a weighted average that accounts for the number of data points that exists for every given value in the data set.<br>**1.1.3c** Measures of center can be used to compare the typical values of the distributions. They provide useful information about whether one distribution is typically larger, smaller, or about the same as another distribution. |

| **Learning Objectives**<br>*Students will be able to ...* | **Essential Knowledge**<br>*Students need to know that ...* |
|---|---|
| **1.1.4** Analyze data distributions with respect to their symmetry or direction of skew. | **1.1.4a** For symmetric distributions, such as the normal distribution, the proportion of data in any range to the left of the mean is equal to the proportion of data in the corresponding range to the right of the mean.<br>**1.1.4b** Skew describes the asymmetry of a distribution. The direction of skew is indicated by the longer tail of data values in an asymmetric distribution.<br>**1.1.4c** When a distribution is skewed, its mean and median will differ. The farther apart the mean and median are in a distribution, the more skewed the distribution will appear. |
| **1.1.5** Analyze data distributions with respect to their variability. | **1.1.5a** Measures of variability quantify the typical spread of a data distribution. They are used to describe how similar the values of a data set are to each other. A distribution with low variability will have data values that are clustered at the center, so the distribution is well characterized by its measures of center. A distribution with high variability will have data values that are spread out from the center, so the distribution is less well characterized by its measures of center.<br>**1.1.5b** The interquartile range is the length of the interval that contains the middle 50% of the values in a distribution.<br>**1.1.5c** The total variation of a distribution can be measured by the sum of the squared deviations from the mean. The variance of a distribution is the average of the squared deviations from the mean.<br>**1.1.5d** The standard deviation is the square root of the variance. The standard deviation can be interpreted as a typical distance of the data values from the mean. |
| **1.1.6** Model a data distribution with a normal distribution. | **1.1.6a** The normal distribution is a model of a data distribution defined by its mean and standard deviation. The normal distribution is bell-shaped and symmetric about the mean. In a normal distribution, the frequency of data values tapers off at one standard deviation above or below the mean.<br>**1.1.6b** When a normal distribution is used to model a data distribution, approximately 68% of the data values fall within one standard deviation of the mean. Approximately 95% of the data values fall within two standard deviations of the mean. Over 99% of the data values fall within three standard deviations of the mean.<br>**1.1.6c** For normally distributed data, the mean and median are the same number, and they correspond to the mode, which is the value in the distribution with the highest frequency. |

**Content Boundary**: In this unit, students are introduced to the normal distribution as a model for some data distributions, similar to how linear functions can be used to model some two-variable data sets. The normal distribution is often used to answer probabilistic questions. Those types of questions should be reserved for the lessons of Key Concept 1.2: Chance events.

**Cross Connection**: Students likely come to this course with a basic understanding of how to calculate some summary statistics, but with limited conceptual understanding about their meaning and utility. A goal of this unit is to expand students' understanding of measures of center and spread. The focus of the unit should be on using these measures to analyze data distributions.

## KEY CONCEPT 1.2: CHANCE EVENTS

**Exploring patterns in random events to anticipate the likelihood of outcomes**

| **Learning Objectives**<br>*Students will be able to …* | **Essential Knowledge**<br>*Students need to know that …* |
|---|---|
| **1.2.1** Create or analyze a data display for a categorical data set. | **1.2.1a** Venn diagrams and contingency tables are common displays of categorical data and are useful for answering questions about probability.<br>**1.2.1b** The intersection of two categories is the set of elements common to both categories.<br>**1.2.1c** The union of two categories is the set of elements found by combining all elements of both categories.<br>**1.2.1d** For categorical data, variability is determined by comparing relative frequencies of categories. |
| **1.2.2** Determine the probability of an event. | **1.2.2a** The sample space is the set of all outcomes of an experiment or random trial. An event is a subset of the sample space.<br>**1.2.2b** Probabilities are numbers between 0 and 1 where 0 means there is no possibility that an event can occur, and 1 means the event is certain to occur. The probability of an event occurring can be described numerically as a ratio of the number of favorable outcomes to the number of total outcomes in a sample space.<br>**1.2.2c** A probability distribution is a function that associates a probability with each possible value or interval of values for a random variable. The sum of the probabilities over all possible values of the independent variable must be 1. |
| **1.2.3** Calculate relative frequencies, joint frequencies, marginal frequencies, or conditional probabilities for a categorical data set. | **1.2.3a** Relative frequencies are the number of times an event occurs divided by the total number of observations. They can be used to estimate probabilities of future events occurring.<br>**1.2.3b** Joint frequencies are events that co-occur for two or more variables. They are the frequencies displayed in cells in a two-way contingency table.<br>**1.2.3c** Marginal frequencies are events that summarize the frequencies across all levels of one variable while holding the second variable constant. They are the row totals and column totals in a two-way contingency table.<br>**1.2.3d** The conditional probability of *B*, given *A* has already occurred, is the proportion of times *B* occurs when restricted to events only in *A*. |
| **1.2.4** Determine if two events are independent. | **1.2.4a** Two events, *A* and *B*, are independent if the occurrence of *A* does not affect the probability of *B*.<br>**1.2.4b** Two events, *A* and *B*, are independent if the probability of *A* and *B* occurring together is the product of their probabilities. |

| **Learning Objectives**<br>*Students will be able to …* | **Essential Knowledge**<br>*Students need to know that …* |
|---|---|
| **1.2.5** Calculate the probability of a range of values of an independent variable, given a mean, standard deviation, and normal distribution. | **1.2.5a** The normal distribution can be used to model a probability distribution that is bell-shaped and symmetric about the mean.<br>**1.2.5b** When a normal distribution is used as a model of a probability distribution, the probability of a data value occurring above the mean is 0.5 and the probability of a data value occurring below the mean is 0.5.<br>**1.2.5c** When a normal distribution is used as a model of a probability distribution, the probability of data occurring within one standard deviation of the mean is approximately 0.68, the probability of data occurring within two standard deviations of the mean is approximately 0.95, and the probability of data occurring within three standard deviations of the mean is approximately 0.997. These proportions can be used to determine the probability of an event occurring in a population. |

**Content Boundary:** Throughout the course framework, the terms *random variable* and *independent variable* are used interchangeably. These terms describe different aspects of the same variable. The term *random variable* describes the process by which the variable was sampled, and *independent variable* is used when the frequency or probability distribution of the variable is of interest.

**Cross Connection:** In this unit, students will explore how the normal distribution can be used to model a probability distribution. This is a slightly different application of the normal distribution than students used in the previous key concept. In those lessons, students were expected to answer questions about the percent of data, or expected percent of data, that occurred within certain ranges. In the lessons of this key concept, students are expected to answer questions about the probability that an event would occur within a given range.

**Cross Connection:** Students will likely have some understanding of probability and randomness from previous courses. It is important for students to understand that, mathematically, the term *random* means that the outcome of a single trial may not be known, although over repeated trials, the proportions of the different outcomes may be predictable.

## KEY CONCEPT 1.3: INFERENCES FROM DATA

**Using probability and statistics to make claims about a population**

| **Learning Objectives**<br>*Students will be able to …* | **Essential Knowledge**<br>*Students need to know that …* |
|---|---|
| **1.3.1** Distinguish between accuracy and precision as measures of statistical variability and statistical bias in measurements. | **1.3.1a** Accuracy is how close the measurements in a measurement process are to the true value being estimated. Accuracy is determined by comparing the center of a sample of measurements to the true value of the measure.<br>**1.3.1b** Precision is how close the measurements in a measurement process are to one another. Precision is determined by examining the variability of a sample of measurements.<br>**1.3.1c** Bias is the tendency of a measurement process to systematically overestimate or underestimate the true measure of a phenomenon. Bias is an indication of the inaccuracy of the measurement process. |
| **1.3.2** Describe how the size of a sample impacts how well it represents the population from which it was drawn. | **1.3.2a** The law of large numbers states that the mean of the results obtained from a large number of trials will tend to become closer to the true value of the phenomenon being measured as more trials are performed. This means we can trust larger samples more than smaller ones.<br>**1.3.2b** The law of large numbers assumes that there is no systematic error of measurement in the sample. |
| **1.3.3** Design a method for gathering data that is appropriate for a given purpose. | **1.3.3a** An experiment is a method of gathering information about phenomena where the independent variable is manipulated by the researcher.<br>**1.3.3b** An observational study is a method of gathering information about phenomena where the independent variable is not under the control of the researcher.<br>**1.3.3c** A survey is a method of gathering information from a sample of people using a questionnaire. |
| **1.3.4** Identify biases in sampling methods for experiments, observational studies, and surveys. | **1.3.4a** Experiments can be subject to systematic bias if the experiment does not sample from the population randomly and does not randomly assign sampling units to experimental and control conditions.<br>**1.3.4b** Observational studies can be subject to sampling bias if the sampling unit being observed is not selected randomly.<br>**1.3.4c** Surveys can be subject to bias from several factors, including sampling bias and response bias. |

**Content Boundary:** A traditionally challenging concept is informally introduced in this key concept: the law of large numbers (Learning Objective 1.3.2). For this concept, students should not be expected to develop a complete understanding. A full understanding of the law of large numbers is beyond the scope of this course. If students go on to take a more advanced statistics course, such as AP Statistics, they will explore this concept more thoroughly.

## Unit 2: Tools and Techniques of Geometric Measurement

**Suggested Timing: Approximately 7 weeks**

This unit introduces students to the basic objects of geometry and the tools used to explore these objects throughout the remainder of the course. The basic objects students investigate in this unit include lines, rays, segments, and angles. These figures serve as the building blocks of more complex objects that students explore in later units. Students continue to expand their understanding of measurement by developing techniques for quantifying and comparing the attributes of geometric objects. The tools they use to analyze objects include straightedges, compasses, rulers, protractors, dynamic geometry software, the coordinate plane, and right triangles. In addition, students use an informal understanding of transformations throughout the unit to justify whether two basic objects are congruent. They formalize transformations and define congruence and similarity through transformations in Unit 3. This unit culminates with an introduction to right triangle trigonometry, which integrates the tools and techniques of the unit into an investigation of new ways to express the relationship between angle measures and side lengths.

Throughout Units 2–4, specific learning objectives require students to prove geometric concepts. Students' proofs can be organized in a variety of formats, such as two-column tables, flowcharts, or paragraphs. The format of a student's proof is not as important as their ability to justify a mathematical claim or provide a counterexample disproving one. They should develop an understanding that a mathematical proof establishes the truth of a statement by combining previously developed truths into a logically consistent argument.

### ENDURING UNDERSTANDINGS

*Students will understand that ...*

- A formal mathematical argument establishes new truths by logically combining previously known facts.
- Measuring features of geometric figures is the process of assigning numeric values to attributes of the figures, which allows the attributes to be compared.
- Pairs of lines in a plane that never intersect or that intersect at right angles have special geometric and algebraic properties.
- Right triangles are simple geometric shapes in which we can relate the measures of acute angles to ratios of their side lengths.

## KEY CONCEPTS

- **2.1: Measurement in geometry –** Using lengths, angles, and distance to describe and compare shapes
- **2.2: Parallel and perpendicular lines –** Determining if and how lines intersect to analyze spatial relationships in the real world
- **2.3: Measurement in right triangles –** Using the relationships between the side lengths and angle measures of right triangles to create new measurements

## KEY CONCEPT 2.1: MEASUREMENT IN GEOMETRY

**Using lengths, angles, and distance to describe and compare shapes**

| **Learning Objectives**<br>*Students will be able to ...* | **Essential Knowledge**<br>*Students need to know that ...* |
|---|---|
| **2.1.1** Describe and correctly label a line, ray, or line segment. | **2.1.1a** For any two distinct points in a plane, there is only one line that contains them.<br>**2.1.1b** A line is straight, has no width, extends infinitely in two directions, and contains infinitely many points. A line can be named by a single lowercase letter, or it can be named by any two distinct points that lie on the line.<br>**2.1.1c** A ray is a portion of a line that has a single endpoint and extends infinitely in one direction. A ray can be named by its endpoint and any other point on the ray, with its endpoint listed first.<br>**2.1.1d** A line segment is a portion of a line between and including two endpoints. A line segment can be named by its two endpoints. |
| **2.1.2** Describe and correctly label an angle. | **2.1.2a** An angle is a geometric figure formed when two lines, line segments, or rays share an endpoint. The point common to both lines, line segments, or rays is called the vertex of the angle.<br>**2.1.2b** An angle can be named by its vertex. An angle can also be named using its vertex and the names of a nonvertex point that lies on each of its sides. For such angle names, the point that indicates the vertex is the second of the three points. |
| **2.1.3** Measure a line segment. | **2.1.3a** The length of a line segment is the distance between its endpoints.<br>**2.1.3b** The length of a line segment is measured using a specified unit of measure. Units of measure can be formal or informal. |
| **2.1.4** Measure an angle. | **2.1.4a** An angle can be measured by determining the amount of rotation one ray would make about the vertex of the angle to coincide with the other ray. The amount of rotation is measured as a fraction of the rotation needed to rotate a full circle.<br>**2.1.4b** An angle can be measured with reference to a circle whose center is the vertex of the angle by determining the fraction of the circular arc between the intersection points of the rays and the circle. The length of the circular arc is measured as a fraction of the circle's circumference.<br>**2.1.4c** An angle can be measured in units of radians, equaling the arc length spanned by the angle when its vertex coincides with the center of a unit circle. |

| **Learning Objectives**<br>*Students will be able to ...* | **Essential Knowledge**<br>*Students need to know that ...* |
|---|---|
| **2.1.5** Prove whether two or more line segments are congruent. | **2.1.5a** Two line segments are congruent if and only if one segment can be translated, rotated, or reflected to coincide with the other segment without changing the length of either line segment.<br>**2.1.5b** Two line segments are congruent if and only if they have the equal lengths. |
| **2.1.6** Prove whether two or more angles are congruent. | **2.1.6a** Two angles are congruent if and only if one angle can be translated, rotated, or reflected to coincide with the other angle without changing the measure of either angle.<br>**2.1.6b** Two angles are congruent if and only if they have equal measures. |
| **2.1.7** Construct a congruent copy of a line segment or an angle. | **2.1.7a** A synthetic geometric construction utilizes only a straightedge and a compass to accurately draw or copy a figure.<br>**2.1.7b** A straightedge is a tool for connecting two distinct points with a line segment.<br>**2.1.7c** A compass is a tool for copying distances between pairs of points. |
| **2.1.8** Calculate the distance between two points. | **2.1.8a** The distance between two points in the plane is the length of the line segment connecting the points.<br>**2.1.8b** The distance between two points in the coordinate plane can be determined by applying the Pythagorean theorem to a right triangle whose hypotenuse is a line segment formed by the two points and whose sides are parallel to each axis. |
| **2.1.9** Solve problems involving segment lengths and/or angle measures. | **2.1.9a** Given line segment $\overline{AC}$ and a point, $B$, that lies on the segment between points $A$ and $C$, the measure of segment $\overline{AC}$ is the sum of the measures of segments $\overline{AB}$ and $\overline{BC}$.<br>**2.1.9b** Given $\angle AOC$ and ray $\overrightarrow{OB}$ that lies between $\overrightarrow{OA}$ and $\overrightarrow{OC}$, the measure of $\angle AOC$ is equal to the sum of the measures of $\angle AOB$ and $\angle BOC$.<br>**2.1.9c** Two angles are called complementary if the sum of their measures is 90°. Two angles are complementary if they form a right angle when adjacent.<br>**2.1.9d** Two angles are called supplementary if the sum of their measures is 180°. Two angles are supplementary if they form a straight angle when adjacent. |

| **Learning Objectives**<br>*Students will be able to ...* | **Essential Knowledge**<br>*Students need to know that ...* |
|---|---|
| **2.1.10** Solve problems involving a segment bisector or an angle bisector. | **2.1.10a** The midpoint of a line segment is the point located on the line segment equidistant from the endpoints.<br>**2.1.10b** In the coordinate plane, the *x*- and *y*- coordinates of the midpoint of a line segment are the arithmetic means of the corresponding coordinates of the endpoints.<br>**2.1.10c** A bisector of an angle is a line, ray, or line segment that contains the vertex of the angle and divides the angle into two congruent adjacent angles.<br>**2.1.10d** Points that lie on the angle bisector are equidistant from the sides of the angle. |

**Content Boundary:** When a mathematical statement includes the phrase "if and only if" to join two sentences, it means that the sentences are logically equivalent. That is, both sentences are true or both sentences are false. These sentences are sometimes referred to as "biconditional statements." Students are expected to know that these statements are both true or both false, but it is not necessary for them to know the term "biconditional" for this course.

## KEY CONCEPT 2.2: PARALLEL AND PERPENDICULAR LINES

**Determining if and how lines intersect to analyze spatial relationships in the real world**

| **Learning Objectives**<br>*Students will be able to ...* | **Essential Knowledge**<br>*Students need to know that ...* |
|---|---|
| **2.2.1** Justify the relationship between the slopes of parallel or perpendicular lines in the coordinate plane using transformations. | **2.2.1a** The relationship between the slopes of parallel lines can be justified by comparing their slope triangles using translation.<br>**2.2.1b** The relationship between the slopes of perpendicular lines can be justified by comparing their slope triangles using rotation by 90°. |
| **2.2.2** Solve problems involving two or more parallel lines, rays, or line segments. | **2.2.2a** Two distinct lines, rays, or line segments in the coordinate plane are parallel if and only if they have the same slope or are both vertical.<br>**2.2.2b** A transversal is a line that intersects a set of lines. Two lines, rays, or line segments intersected by a transversal will be parallel if and only if the same-side interior angles formed by the lines and the transversal are supplementary.<br>**2.2.2c** Two lines intersected by a transversal will be parallel if and only if the corresponding angles, alternate interior angles, or alternate exterior angles formed by the lines and the transversal are congruent. |
| **2.2.3** Construct a line, ray, or line segment parallel to another line, ray, or line segment that passes through a point not on the given line, ray, or line segment. | **2.2.3a** Given a line and a point not on the given line, there is exactly one line through the point that will be parallel to the given line.<br>**2.2.3b** Two parallel lines, rays, or line segments in the coordinate plane will have equal slopes and contain no common points. |
| **2.2.4** Solve problems involving the triangle sum theorem. | **2.2.4a** The sum of the interior angles of a triangle in a plane is 180°. |
| **2.2.5** Solve problems involving two or more perpendicular lines, rays, or line segments. | **2.2.5a** A line, ray, or line segment is perpendicular to another line, ray, or line segment if and only if they form right angles at the point where the two figures intersect.<br>**2.2.5b** A line, ray, or line segment is perpendicular to another line, ray, or line segment in the coordinate plane if and only if the two figures intersect and their slopes are opposite reciprocals of each other, or if one is vertical and other is horizontal. |
| **2.2.6** Construct the perpendicular bisector of a line segment. | **2.2.6a** The perpendicular bisector of a line segment intersects the line segment at its midpoint and forms four right angles with the line segment.<br>**2.2.6b** The perpendicular bisector of a line segment is determined by identifying two points in a plane that are equidistant from the endpoints of the line segment and constructing a line, ray, or line segment through those two points.<br>**2.2.6c** Every point that lies on the perpendicular bisector of a line segment is equidistant from the endpoints of the line segment. |

| **Learning Objectives**<br>*Students will be able to ...* | **Essential Knowledge**<br>*Students need to know that ...* |
|---|---|
| **2.2.7** Construct a line, ray, or line segment perpendicular to another line, ray, or line segment. | **2.2.7a** A horizontal line, ray, or line segment in the coordinate plane is perpendicular to a vertical line, ray, or line segment if they intersect.<br>**2.2.7b** Two perpendicular lines, rays, or line segments in the coordinate plane will intersect and have slopes that are opposite reciprocals of each other, or one will be vertical and the other will be horizontal.<br>**2.2.7c** Applying the perpendicular bisector construction to a point on a line, ray, or line segment is sufficient to construct a line, ray, or line segment perpendicular to the given line, ray, or line segment. |

**Content Boundary:** Some learning objectives in this key concept require students to create both synthetic and analytic arguments for geometric relationships. Pre-AP expects students to use tools and techniques of synthetic geometry to determine, justify, or explain relationships of figures studied in a plane without coordinates and to use tools and techniques of analytic geometry to determine, justify, or explain relationships of figures studied in the coordinate plane. Students are expected to develop proficiency in both realms and to move fluently between them. However, it is not necessary that they use the terms *synthetic* and *analytic*.

**Cross Connection:** In Pre-AP Algebra 1, students made extensive use of *slope triangles* – right triangles whose legs are parallel to the axes of the coordinate plane – to calculate the slope of a non-vertical and non-horizontal line. In this course, students connect their prior knowledge of slope triangles with their understanding of geometric transformations to gain new insights into the relationships between the slopes of parallel and perpendicular lines.

## KEY CONCEPT 2.3: MEASUREMENT IN RIGHT TRIANGLES

**Using the relationships between the side lengths and angle measures of right triangles to create new measurements**

| **Learning Objectives**<br>*Students will be able to ...* | **Essential Knowledge**<br>*Students need to know that ...* |
|---|---|
| **2.3.1** Prove whether two right triangles are similar using informal similarity transformations. | **2.3.1a** Two right triangles are similar if and only if one triangle can be translated, reflected, and/or rotated so it coincides with the other after dilating one triangle by a scale factor.<br>**2.3.1b** Two right triangles are similar if and only if their corresponding angles have equal measures.<br>**2.3.1c** Two right triangles are similar if and only if their corresponding side lengths are in proportion. |
| **2.3.2** Determine the coordinates of a point on a line segment. | **2.3.2a** The coordinates of a point along a line segment in the coordinate plane that divides the line segment into a given ratio can be determined using similar triangles. |
| **2.3.3** Prove the Pythagorean theorem using similar right triangles. | **2.3.3a** An altitude drawn from the right angle of a right triangle to the hypotenuse creates similar right triangles.<br>**2.3.3b** When an altitude is constructed from the right angle to the hypotenuse of a right triangle, the proportions of the side lengths of the similar right triangles formed can be used to prove the Pythagorean theorem. |
| **2.3.4** Associate the measures of an acute angle, $\angle A$, in a right triangle to ratios of the side lengths. | **2.3.4a** The sine of the measure of $\angle A$ is the ratio of the length of the side opposite the angle and the length of the hypotenuse.<br>**2.3.4b** The cosine of the measure of $\angle A$ is the ratio of the length of the side adjacent to the angle and the length of the hypotenuse.<br>**2.3.4c** The tangent of the measure of $\angle A$ is the ratio of the length of the side opposite the angle and the length of the side adjacent to the angle. |
| **2.3.5** Explain why a trigonometric ratio depends only on an angle measure of a right triangle and not on the side lengths. | **2.3.5a** Trigonometric ratios are functions whose input is an acute angle measure and whose output is a ratio of two side lengths in a right triangle.<br>**2.3.5b** The ratio of the lengths of two sides of a right triangle will equal the ratio of the lengths of the corresponding sides of a similar right triangle. Therefore, the ratios of the sides depend only on the angle measure. |
| **2.3.6** Determine an acute angle measure in a right triangle, given a ratio of its side lengths, using an understanding of inverses. | **2.3.6a** For acute angles in a right triangle, the angle measure and the ratio of the lengths of any two specific sides have a one-to-one correspondence.<br>**2.3.6b** Given a ratio of any two side lengths in a right triangle, it is possible to determine the acute angle measures of the right triangle. |

| **Learning Objectives**<br>*Students will be able to ...* | **Essential Knowledge**<br>*Students need to know that ...* |
|---|---|
| **2.3.7** Model contextual scenarios using right triangles. | **2.3.7a** Contextual scenarios that involve nonvertical and nonhorizontal segments or the distance between two points that do not lie on a vertical or horizontal line can be modeled by right triangles.<br>**2.3.7b** Trigonometric ratios can be used to solve problems or model scenarios involving angles of elevation or depression. |

**Content Boundary:** Formally defining inverse trigonometric functions is beyond the scope of this course. However, students should understand that because each acute angle in a right triangle is uniquely associated with a specific ratio of side lengths, then a ratio of side lengths can be used to determine a specific acute angle in a right triangle. That is, students should be expected to "go forward" by determining the sine, cosine, or tangent of an acute angle and to "go backward" by determining the acute angle whose sine, cosine, or tangent ratio is given. Students are expected to use a scientific calculator to determine an angle measure, given a trigonometric ratio.

## Unit 3: Measurement in Congruent and Similar Figures

**Suggested Timing: Approximately 7 weeks**

Informal transformations are the way we, as humans, compare two objects to see if they are congruent. We turn, twist, and flip objects to see if one can lay exactly on the other without bending, stretching, or breaking either object. When they match, we say the objects are congruent. If they do not match, but they have the same shape and the same scaled measurements, we say the objects are similar. Transformations in geometry give us language to describe these turns, twists, flips, and scaling precisely and systematically. This unit formalizes the concept of congruence and similarity of planar objects by identifying the essential components of rigid motion and similarity transformations. Students are expected to become proficient with transformations that involve coordinates as well as with transformations that do not involve coordinates. Throughout the course, transformations are presented as functions. This connection further develops students' understanding of functions and connects the statistics and geometry units of the course. It also creates a bridge between Algebra 1 and Algebra 2 since concept of function permeates and links nearly all aspects of high school mathematics. Students develop further insights into congruence and similarity by exploring which transformations affect angle measures and distances between pairs of points and which do not. Students apply their understandings of transformations, congruence, and similarity to solve problems involving polygons and circles.

Throughout Units 2–4, specific learning objectives require students to prove geometric concepts. Students' proofs can be organized in a variety of formats, such as two-column tables, flowcharts, or paragraphs. The format of a student's proof is not as important as their ability to justify a mathematical claim or provide a counterexample disproving one. They should develop an understanding that a mathematical proof establishes the truth of a statement by combining previously developed truths into a logically consistent argument.

### ENDURING UNDERSTANDINGS

*Students will understand that ...*

- Transformations are functions that can affect the measurements of a geometric figure.
- Congruent figures have equal corresponding angle measures and equal distances between corresponding pairs of points.
- Similar figures have equal corresponding angle measurements, and the distances between corresponding pairs of points are proportional.
- The geometry of a circle is completely determined by its radius.

## KEY CONCEPTS

- **3.1: Transformations of points in a plane –** Defining transformations to describe the movement of points and shapes
- **3.2: Congruent and similar polygons –** Using transformations to compare figures with the same size or same shape
- **3.3: Measurement of lengths and angles in circles –** Using measurements in circles to make sense of round flat objects in the physical world

## KEY CONCEPT 3.1: TRANSFORMATIONS OF POINTS IN A PLANE

**Defining transformations to describe the movement of points and shapes**

| **Learning Objectives**<br>*Students will be able to ...* | **Essential Knowledge**<br>*Students need to know that ...* |
|---|---|
| **3.1.1** Perform transformations on points in a plane. | **3.1.1a** Transformations describe motions in the plane. Analyzing these transformations indicates if and how these motions affect lengths and angle measures of figures. Congruence and similarity are defined in terms of measurements that are preserved by transformations.<br>**3.1.1b** A transformation is a function whose inputs and outputs are points in the plane. A set of all input points of a transformation is called a preimage; a set of all output points of the preimage is called an image.<br>**3.1.1c** A rigid motion transformation preserves both the distance between pairs of points and the angle measures. A similarity transformation preserves angle measures but not necessarily distances between pairs of points. |
| **3.1.2** Express transformations using function notation. | **3.1.2a** Given a transformation $T$ and two points, $A$ and $B$, the notation $T(A) = B$ means that the image of point $A$ under transformation $T$ is point $B$. The transformation is said to map point $A$ to point $B$.<br>**3.1.2b** Algebra can be used to express how a transformation affects the $x$- and $y$-coordinates of points. All transformations can be represented using function notation, but some transformations are difficult to define as algebraic expressions. |
| **3.1.3** Prove that a rigid motion transformation maps an object to a congruent object. | **3.1.3a** A rigid motion transformation is a transformation that preserves distances between pairs of points as well as angle measures.<br>**3.1.3b** A translation is a transformation that maps each point in the plane to an image that is a specified distance in a specified direction from the preimage.<br>**3.1.3c** A reflection is a transformation that maps each point in the plane to its mirror image across a line called the axis of symmetry.<br>**3.1.3d** A rotation is a transformation that maps each point in the plane to an image that is turned by a specified angle about a fixed point called the center of rotation. |

| **Learning Objectives**<br>*Students will be able to ...* | **Essential Knowledge**<br>*Students need to know that ...* |
|---|---|
| **3.1.4** Solve problems involving rigid motion transformations. | **3.1.4a** Applying one or more translations, rotations, and reflections maps an object to a congruent object.<br>**3.1.4b** Any transformation that preserves distance between points and angle measures can be written as a sequence of translations, reflections, and/or rotations.<br>**3.1.4c** If two figures are congruent, there must exist a sequence of one or more rigid motion transformations that maps one figure to the other. |
| **3.1.5** Prove that a similarity transformation maps an object to a similar object. | **3.1.5a** A similarity transformation is a sequence of a dilation and/or one or more rigid motion transformations.<br>**3.1.5b** A dilation from a fixed point, called the center, with a scale factor $k$ is a transformation that maps each point in the plane to an image whose distance from the center is $k$ times the distance between the center and the preimage, in the same direction as the preimage.<br>**3.1.5c** Dilations of figures do not affect the angle measures of a figure. |
| **3.1.6** Solve problems involving similarity transformations. | **3.1.6a** Dilating the plane by a scale factor $k$ with center $(0, 0)$ will scale each coordinate by $k$.<br>**3.1.6b** A dilation maps a line not passing through the center of the dilation to a parallel line and maps a line passing through the center of dilation to itself.<br>**3.1.6c** The scale factor of a dilation can be determined by dividing a length from the image by its corresponding length in the preimage.<br>**3.1.6d** The perimeter of the image of a figure is the perimeter of the preimage scaled by the same scale factor as the dilation. |

**Content Boundary:** Students are expected to use algebra to express translations in the coordinate plane, reflections across the $x$-axis, the $y$-axis, and the line $y = x$, and rotations about the origin, clockwise or counterclockwise, by angles of 90° and 180°. Students are also expected to identify axes of symmetry and angles of rotation beyond those listed above. However, using algebra to express reflections across lines other than those listed, or rotations about angles other than 90° or 180° is beyond the scope of the course. It is most important that students understand that some transformations are difficult to express using algebra, but that function notation can be used to communicate the relationship between the inputs and outputs of any transformation.

## KEY CONCEPT 3.2: CONGRUENT AND SIMILAR POLYGONS

**Using transformations to compare figures with the same size or same shape**

| **Learning Objectives**<br>*Students will be able to ...* | **Essential Knowledge**<br>*Students need to know that ...* |
|---|---|
| **3.2.1** Prove that two triangles are congruent by comparing their side lengths and angle measures. | **3.2.1a** If the three sides and three angles of a triangle are congruent to the three sides and three angles of another triangle, then the two triangles are congruent.<br>**3.2.1b** If two triangles are congruent, then all six corresponding parts of the triangles are also congruent. |
| **3.2.2** Prove that two triangles are congruent by comparing specific combinations of side lengths and angle measures. | **3.2.2a** If three sides of one triangle are congruent to three sides of another triangle, then the triangles are congruent (SSS).<br>**3.2.2b** If two sides of a triangle and the interior angle they form are congruent to two sides of another triangle and the interior angle they form, then the triangles are congruent (SAS).<br>**3.2.2c** If two angles and the side adjacent to both angles of a triangle are congruent to two angles and the side adjacent to both angles in another triangle, then the triangles are congruent (ASA). |
| **3.2.3** Prove that two triangles are similar. | **3.2.3a** Two triangles are similar if and only if they have three pairs of congruent angles.<br>**3.2.3b** Two triangles are similar if and only if the lengths of their corresponding sides are in proportion.<br>**3.2.3c** Two triangles are similar if and only if one can be mapped to coincide with the other after applying a similarity transformation. |
| **3.2.4** Prove theorems about parallelograms. | **3.2.4a** Proofs about parallelograms are based on relationships among their sides, angles, and diagonals.<br>**3.2.4b** A line segment between two opposite vertices in a parallelogram forms two congruent triangles that share a common side.<br>**3.2.4c** For a parallelogram in the coordinate plane, the slopes of the sides and diagonals can be used to prove statements about the parallelogram. |

**Content Boundary:** This key concept is traditionally the major focus of high school geometry courses. It is certainly valuable that students prove theorems about congruent and similar triangles and quadrilaterals. Students are expected to use a variety of formats to construct mathematical arguments including but not limited to two-column proofs and paragraph proofs. The format of a student's proof is not as important as their ability to justify or provide a counterexample to a mathematical claim.

## KEY CONCEPT 3.3: MEASUREMENT OF LENGTHS AND ANGLES IN CIRCLES

**Using measurements in circles to make sense of round flat objects in the physical world**

| **Learning Objectives**<br>*Students will be able to ...* | **Essential Knowledge**<br>*Students need to know that ...* |
|---|---|
| **3.3.1** Determine whether a particular point lies on a given circle. | **3.3.1a** A point in the coordinate plane lies on a circle if its coordinates satisfy the equation of a circle.<br>**3.3.1b** All points that lie on a circle are equidistant from the center of the circle. |
| **3.3.2** Translate between the geometric and algebraic representations of a circle. | **3.3.2a** A circle is the set of all points equidistant from a given point.<br>**3.3.2b** In the coordinate plane, the graph of the equation $(x-h)^2+(y-k)^2=r^2$ is the set of all points located $r$ units from the point $(h, k)$. This is a circle with radius $r$ and center $(h, k)$. |
| **3.3.3** Prove that any two circles are similar. | **3.3.3a** Every circle can be expressed as the image of any other circle under a similarity transformation. |
| **3.3.4** Determine the measure of a central angle or the circular arc it intercepts. | **3.3.4a** A central angle is an angle whose vertex is the center of a circle and whose sides are, or contain, two radii of the circle.<br>**3.3.4b** The measure of an arc is defined as the measure of the central angle that intercepts the arc. |
| **3.3.5** Determine the measure of an inscribed angle or the circular arc it intercepts. | **3.3.5a** An inscribed angle is an angle whose vertex lies on a circle and whose sides contain chords of the circle.<br>**3.3.5b** The measure of an inscribed angle is half the measure of the arc it intercepts. Equivalently, the measure of the intercepted arc is twice the measure of the inscribed angle.<br>**3.3.5c** Inscribed angles that intercept the same arc have equal angle measures. |
| **3.3.6** Determine the length of a circular arc. | **3.3.6a** The length of a circular arc depends on the measure of the central angle that intercepts the arc and the radius of the circle.<br>**3.3.6b** The ratio of the length of a circular arc and the circumference of the circle is equal to the ratio of the measure of the central angle that intercepts the arc and the angle measure of a full circle. |
| **3.3.7** Construct a line, ray, or line segment tangent to a circle. | **3.3.7a** A line, ray, or line segment tangent to a circle intersects the circle at exactly one point.<br>**3.3.7b** A line, ray, or line segment tangent to a circle is perpendicular to a radius of the circle at the point of intersection.<br>**3.3.7c** In the coordinate plane, the slope of the line, ray, or line segment tangent to the circle and the slope of the radius that intersects this tangent line, ray, or line segment will be opposite reciprocals, or one will be vertical and the other will be horizontal. |

| **Learning Objectives**<br>*Students will be able to ...* | **Essential Knowledge**<br>*Students need to know that ...* |
|---|---|
| **3.3.8** Solve a system of equations consisting of a linear equation and the equation of a circle. | **3.3.8a** The intersection of a line and a circle corresponds to an algebraic solution of the system of their corresponding equations.<br>**3.3.8b** An algebraic solution to a system of equations is an ordered pair that makes all equations true simultaneously. The system may have zero, one, or two solutions. |

**Content Boundary:** It is beyond the scope of the course for students to know that a unit circle has a radius of length 1, or to know the coordinates of points on the circle that correspond to special reference angles.

**Cross Connection:** The length of a circular arc, as defined through Learning Objective 3.3.6, explicitly connects students' prior knowledge of ratios to their current study of geometry. It is more important for students to understand that an arc length is proportional to the circumference of the circle than it is for them to memorize a formula relating arc length and central angle measure.

## Unit 4: Measurement in Two and Three Dimensions

**Suggested Timing: Approximately 7 weeks**

This unit deepens students' understanding of measurement by expanding the concept of measurement to two dimensions through the areas of planar figures and to three dimensions through volumes of solid figures. One reason for studying area is that it often represents quantities that are otherwise difficult to compute. For example, the area under the graph of an object's speed corresponds to its total distance traveled. Therefore, techniques for calculating area can be adapted to find other quantities. Students likely have prior experience with calculating the areas of conventional figures and composites of those figures. Students may also have experience calculating the volumes of conventional solids. The unit introduces students to Cavalieri's principle, which relates the area of a figure to its cross-sectional lengths and the volume of a solid to its cross-sectional areas. The focus of the unit is on justifying area and volume formulas with which students are already familiar and using area and volume to model real-world physical scenarios.

Throughout Units 2-4, specific learning objectives require students to prove geometric concepts. Students' proofs can be organized in a variety of formats, such as two-column tables, flowcharts, or paragraphs. The format of a student's proof is not as important as their ability to justify a mathematical claim or provide a counterexample disproving one. They should develop an understanding that a mathematical proof establishes the truth of a statement by combining previously developed truths into a logically consistent argument.

### ENDURING UNDERSTANDINGS

*Students will understand that ...*

- The area of a figure depends on its height and its cross-sectional widths.
- The volume of a solid depends on its height and its cross-sectional areas.
- The geometry of a sphere is completely determined by its radius.

## KEY CONCEPTS

- **4.1: Area as a two-dimensional measurement –** Connecting one- and two-dimensional measurements to develop an understanding of area as a measurement of flat coverage
- **4.2: Volume as a three-dimensional measurement –** Connecting two- and three-dimensional measurements to develop an understanding of volume as a measurement of space occupied
- **4.3: Measurements of spheres –** Measuring areas and volumes of spheres to make sense of round objects in the physical world

## KEY CONCEPT 4.1: AREA AS A TWO-DIMENSIONAL MEASUREMENT

**Connecting one- and two-dimensional measurements to develop an understanding of area as a measurement of flat coverage**

| **Learning Objectives**<br>*Students will be able to ...* | **Essential Knowledge**<br>*Students need to know that ...* |
|---|---|
| **4.1.1** Use Cavalieri's principle to solve problems involving the areas of figures. | **4.1.1a** If two figures have congruent bases and equal heights, and line segments in the interiors of those figures that are parallel to, and equal distances from the base are congruent, then the figures will have equal area. |
| **4.1.2** Determine the area of a sector. | **4.1.2a** The area of a sector depends on the measure of the central angle that forms the sector and the radius of the circle.<br>**4.1.2b** The ratio of the area of a sector and the area of the circle is equal to the ratio of the measure of the central angle that forms the sector and the angle measure of a full circle. |
| **4.1.3** Determine the effect of a similarity transformation on the area of a figure. | **4.1.3a** The area of the image of a figure is the area of the preimage scaled by the square of the scale factor of the dilation. |

**Content Boundary:** Prior to this course, students will have explored the area formulas for planar figures, such as triangles, quadrilaterals, other polygons, and circles. In this course, students deepen their understanding of area by connecting length measures within a figure to its area, and by using area to solve real-world problems.

**Cross Connection:** The area of a sector, as defined through Learning Objective 4.1.2, explicitly connects students' prior knowledge of ratios to their current study of geometry. As with the length of a circular arc in Unit 3, it is more important that students understand that the area of a sector is proportional to the area inside the circle than it is for them to memorize the related formula.

## KEY CONCEPT 4.2: VOLUME AS A THREE-DIMENSIONAL MEASUREMENT

**Connecting two- and three-dimensional measurements to develop an understanding of volume as a measurement of space occupied**

| **Learning Objectives**<br>*Students will be able to ...* | **Essential Knowledge**<br>*Students need to know that ...* |
|---|---|
| **4.2.1** Justify the volume formula for a right prism. | **4.2.1a** The cross section of a right prism is the polygon formed by the intersection of the solid with a plane parallel to its base.<br>**4.2.1b** The volume of a right prism is equal to the product of the height of the solid and the area of its base. |
| **4.2.2** Justify the volume formula for pyramids. | **4.2.2a** The cross section of a pyramid is the polygon formed by the intersection of the solid with a plane parallel to its base.<br>**4.2.2b** The volume of a pyramid is equal to one-third of the volume of its associated prism. That is, the volume of a pyramid is one-third of the product of the height of the solid and the area of its base. |
| **4.2.3** Justify the volume formula for a right cylinder. | **4.2.3a** The cross section of a right cylinder is the circle formed by the intersection of the solid with a plane parallel to its base.<br>**4.2.3b** The volume of a right cylinder is equal to the product of the height of the solid and the area of its base. |
| **4.2.4** Justify the volume formula for a cone. | **4.2.4a** The cross section of a cone is the circle formed by the intersection of the solid with a plane parallel to its base.<br>**4.2.4b** The volume of a cone is equal to one-third of the volume of its associated cylinder. That is, the volume of a cone is one-third of the product of the height of the solid and the area of its base. |
| **4.2.5** Use Cavalieri's principle to solve problems involving volumes of solids. | **4.2.5a** If two solid figures have congruent bases and equal heights, and cross sections that are parallel to and equal distances from each base are congruent, then the solids have equal volume. |
| **4.2.6** Solve contextual problems involving volume of solid figures. | **4.2.6a** Physical objects in many real-world scenarios can be modeled by solid geometric figures such as prisms, pyramids, cylinders, and cones. |

**Cross Connection:** The concept of Cavalieri's principle, which students use to solve problems involving volumes of solids, connects the area of a cross section of a solid to the volume of that solid. Understanding the relationship between the area of a cross section and the volume of a solid will help students who progress to AP Calculus make sense of why finding volumes of solids is an application of integration.

## KEY CONCEPT 4.3: MEASUREMENTS OF SPHERES

**Measuring areas and volumes of spheres to make sense of round objects in the physical world**

| **Learning Objectives**<br>*Students will be able to ...* | **Essential Knowledge**<br>*Students need to know that ...* |
|---|---|
| **4.3.1** Define spheres in terms of distance. | **4.3.1a** A sphere is an object in three-dimensional space that is the set of all points equidistant from a given point, called its center. |
| **4.3.2** Justify the surface area formula for a sphere. | **4.3.2a** The surface area of a sphere is given by the formula $SA = 4\pi r^2$, where $r$ represents the length of the radius of the sphere. |
| **4.3.3** Justify the volume formula for a solid sphere. | **4.3.3a** The volume of a solid sphere is given by the formula $V = \frac{4}{3}\pi r^3$, where $r$ represents the length of the radius of the sphere. |
| **4.3.4** Solve contextual problems using spheres. | **4.3.4a** Round physical objects in real-world scenarios can be modeled by spheres. |

**Content Boundary:** It is likely that students have some familiarity with the surface area and volume formulas for a sphere. The focus of this key concept is for students to develop an informal understanding of the derivation of the surface area and volume formulas for spheres and to use spheres to model physical scenarios.

## Pre-AP Geometry with Statistics Model Lessons

Model lessons in Pre-AP Geometry with Statistics are developed in collaboration with geometry and statistics educators across the country and are rooted in the course framework, shared principles, and areas of focus. Model lessons are carefully designed to illustrate on-grade-level instruction. Pre-AP strongly encourages teachers to internalize the lessons and then offer the supports, extensions, and adaptations necessary to help all students acheive the lesson goals.

The purpose of these model lessons is twofold:

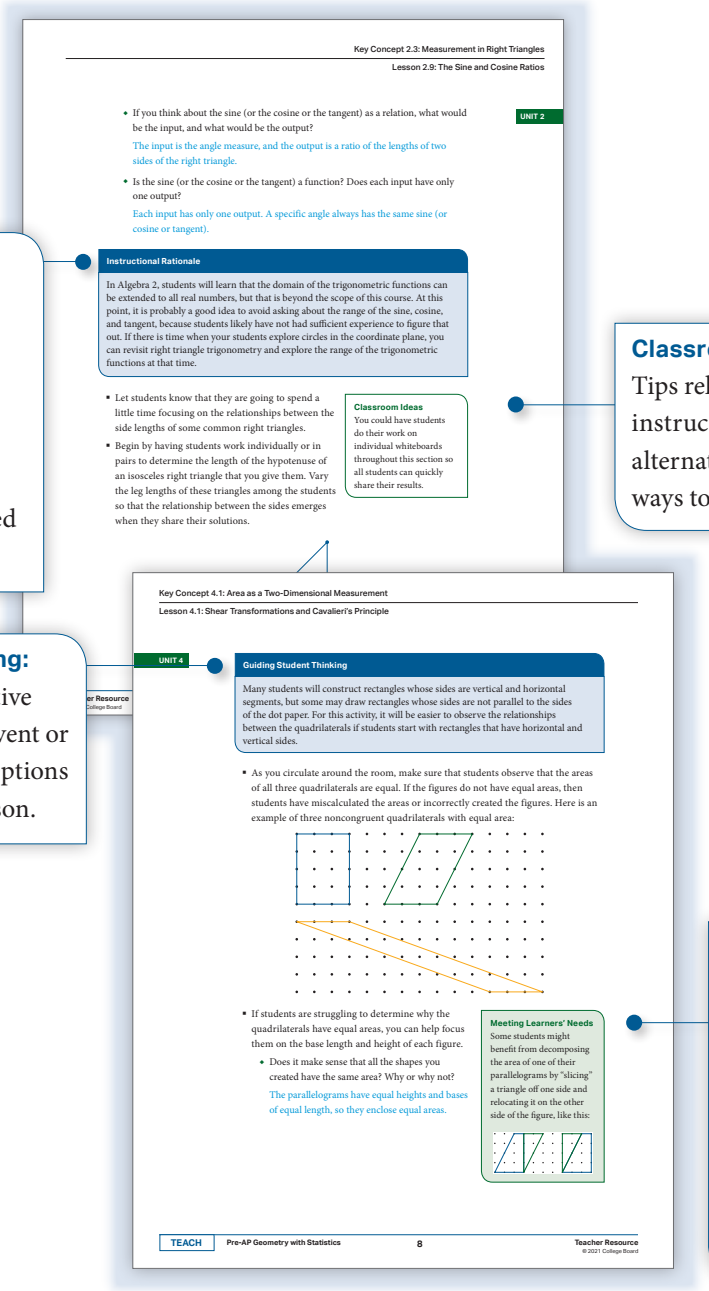
- **Robust instructional support for teachers:** Pre-AP Geometry with Statistics model lessons are comprehensive lesson plans that, along with accompanying student resources, embody the Pre-AP approach to teaching and learning. Model lessons provide clear and substantial instructional guidance to support teachers as they engage students in the shared principles and areas of focus.
- **Key instructional strategies:** Commentary and analysis embedded in each lesson highlight not just what students and teachers do in the lesson, but also how and why they do it. This educative approach provides a way for teachers to gain unique insight into key instructional moves that are powerfully aligned with the Pre-AP approach to teaching and learning. In this way, each model lesson works to support teachers in the moment of use with students in their classroom.

Teachers have the option to use any or all model lessons alongside their own locally developed instructional resources. Model lessons target content areas that tend to be challenging for teachers and students. While the lessons are distributed throughout all four units, they are concentrated more heavily in the beginning of the course to support teachers and students in establishing a strong foundation in the Pre-AP approach to teaching and learning.

## SUPPORT FEATURES IN MODEL LESSONS

The following support features recur throughout the Pre-AP Geometry with Statistics lessons, to promote teacher understanding of the lesson design and provide direct-to-teacher strategies for adapting lessons to meet their students' needs:

- Instructional Rationale
- Guiding Student Thinking
- Meeting Learners' Needs
- Classroom Ideas



Key Concept 2.3: Measurement in Right Triangles

Lesson 2.9: The Sine and Cosine Ratios

UNIT 2

- If you think about the sine (or the cosine or the tangent) as a relation, what would be the input, and what would be the output?

  The input is the angle measure, and the output is a ratio of the lengths of two sides of the right triangle.

- Is the sine (or the cosine or the tangent) a function? Does each input have only one output?

  Each input has only one output. A specific angle always has the same sine (or cosine or tangent).

**Instructional Rationale**

In Algebra 2, students will learn that the domain of the trigonometric functions can be extended to all real numbers, but that is beyond the scope of this course. At this point, it is probably a good idea to avoid asking about the range of the sine, cosine, and tangent, because students likely have not had sufficient experience to figure that out. If there is time when your students explore circles in the coordinate plane, you can revisit right triangle trigonometry and explore the range of the trigonometric functions at that time.

- Let students know that they are going to spend a little time focusing on the relationships between the side lengths of some common right triangles.
- Begin by having students work individually or in pairs to determine the length of the hypotenuse of an isosceles right triangle that you give them. Vary the leg lengths of these triangles among the students so that the relationship between the sides emerges when they share their solutions.

**Classroom Ideas**

You could have students do their work on individual whiteboards throughout this section so all students can quickly share their results.

Key Concept 4.1: Area as a Two-Dimensional Measurement

Lesson 4.1: Shear Transformations and Cavalieri's Principle

UNIT 4

**Guiding Student Thinking**

Many students will construct rectangles whose sides are vertical and horizontal segments, but some may draw rectangles whose sides are not parallel to the sides of the dot paper. For this activity, it will be easier to observe the relationships between the quadrilaterals if students start with rectangles that have horizontal and vertical sides.

- As you circulate around the room, make sure that students observe that the areas of all three quadrilaterals are equal. If the figures do not have equal areas, then students have miscalculated the areas or incorrectly created the figures. Here is an example of three noncongruent quadrilaterals with equal area:

- If students are struggling to determine why the quadrilaterals have equal areas, you can help focus them on the base length and height of each figure.
  - Does it make sense that all the shapes you created have the same area? Why or why not?

    The parallelograms have equal heights and bases of equal length, so they enclose equal areas.

**Meeting Learners' Needs**

Some students might benefit from decomposing the area of one of their parallelograms by "slicing" a triangle off one side and relocating it on the other side of the figure, like this:

TEACH Pre-AP Geometry with Statistics 8 Teacher Resource © 2021 College Board

**Instructional Rationale:** Insight into the strategic design and purpose of the instructional choices, flow, and scaffolding within the model lesson. Rationales often describe how a concept is continued later in the lesson or unit.

**Classroom Ideas:** Tips related to the logistics of the instruction, such as suggestions for alternative presentation methods or ways to alleviate pacing concerns.

**Guiding Student Thinking:** Ways to facilitate productive student thinking and prevent or address student misconceptions in critical areas of the lesson.

**Meeting Learners' Needs:** Optional differentiation strategies to address diverse learning needs, such as ideas for just-in-time skill building during a lesson or ways to break a task into smaller tasks, if needed, to make it more accessible.

# Pre-AP Geometry with Statistics Assessments for Learning

Pre-AP Geometry with Statistics assessments function as a component of the teaching and learning cycle. Progress is not measured by performance on any single assessment. Rather, Pre-AP Geometry with Statistics offers a place to practice, to grow, and to recognize that learning takes time. The assessments are updated and refreshed periodically.

## LEARNING CHECKPOINTS

Based on the Pre-AP Geometry with Statistics Course Framework, the learning checkpoints require students to examine data, models, diagrams, and short texts—set in authentic contexts—in order to respond to a targeted set of questions that measure students' application of the key concepts and skills from the unit. All eight learning checkpoints are automatically scored, with results provided through feedback reports that contain explanations of all questions and answers as well as individual and class views for educators. Teachers also have access to assessment summaries on Pre-AP Classroom, which provide more insight into the question sets and targeted learning objectives for each assessment event.

The following tables provide a synopsis of key elements of the Pre-AP Geometry with Statistics learning checkpoints.

| | |
|---|---|
| **Format** | Two learning checkpoints per unit<br>Digitally administered with automated scoring and reporting<br>Questions target both concepts and skills from the course framework |
| **Time Allocated** | Designed for one 45-minute class period per assessment |
| **Number of Questions** | 10–12 questions per assessment<br>▪ 7–9 four-option multiple choice<br>▪ 3–5 technology-enhanced questions |

| Domains Assessed | |
|---|---|
| Learning Objectives | Learning objectives within each key concept from the course framework |
| Skills | Three skill categories aligned to the Pre-AP mathematics areas of focus are assessed regularly across all eight learning checkpoints:<br>▪ greater authenticity of applications and modeling<br>▪ engagement in mathematical argumentation<br>▪ connections among multiple representations |

| | |
|---|---|
| **Question Styles** | Question sets consist of two to three questions that focus on a single stimulus or group of related stimuli, such as diagrams, graphs, or tables.<br>Questions embed mathematical concepts in real-world contexts.<br>*Please see page 64 for a sample question set that illustrates the types of questions included in Pre-AP learning checkpoints and the Pre-AP final exam.* |

## PERFORMANCE TASKS

Each unit includes one performance-based assessment designed to evaluate the depth of student understanding of key concepts and skills that are not easily assessed in a multiple-choice format.

These tasks, developed for high school students across a broad range of readiness levels, are accessible while still providing sufficient challenge and the opportunity to practice the analytical skills that will be required in AP mathematics courses and for college and career readiness. Teachers participating in the official Pre-AP Program will receive access to online learning modules to support them in evaluating student work for each performance task.

| | |
|---|---|
| **Format** | One performance task per unit<br>Administered in print<br>Educator scored using scoring guidelines |
| **Time Allocated** | Approximately 45 minutes or as indicated |
| **Number of Questions** | An open-response task with multiple parts |

| Domains Assessed | |
|---|---|
| Key Concepts | Key concepts and prioritized learning objectives from the course framework |
| Skills | Three skill categories aligned to the Pre-AP mathematics areas of focus:<br>▪ greater authenticity of applications and modeling<br>▪ engagement in mathematical argumentation<br>▪ connections among multiple representations |

## PRACTICE PERFORMANCE TASKS

Practice performance tasks in each unit provide students with the opportunity to practice applying skills and knowledge in a context similar to a performance task, but in a more scaffolded environment. These tasks include strategies for adapting instruction based on student performance and ideas for modifying or extending tasks based on students' needs.

### Performance Assessments At-a-Glance

| Unit | Performance Assessment | Title | Teacher Access | Student Access |
|---|---|---|---|---|
| Unit 3 Measurement in Congruent and Similar Figures | Practice Performance Task | Transformations in the Coordinate Plane | Teacher Resources: Units 3 & 4 | Student Resources: Unit 3 |
| | Performance Task | Olga's Walkie-Talkie | | Teacher-distributed handout |
| Unit 4 Measurement in Two and Three Dimensions | Practice Performance Task | Digging a Ditch | Teacher Resources: Units 3 & 4 | Student Resources: Unit 4 |
| | Performance Task | Star Energy | | Teacher-distributed handout |

## FINAL EXAM

Pre-AP Geometry with Statistics includes a final exam featuring multiple-choice and technology-enhanced questions as well as an open-response question. The final exam is a summative assessment designed to measure students' success in learning and applying the knowledge and skills articulated in the Pre-AP Geometry with Statistics Course Framework. The final exam's development follows best practices such as multiple levels of review by educators and experts in the field for content accuracy, fairness, and sensitivity. The questions on the final exam have been pretested, and the resulting data are collected and analyzed to ensure that the final exam is fair and represents an appropriate range of the knowledge and skills of the course.

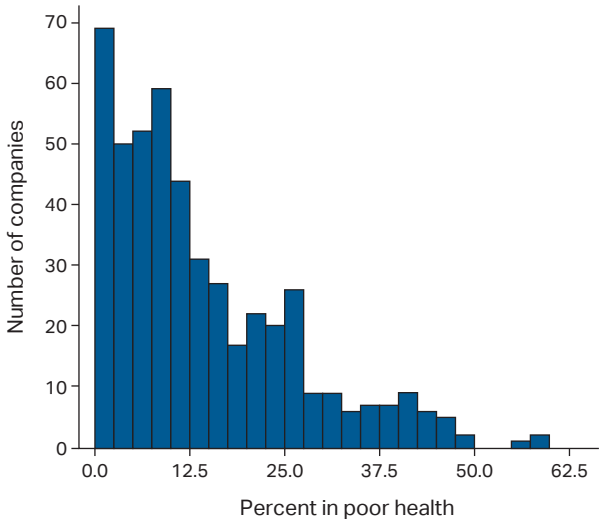
The final exam is designed to be delivered on a secure digital platform in a classroom setting. Educators have the option of administering the final exam in a single extended session or two shorter consecutive sessions to accommodate a range of final exam schedules.

Multiple-choice and technology-enhanced questions are delivered digitally and scored automatically with detailed score reports available to educators. This portion of the final exam is designed to build on the question styles and formats of the learning checkpoints; thus, in addition to their formative purpose, the learning checkpoints provide practice and familiarity with the final exam. The open-response question, modeled after the performance tasks, is delivered as part of the digital final exam but is designed to be scored separately by educators using scoring guidelines that are designed and vetted with the question.

## SAMPLE ASSESSMENT QUESTIONS

The following questions are representative of what students and educators will encounter on the learning checkpoints and final exam.

1.




A historian was interested in researching the health of Virginia soldiers near the start of the American Civil War. The data collected for each company, or small group, of soldiers were the percentage of soldiers in each company that were in poor health. A histogram of the data is shown in the figure.

Which of the following statements is most consistent with the distribution of soldiers in poor health?

(A) The total number of companies that reported a percent of soldiers in poor health that was above the mean is equal to the total number of companies that reported a percent of soldiers in poor health below the mean.

(B) The mean and median values for percent of soldiers in poor health are similar.

(C) The variance and the interquartile range of values for percent of soldiers in poor health are similar.

(D) The median value for the percent of soldiers in poor health is less than the mean value.

**Assessment Focus**

In question 1, students must analyze the distribution of data to determine the relationship between the mean and median based on the skew in the distribution.

**Correct Answer: D**

**Learning Objective:**

**1.1.5** Analyze data distributions with respect to their variability.

**Area of Focus:** Greater Authenticity of Applications and Modeling

2.


Two friends, Matt and Kylie, use different skateboard ramps at their local skate park, as represented in the diagram. They want to know which ramp is steeper. Matt says his ramp has a vertical support (shown) that is 3-feet tall and a ramp length of 5 feet. Kylie measured the angle that the bottom of her ramp made with the ground (angle A) and found it to be 40°.

Based on the known measurements, which ramp is steeper?

(A) Matt's ramp has a slope of $\frac{3}{5}$, which is greater than Kylie's ramp slope of tan(40°).

(B) Matt's ramp has a slope of $\frac{3}{4}$, which is greater than Kylie's ramp slope of sin(40°).

(C) Kylie's ramp has a slope of cos(40°), which is greater than Matt's ramp slope of $\frac{3}{5}$.

(D) Kylie's ramp has a slope of tan(40°), which is greater than Matt's ramp slope of $\frac{3}{4}$.

**Assessment Focus**

Question 2 requires students to apply the Pythagorean theorem and use the tangent of an angle to determine and compare the slopes of two different segments. Students use these slopes as evidence to support a claim.

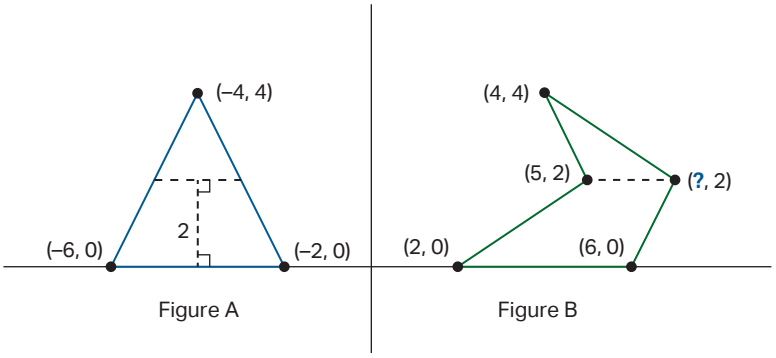
**Correct Answer: D**

**Learning Objective:**

**2.3.4** Associate the measures of an acute angle, $\angle A$, in a right triangle to ratios of the side lengths.

**Area of Focus:** Engagement in Mathematical Argumentation

3. The following two figures have the same area.




In the diagram of Figure B, one $x$-coordinate is represented with a question mark (?). Sahir claims that this missing $x$-coordinate is 7. Which of the following reasons best supports his claim?

(A) The horizontal distance between points on the sides of Figure B is 2, therefore the missing $x$-coordinate is 2 more than 5.

(B) The horizontal distances at the same height between two points of Figure A and two points of Figure B are equal.

(C) The slope of the line segment whose endpoints are (6, 0) and (7, 2) is 2.

(D) If points on Figures A and B have the same $y$-coordinate, then the points will have the same horizontal distance from the $y$-axis.

**Assessment Focus**

Question 3 requires students to apply Cavalieri's principle to support a claim about a missing coordinate in a shape.

**Correct Answer: B**

**Learning Objective:**

**4.1.1** Use Cavalieri's principle to solve problems involving the areas of figures.

**Area of Focus:** Engagement in Mathematical Argumentation

# Pre-AP Geometry with Statistics Course Designation

Schools can earn an official Pre-AP Geometry with Statistics course designation by meeting the requirements summarized below. Pre-AP Course Audit Administrators and teachers will complete a Pre-AP Course Audit process to attest to these requirements. All schools offering courses that have received a Pre-AP Course Designation will be listed in the Pre-AP Course Ledger, in a process similar to that used for listing authorized AP courses.

### PROGRAM REQUIREMENTS

- The school ensures that Pre-AP frameworks and assessments serve as the foundation for all sections of the course at the school. This means that the school must not establish any barriers (e.g., test scores, grades in prior coursework, teacher or counselor recommendation) to student access and participation in the Pre-AP Geometry with Statistics coursework.
- Teachers have read the most recent *Pre-AP Geometry with Statistics Course Guide*.
- Teachers administer each performance task and at least one of two learning checkpoints per unit.
- Teachers and at least one administrator per site complete a Pre-AP Summer Institute or the Online Foundational Module Series. Teachers complete at least one Online Performance Task Scoring Module.
- Teachers align instruction to the Pre-AP Geometry with Statistics Course Framework and ensure their course meets the curricular requirements summarized below.
- The school ensures that the resource requirements summarized below are met.

### CURRICULAR REQUIREMENTS

- The course provides opportunities for students to develop understanding of the Pre-AP Geometry with Statistics key concepts and skills articulated in the course framework through the four units of study.
- The course provides opportunities for students to engage in the Pre-AP shared instructional principles.
  - close observation and analysis
  - evidence-based writing
  - higher-order questioning
  - academic conversation

- The course provides opportunities for students to engage in the three Pre-AP mathematics areas of focus. The areas of focus are:
  - greater authenticity of applications and modeling
  - engagement in mathematical argumentation
  - connections among multiple representations
- The instructional plan for the course includes opportunities for students to continue to practice and develop disciplinary skills.
- The instructional plan reflects time and instructional methods for engaging students in reflection and feedback based on their progress.
- The instructional plan reflects making responsive adjustments to instruction based on student performance.

**RESOURCE REQUIREMENTS**

- The school ensures that participating teachers and students are provided computer and internet access for completion of course and assessment requirements.
- Teachers should have consistent access to a video projector for sharing web-based instructional content and short web videos.

## Accessing the Digital Materials

Pre-AP Classroom is the online application through which teachers and students can access Pre-AP instructional resources and assessments. The digital platform is similar to AP Classroom, the online system used for AP courses.

Pre-AP coordinators receive access to Pre-AP Classroom via an access code delivered after orders are processed. Teachers receive access after the Pre-AP Course Audit process has been completed.

Once teachers have created course sections, student can enroll in them via access code. When both teachers and students have access, teachers can share instructional resources with students, assign and score assessments, and complete online learning modules; students can view resources shared by the teacher, take assessments, and receive feedback reports to understand progress and growth.

# Unit 3

# Unit 3
# Measurement in Congruent and Similar Figures

## Overview

**SUGGESTED TIMING: APPROXIMATELY 7 WEEKS**

Informal transformations are the way we, as humans, compare two objects to see if they are congruent. We turn, twist, and flip objects to see if one can lay exactly on the other without bending, stretching, or breaking either object. When they match, we say the objects are congruent. If they do not match, but they have the same shape and the same scaled measurements, we say the objects are similar. Transformations in geometry give us language to describe these turns, twists, flips, and scaling precisely and systematically. This unit formalizes the concept of congruence and similarity of planar objects by identifying the essential components of rigid motion and similarity transformations. Students are expected to become proficient with transformations that involve coordinates as well as with transformations that do not involve coordinates. Throughout the course, transformations are presented as functions. This connection further develops students' understanding of functions and connects the statistics and geometry units of the course. It also creates a bridge between Algebra 1 and Algebra 2 since concept of function permeates and links nearly all aspects of high school mathematics. Students develop further insights into congruence and similarity by exploring which transformations affect angle measures and distances between pairs of points and which do not. Students apply their understandings of transformations, congruence, and similarity to solve problems involving polygons and circles.

Throughout Units 2-4, specific learning objectives require students to prove geometric concepts. Students' proofs can be organized in a variety of formats, such as two-column tables, flowcharts, or paragraphs. The format of a student's proof is not as important as their ability to justify a mathematical claim or provide a counterexample disproving one. They should develop an understanding that a mathematical proof establishes the truth of a statement by combining previously developed truths into a logically consistent argument.

UNIT 3

## ENDURING UNDERSTANDINGS

This unit focuses on the following enduring understandings:

- Transformations are functions that can affect the measurements of a geometric figure.
- Congruent figures have equal corresponding angle measures and equal distances between corresponding pairs of points.
- Similar figures have equal corresponding angle measurements, and the distances between corresponding pairs of points are proportional.
- The geometry of a circle is completely determined by its radius.

## KEY CONCEPTS

This unit focuses on the following key concepts:

- 3.1: Transformations of Points in a Plane
- 3.2: Congruent and Similar Polygons
- 3.3: Measurement of Lengths and Angles in Circles

## UNIT RESOURCES

The tables below outline the resources provided by Pre-AP for this unit.

**Lessons for Key Concept 3.1: Transformations of Points in a Plane**

| Lesson Title | Learning Objectives Addressed | Essential Knowledge Addressed | Suggested Timing | Areas of Focus |
|---|---|---|---|---|
| Lesson 3.1: Symmetries of Objects | 3.1.3 | 3.1.3a, 3.1.3b, 3.1.3c, 3.1.3d | ~45 minutes | Greater Authenticity of Applications and Modeling, Engagement in Mathematical Argumentation |
| Lesson 3.2: Rigid Motions Without Coordinates | 3.1.1, 3.1.2, 3.1.4 | 3.1.1a, 3.1.1b, 3.1.1c, 3.1.2a, 3.1.2b, 3.1.4a, 3.1.4b, 3.1.4c | ~45 minutes | Connections Among Multiple Representations, Engagement in Mathematical Argumentation |

| | | | | |
|---|---|---|---|---|
| Lesson 3.3: Rigid Motions in the Coordinate Plane | 3.1.1, 3.1.2, 3.1.3, 3.1.4 | 3.1.1a, 3.1.1b, 3.1.1c, 3.1.2a, 3.1.2b, 3.1.3a, 3.1.3b, 3.1.3c, 3.1.3d, 3.1.4a, 3.1.4b, 3.1.4c | ~90 minutes | Engagement in Mathematical Argumentation, Connections Among Multiple Representations |
| Lesson 3.4: Dilations | 3.1.5, 3.1.6 | 3.1.5a, 3.1.5b, 3.1.5c, 3.1.6a, 3.1.6b, 3.1.6c | ~90 minutes | Engagement in Mathematical Argumentation, Connections Among Multiple Representations |
| Lesson 3.5: Defining Congruence and Similarity Through Transformation | 3.1.1, 3.1.3, 3.1.4, 3.1.5, 3.1.6 | 3.1.1a, 3.1.1b, 3.1.1c, 3.1.3a, 3.1.3b, 3.1.3c, 3.1.3d, 3.1.4a, 3.1.4b, 3.1.4c, 3.1.5a, 3.1.5b, 3.1.5c, 3.1.6a, 3.1.6c, 3.1.6d | ~45 minutes | Engagement in Mathematical Argumentation |
| | All learning objectives and essential knowledge statements for this key concept are addressed with the provided materials. | | | |

**Practice Performance Task for Unit 3 (~45 minutes)**

This practice performance task assesses learning objectives and essential knowledge statements addressed up to this point in the unit.

**Lessons for Key Concept 3.2: Congruent and Similar Polygons**

There are no provided Pre-AP lessons for this key concept. As with all key concepts, this key concept is addressed in a learning checkpoint.

**Learning Checkpoint 1: Key Concepts 3.1–3.2 (~45 minutes)**

This learning checkpoint assesses learning objectives and essential knowledge statements from Key Concepts 3.1 and 3.2. For sample items and learning checkpoint details, visit Pre-AP Classroom.

UNIT 3

| Lessons for Key Concept 3.3: Measurement of Lengths and Angles in Circles | | | | |
|---|---|---|---|---|
| **Lesson Title** | **Learning Objectives Addressed** | **Essential Knowledge Addressed** | **Suggested Timing** | **Areas of Focus** |
| Lesson 3.6: The Equation of a Circle | 3.3.1, 3.3.2 | 3.3.1a, 3.3.1b, 3.3.2a, 3.3.2b | ~90 minutes | Connections Among Multiple Representations, Engagement in Mathematical Argumentation |
| Lesson 3.7: Intersections of Circles and Lines | 3.3.8 | 3.3.8a, 3.3.8b | ~45 minutes | Connections Among Multiple Representations, Greater Authenticity of Applications and Modeling |
| Lesson 3.8: Lines Tangent to a Circle | 3.3.7, 3.3.8 | 3.3.7a, 3.3.7b, 3.3.7c, 3.3.8a, 3.3.8b | ~45 minutes | Connections Among Multiple Representations, Greater Authenticity of Applications and Modeling |
| | The following Key Concept 3.3 learning objectives and essential knowledge statements are not addressed in Pre-AP lessons. Address these in teacher-developed materials.<br>▪ Learning Objectives: 3.3.3, 3.3.4, 3.3.5, 3.3.6<br>▪ Essential Knowledge Statements: 3.3.3a, 3.3.4a, 3.3.4b, 3.3.5a, 3.3.5b, 3.3.5c, 3.3.6a, 3.3.6b | | | |

**Learning Checkpoint 2: Key Concept 3.3 (~45 minutes)**

This learning checkpoint assesses learning objectives and essential knowledge statements from Key Concept 3.3. For sample items and learning checkpoint details, visit Pre-AP Classroom.

**Performance Task for Unit 3 (~45 minutes)**

This performance task assesses learning objectives and essential knowledge statements from the entire unit.

UNIT 3

LESSON 3.1

# Symmetries of Objects

## OVERVIEW

### LESSON DESCRIPTION

**Part 1: Exploring Symmetry in Logos**
Students explore various corporate logos and symbols to identify the types of reflection and rotational symmetries they exhibit.

**Part 2: Investigating Tessellations**
Students explore translation symmetry through contextual examples of tessellations of the plane.

**Part 3: Defining Symmetries**
Students formally define reflection, rotational, and translation symmetries using a Frayer graphic organizer.

### CONTENT FOCUS

This lesson utilizes the symmetry of familiar figures, such as corporate logos and tile floor patterns, to formalize rigid motion transformations. Earlier in the course, students made use of reflections, rotations, and translations when they explained why two segments or angles were congruent. These three transformations are used in later lessons of this unit as the basis for a definition of congruence. The goal of this lesson is for students to identify which transformations can be performed on an object so that it appears unchanged.

### AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Engagement in Mathematical Argumentation

### SUGGESTED TIMING

~45 minutes

### LESSON SEQUENCE

- This lesson is part of a lesson sequence (~315 minutes) that includes Lessons 3.1 through 3.5.

### MATERIALS

- protractor
- straightedge or ruler

### HANDOUTS

**Lesson**

- 3.1.A: Logos
- 3.1.B: Tessellations of the Plane
- 3.1.C: Vocabulary Graphic Organizer

## COURSE FRAMEWORK CONNECTIONS

| Enduring Understandings | |
|---|---|
| ▪ Transformations are functions that can affect the measurements of a geometric figure. | |
| **Learning Objectives** | **Essential Knowledge** |
| **3.1.3** Prove that a rigid motion transformation maps an object to a congruent object. | **3.1.3a** A rigid motion transformation is a transformation that preserves distances between pairs of points as well as angle measures.<br>**3.1.3b** A translation is a transformation that maps each point in the plane to an image that is a specified distance in a specified direction from the preimage.<br>**3.1.3c** A reflection is a transformation that maps each point in the plane to its mirror image across a line called the axis of symmetry.<br>**3.1.3d** A rotation is a transformation that maps each point in the plane to an image that is turned by a specified angle about a fixed point called the center of rotation. |

UNIT 3

## FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

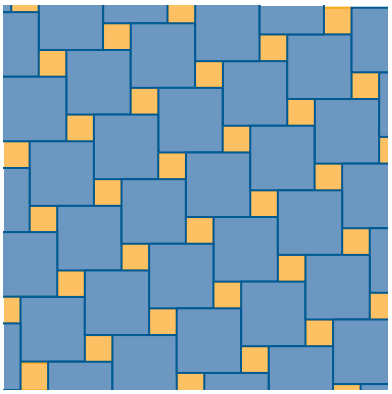
For each figure below, write two sentences that describe the symmetry of the figure, your evidence that the figure has that type of symmetry, and the important characteristics of the symmetry you observe.

(a)

(Alamy)

(b)

(c)



## PART 1: EXPLORING SYMMETRY IN LOGOS

The lesson begins with students examining various logos and symbols and thinking about the kinds of symmetry each exhibits. This part of the lesson focuses on rotational and reflection symmetries. Having grown up in a consumer culture, most students will have developed a familiarity with a wide variety of corporate logos. This familiarity provides an opportunity for students to connect geometric concepts with their real-world experiences.

The activity in this lesson is not intended to encourage the use of any particular brand, product, or service. Instead, it invites students to think deeper about the symbols that surround them. Every logo, symbol, and icon is designed to be visually unique and to relay specific information. In this activity, students can closely observe and analyze these symbols through a geometric lens to identify any patterns they contain.

- You can begin this part of the lesson by asking students to engage in a quickwrite in which they write for two or three minutes about what it means for an object to be symmetric. After students have some time to write, ask a few students to share their thoughts.

  Responses will vary. It is likely that students will state that some objects are symmetric because one part of the object is a mirror image of another part of the object. Some students might recognize that other objects are symmetric because they look the same when they are rotated.

**Instructional Rationale**

Quickwrite activities are a good way to encourage writing in math classes. They provide students with time to gather and organize their thoughts on paper before sharing. Students who are usually reserved in class discussions can rely on what they wrote to contribute to the discussion.

- After students have shared their thoughts about symmetry, summarize their responses. The main goal is for students to understand that an object exhibits symmetry if it appears unchanged after a transformation.
- Take some time to show the class some examples of symmetric objects in nature, such as the butterfly and snowflake shown on the next page. You can invite students to share other examples of symmetries they have observed in their own experiences, such as in nature, art, or architecture.

(Getty Images / DEA / M. Borchi)

(Getty Images / Fred Widall)

- You can use the image of the butterfly to allow students to explain how it exhibits symmetry. Students should identify that the left and right sides of the butterfly are mirror images.
- Explain to students that an object has *reflection symmetry* if there is a line, called an *axis of symmetry*, about which either side can be reflected and the object appears unchanged.
- You can use the snowflake image to have students think about and discuss how it exhibits a different kind of symmetry than the butterfly image. Students may quickly notice its reflection symmetry over six different axes of symmetry. You may have to direct them to think about rotating the snowflake. Students should identify that the snowflake looks the same when it is rotated by 60°, clockwise or counterclockwise.
- Explain to students that an object has *rotational symmetry* if there is a point, called the *center of rotation*, about which the object can be rotated by a specific angle and appear unchanged.
- Distribute **Handout 3.1.A: Logos** and have students work with a partner to identify what kind of symmetry, if any, each logo or symbol exhibits. Students should identify an axis of symmetry if the logo or symbol has reflection symmetry and a center of rotation if the object has rotational symmetry. You can distribute protractors so

**Meeting Learners' Needs**

If students struggle with identifying the location of an axis of symmetry, you can ask them to think about how they could fold the logo so its sides would match exactly. If students struggle to identify an angle of rotation, you can have them cut out and manually turn a copy of the logo until it looks like the original logo. Sometimes drawing several rays from the center of rotation through points that would match up before and after a rotation can help students identify the angle of rotation.

students can identify one or more angles of rotation for logos or symbols that exhibit rotational symmetry.

**Guiding Student Thinking**

Students might recognize that all of the logos and symbols have 360° rotational symmetry. You can acknowledge that fact and then encourage them to find angles of rotation that are less than 360° that leave the logo or symbol unchanged.

- As you circulate around the room while students work, encourage them to use proper vocabulary, like "rotate about" and "reflect over" rather than "turn" and "flip."
- To debrief this part of the lesson, have pairs of students share with the class a symmetry they observed in a logo or symbol, along with its axis of symmetry or angle of rotation. For logos and symbols that have more than one type of symmetry, angle of rotation, or axis of symmetry, have another pair of students identify a different symmetry they observed in the logo or symbol along with its axis of symmetry or angle of rotation. Allow time for students to disagree with or challenge each other's claims. This discussion can encourage academic conversation that provides opportunities for students to critique each other's reasoning and also to get insight into another student's point of view.

**Meeting Learners' Needs**

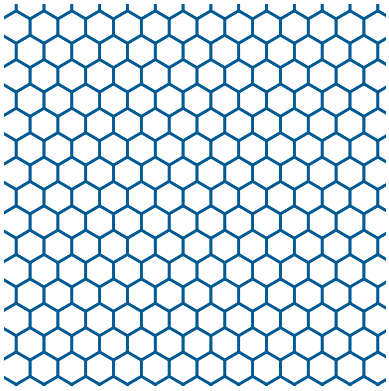
Many of the logos and symbols on **Handout 3.1.A: Logos** exhibit both rotational and reflection symmetry. You can challenge students to find as many symmetries as possible.

## PART 2: INVESTIGATING TESSELLATIONS

In this part of the lesson, students investigate translations by examining tessellations of a plane. A tessellation is a pattern of repeating polygons or other planar shapes that completely cover a plane. The tessellations that students examine in this part of the lesson can be achieved by only translating a polygonal tile. More complex tessellations, especially those that appear in art, often make use of rotations or reflections as well as translations.

- Let students know that there is another type of symmetry they will explore. Display for students the photograph of a bee's honeycomb, as well as the diagram of the geometric representation of a honeycomb. You can explain to students that these repeating, nonoverlapping patterns are called *tessellations*. Mathematically, the pattern is assumed to extend indefinitely in all four directions.

(Getty Images / Frank Bienewald)

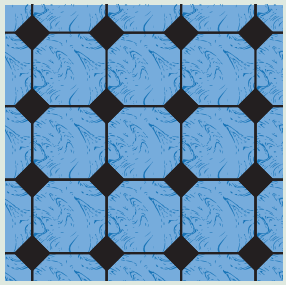


- Ask students to closely observe and analyze the images.
  - What do you notice? What do you wonder? How are these images the same as, and different from, the logos we explored earlier?
- These images provide another opportunity for a quickwrite activity, allowing students sufficient time to process their thoughts. Once students have had a few minutes to think or to write, ask for volunteers to share what they notice and what they wonder.

  Students may observe that each image is composed of many identical hexagons and that each exhibits both reflection and rotational symmetries. Students may wonder how big the bees' honeycomb is, or how deep each of the hexagonal cells is. Students may identify that these images are similar because they exhibit the same types of symmetry and are made of many copies of the same figure (a regular hexagon).
- You may have to direct students' attention to observe that the images also exhibit a different kind of symmetry from any of the corporate logos. If the hexagon pattern continues indefinitely in all directions, then it will appear unchanged when it is moved up, down, left, or right. This kind of symmetry is called *translation symmetry*. When a figure exhibits translation symmetry, there is a single shape that is the basis for the repeating pattern. In the honeycomb figure, the pattern is formed by translating a hexagon.

**Meeting Learners' Needs**

Some students may struggle to identify the translated shape in some of the more complex tessellations, because some of the repeated shapes are a combination of simpler shapes, like in this tile floor common in many bathrooms:



This pattern can be thought of as a tessellation of an octagon adjacent to a square:

- Distribute **Handout 3.1.B: Tessellations of the Plane**, and have students work with a partner to identify the shape that was translated to form each repeated pattern. You can let students know that when a shape is used to cover a flat surface with no gaps or overlapping regions, it is described mathematically as a tessellation.
- As you circulate around the room while students work, look for student pairs who explained that the shape could be rotated to form the figure. While some of the patterns on **Handout 3.1.B: Tessellations of the Plane** can be thought of as tessellations formed using rotations, every pattern on the handout can be formed using only translation. Those students who state that the patterns can be formed using rotation are not incorrect, but they have not completed the task using the given guidelines. Challenge them to find a shape that makes each pattern without having to rotate it. Often they will have to identify a more complex base figure.
- To debrief this part of the lesson, have student pairs share the repeated shape they identified as the one used to form each pattern. There might be an opportunity for students to engage in an academic conversation if pairs of students identified different repeated shapes for the same pattern.

## PART 3: DEFINING SYMMETRIES

In this part of the lesson, students formally define rotations, reflections, and translations. These transformations form the basis of the definition of congruence that students will develop later in the unit.

- Let students know that they will be creating a graphic organizer for each of the three transformations (rotations, reflections, and translations) they explored in the lesson. For each graphic organizer, they will need to write a definition, identify important characteristics, and provide examples and nonexamples of the transformation.

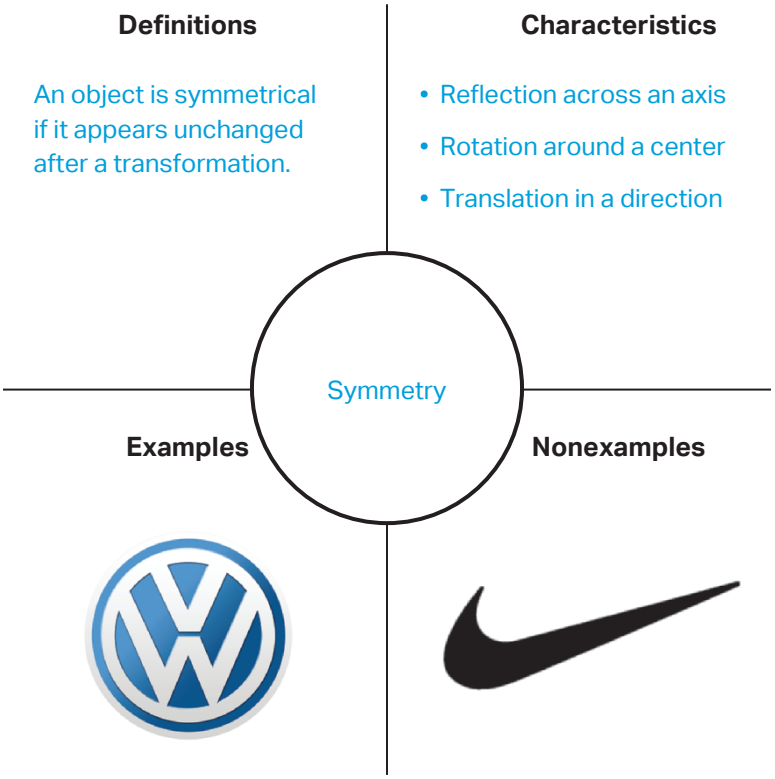
**Classroom Ideas**

You can use a Frayer model organizer like the one shown on **Handout 3.1.C: Vocabulary Graphic Organizer**, or you can have students fold standard pieces of paper horizontally and vertically to create their own versions of the model's four quadrants.

**Guiding Student Thinking**

Some students may want to use this activity to define the symmetries rather than the transformations they explored in the class. In general, people have an intuitive sense of symmetry. Mathematics is the tool that we use to quantify the symmetries of an object. An object is symmetrical if there is a transformation that maps the object to itself. So, it is critical to distinguish between the transformations that appear to leave an object unchanged from those that do not.

- If you think it will be helpful to model for the class, you can create a Frayer organizer for "symmetry" like the one shown here:




(Alamy)

Have students work in pairs to write first drafts of their graphic organizers. When most pairs have written their first drafts, have student pairs form groups of four. Each group of four should critique their first drafts and make decisions about what to keep and what to revise for a final version.

- As you circulate around the room while students work, look for particularly good definitions that students have developed to share with the class. Also look for opportunities to help students revise their definitions to eliminate any misconceptions or mistakes.
- Be sure that students address—either in their definitions or in the characteristics of each transformation—the necessary information for each transformation. That is, a reflection requires

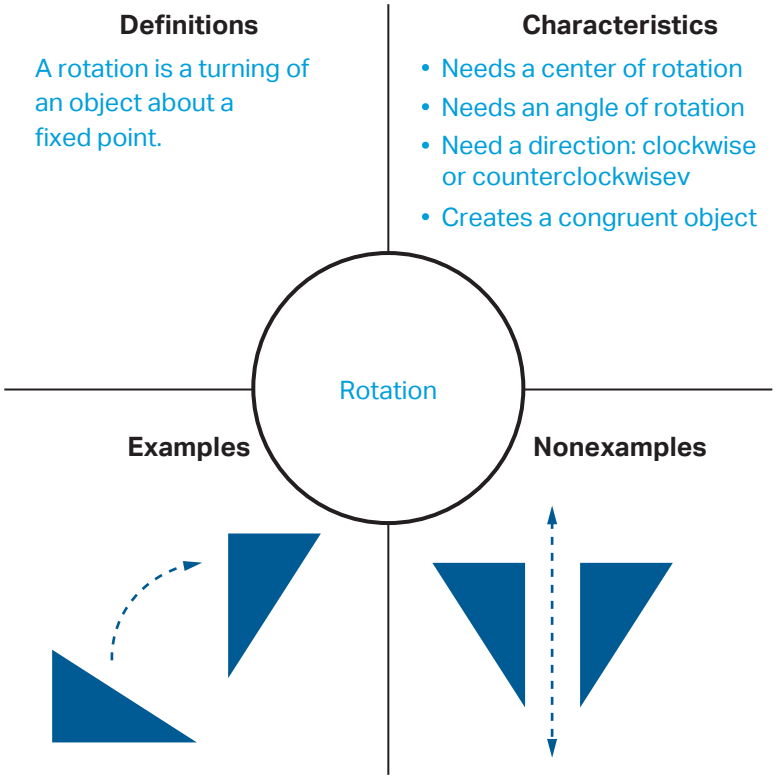
**Meeting Learners' Needs**
It might not be obvious to students that translations require both a direction and distance. You may have to point out to students that the tessellations they explored translated the shape repeatedly by an exact distance vertically and/or horizontally to precisely align the sides of the figure.

an axis of symmetry, a rotation requires a center and an angle of rotation, and a translation requires a direction and a distance. This information will be crucially important for students as they produce or verify congruent shapes with rigid motion transformations in future lessons.

**Instructional Rationale**

It is important for students to express their own understanding about the meanings of the terms rotation, reflection, and translation. You can use this activity as an opportunity to formatively assess what students do or do not know about these transformations. This allows you to correct students' misconceptions in a low-risk environment before they participate in more formal assessments.

- Students should revise their vocabulary organizer as a culminating activity for the lesson. If you have a word wall in your classroom, you can post your own version of each graphic organizer.



## Reflection

**Definitions**

A reflection is a flipping of an object to create a mirror image

**Characteristics**

- Needs an axis of symmetry
- Creates a congruent object

**Examples**

**Nonexamples**

## Translation

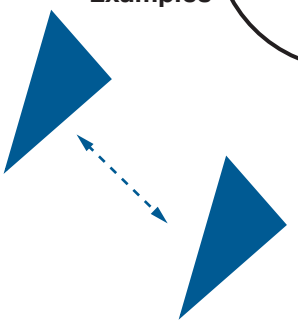
**Definitions**

A translation is a sliding of an object in the plane without turning or flipping it.

**Characteristics**

- Needs a direction and a distance
- Creates a congruent object

**Examples**



**Nonexamples**

## ASSESS AND REFLECT ON THE LESSON

### FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

For each figure below, write two sentences that describe the symmetry of the figure, your evidence that the figure has that type of symmetry, and the important characteristics of the symmetry you observe.

(a)

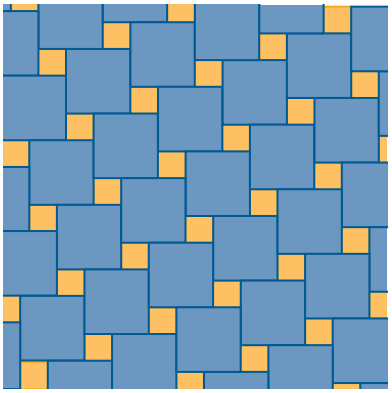
The figure has reflection symmetry because the left and right sides can be flipped and the resulting figure will be congruent to the original. The axis of symmetry for this figure is a vertical line through the center of the figure.

(b)

The figure has rotational symmetry because the figure can be rotated so that the resulting figure will be congruent to the original. The center of rotation is the center of the figure, and the angle of rotation is 120°.

**Guiding Student Thinking**

Encourage students who suggest that this figure has reflection symmetry, which is not true, to also write about the rotational symmetry of the figure.

(c) 

The figure has translation symmetry because it can be slid in several directions so that the resulting figure will be congruent to the original. Copies of one shape made by a small square and an adjacent large square tile are repeatedly slid to fit together to create the entire pattern.

## HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

### Handout 3.1.A: Logos

1. Reflection symmetry over a vertical line through the center
2. Reflection symmetry over a vertical line through the center
3. No symmetry
4. Reflection symmetry over a line through any pair of opposite vertices or through the midpoints of any pair of opposite sides; rotational symmetry about the center through an angle of 45°
5. No symmetry
6. No symmetry
7. No symmetry
8. Reflection symmetry over a vertical line through the center
9. Reflection symmetry over every line through the center; rotational symmetry about the center through any angle measure
10. Rotational symmetry about the center through an angle of 180°

11. Rotational symmetry about the center through an angle of 120°; reflection symmetry over a line through the center and the top vertex, the bottom left vertex, or the bottom right vertex
12. Reflection symmetry over a horizontal line through the center

### Handout 3.1.B: Tessellations of the Plane

1. any two equilateral triangles that share a side, such as two stacked equilateral triangles 
2. irregular polygon 
3. square adjacent to octagon 
4. regular hexagon and two adjacent equilateral triangles 
5. two adjacent rectangles that form an “L” shape 
6. three adjacent parallelograms 
7. four irregular pentagons adjacent to each other 
8. regular hexagon, two equilateral triangles and three squares 

UNIT 3

LESSON 3.2

# Rigid Motions Without Coordinates

## OVERVIEW

### LESSON DESCRIPTION

**Part 1: Performing a Rotation**
Students learn how to perform a rotation on points in a synthetic plane using a paper-and-pencil approach or with dynamic geometry. Students understand that performing a rotation requires a center of rotation and an angle measure.

**Part 2: Performing a Reflection**
Students learn how to perform a reflection on points in a synthetic plane using a paper-and-pencil approach or with dynamic geometry. Students understand that performing a reflection requires an axis of symmetry.

**Part 3: Performing a Translation**
Students learn how to perform a translation on points in a synthetic plane using a paper-and-pencil approach or with dynamic geometry. Students understand that performing a translation requires a distance and a direction.

### CONTENT FOCUS

This lesson uses students' understanding of rigid motion transformations to explore how to perform transformations on objects in the synthetic plane. Students learn additional terminology and notation for transformations and develop the understanding that a rigid motion transformation is a function.

### AREAS OF FOCUS

- Connections Among Multiple Representations
- Engagement in Mathematical Argumentation

### SUGGESTED TIMING

~45 minutes

### LESSON SEQUENCE

- This lesson is part of a lesson sequence (~315 minutes) that includes Lessons 3.1 through 3.5.

### MATERIALS

- ruler or straightedge
- compass
- protractor
- unlined paper
- patty paper
- access to **Desmos.com**

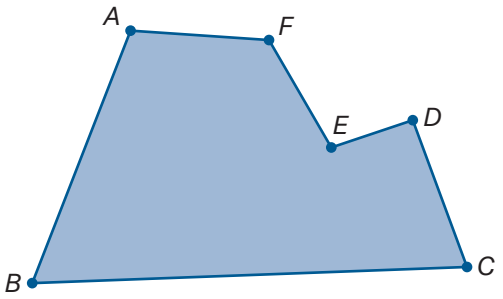
## COURSE FRAMEWORK CONNECTIONS

| Enduring Understandings | |
|---|---|
| ▪ Transformations are functions that can affect the measurements of a geometric figure. | |
| **Learning Objectives** | **Essential Knowledge** |
| **3.1.1** Perform transformations on points in a plane. | **3.1.1a** Transformations describe motions in the plane. Analyzing these transformations indicates if and how these motions affect lengths and angle measures of figures. Congruence and similarity are defined in terms of measurements that are preserved by transformations.<br>**3.1.1b** A transformation is a function whose inputs and outputs are points in the plane. A set of all input points of a transformation is called a preimage; a set of all output points of the preimage is called an image.<br>**3.1.1c** A rigid motion transformation preserves both the distance between pairs of points and the angle measures. A similarity transformation preserves angle measures but not necessarily distances between pairs of points. |
| **3.1.2** Express transformations using function notation. | **3.1.2a** Given a transformation $T$ and two points, $A$ and $B$, the notation $T(A) = B$ means that the image of point $A$ under transformation $T$ is point $B$. The transformation is said to map point $A$ to point $B$.<br>**3.1.2b** Algebra can be used to express how a transformation affects the $x$- and $y$-coordinates of points. All transformations can be represented using function notation, but some transformations are difficult to define as algebraic expressions. |

| Learning Objectives | Essential Knowledge |
|---|---|
| **3.1.4** Solve problems involving rigid motion transformations. | **3.1.4a** Applying one or more translations, rotations, and reflections maps an object to a congruent object.<br>**3.1.4b** Any transformation that preserves distance between points and angle measures can be written as a sequence of translations, reflections, and/or rotations.<br>**3.1.4c** If two figures are congruent, there must exist a sequence of one or more rigid motion transformations that maps one figure to the other. |

**FORMATIVE ASSESSMENT GOAL**

This lesson should prepare students to complete the following formative assessment activity.




Use the polygon above to perform each of the following transformations.

(a) Rotate hexagon *ABCDEF* about point *C* by 100° clockwise.

(b) Reflect hexagon *ABCDEF* over the line containing segment $\overline{AF}$.

(c) Translate hexagon *ABCDEF* so *B*' coincides with *D*.

## PART 1: PERFORMING A ROTATION

In this part of the lesson, students experiment with rotating a single point and then a triangle around a fixed point. Throughout the lesson, students learn notation for transformations and begin to relate transformations with the functions they learned about in prior courses.

**Instructional Rationale**

Each of the three parts of the lesson can be done using a paper-and-pencil approach or with dynamic geometry. Both approaches have advantages and disadvantages. A paper-and-pencil approach allows students to actually perform the transformations themselves through physical interactions with the figures and geometry tools. However, this could be difficult for students with manual dexterity challenges. It also introduces the possibility that students could be incomplete or imprecise when describing the required components of the transformation, such as specifying the axis of symmetry or the angle of rotation, since the student may not see the need to indicate those features. A dynamic geometry approach requires more precision than a paper-and-pencil approach, because the dynamic geometry program will only perform the transformation if all the necessary components are defined. The transformations performed by dynamic geometry software can be changed and modified easily, whereas students using a paper-and-pencil approach would have to redraw a figure. You will have to choose the best available method for your students and school.

- You can start this part of the lesson by asking students to explain what a rotation is and what the necessary components of a rotation are.

  A rotation is a transformation in which a figure is turned. A rotation requires a center of rotation, a direction of rotation (clockwise or counterclockwise), and a specific angle measure.

- Let students know that they will first learn how to rotate a single point and then how to rotate a triangle.
- If you are using a paper-and-pencil approach, distribute paper, protractors, and rulers or straightedges to students. If you are using dynamic geometry software, have students open the program. All screenshots in this lesson are taken from the Desmos Geometry tool (**desmos.com/geometry**).

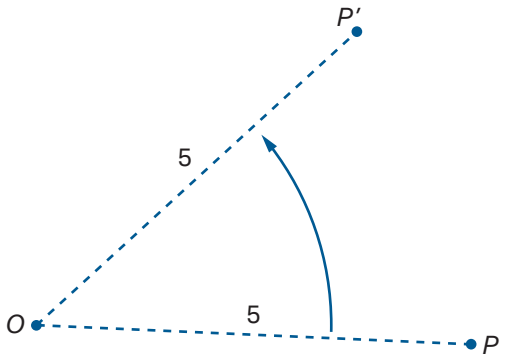
**Classroom Ideas**

You could assign students to pairs in which one student uses a paper-and-pencil approach while the other uses a dynamic geometry program. The students can then compare and contrast their approaches and their resulting figures.

- Let students know that their task is:

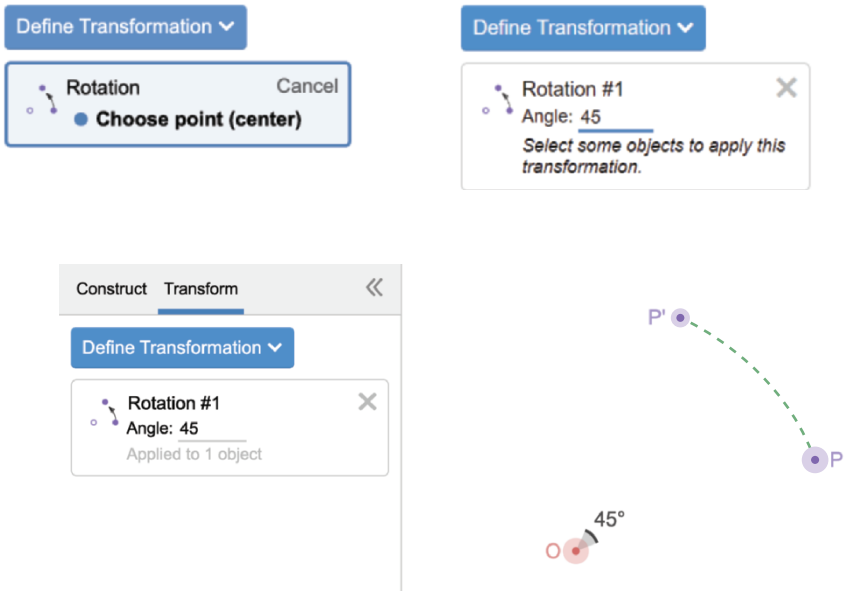
  > Rotate point $P$ about point $O$ by an angle of 45° counterclockwise and label the rotated point $P'$.

  You could display the task in a central location in the room to allow students to reference it as needed.
- Allow students some time to work on the task. Some students will find it challenging to complete the task without more explicit instruction. If you have not already paired students, you can form pairs of students who would benefit from a shared exploration to figure out how to accomplish the task.
- Students using a pencil-and-paper approach will need to choose and label a point $P$ that will be translated and a point $O$ that serves as the center of rotation. Students will need a protractor to determine an angle of 45° counterclockwise about point $O$. Students will also need to determine the distance between points $P$ and $O$, either directly with a ruler or indirectly with a compass. Measuring an angle with a protractor or a distance with a ruler is imprecise and will likely introduce some error. Because rotations preserve the distance between pairs of points, the distance between points $P'$ and $O$ should be equal to the distance between points $P$ and $O$.




- Students using dynamic geometry software will also need to choose and label a point $P$ to be rotated and a point $O$ to serve as the center of rotation. Then they should navigate to the "Transform" tab, pull down the "Define Transformation" menu, and select "Rotation." After selecting the Rotation option, the program will prompt them to choose a point to serve as the center of rotation. Next, the program allows students to choose the angle of rotation. The default angle in Desmos Geometry is 90°, so they will have to change it to 45°. The menu also tells students that they should select the objects on which to apply the transformation. Once students choose point $P$, the program will show an "APPLY" button, which

will perform the transformation when clicked. The program does not automatically label the transformed point. Students will have to select the "Construct" tab and label the point as usual.

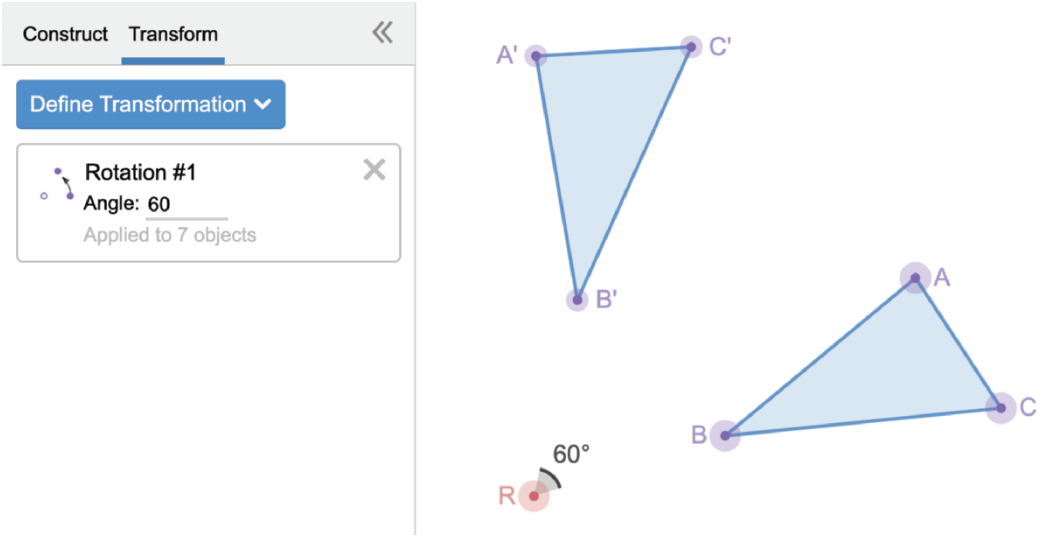



- Let students know that if a transformation is denoted by $T$, and $P$ is a point in the plane, then the transformation $T$ maps the point $P$ to a point labeled $P'$. This can be written in function notation as $T(P) = P'$. In this case, the transformation $T$ is the rotation about point $O$ by 45° counterclockwise. The input and output points of a transformation have specific names. The point $P$ is called the *preimage*, and $P'$ is called the *image*.
- Once students have completed the rotation of a single point, give them another task. Explain that their next task is:

  > Rotate triangle $\Delta ABC$ about point $R$ by a 60° angle counterclockwise and label the rotated triangle as $\Delta A'B'C'$.

  Challenge students to write a statement using function notation that relates the preimage and image of each vertex of the triangle.

  If $T$ is the transformation defined as a 60° rotation counterclockwise about point $R$, then $T(A) = A'$, $T(B) = B'$, and $T(C) = C'$.




- If you choose, you can repeat the rotation exercise with a more complex shape and a different angle measure. It is not necessary for students to master rotating a figure yet, but they should become familiar with defining and using the necessary components of this transformation.

## PART 2: PERFORMING A REFLECTION

In this part of the lesson, students experiment with reflecting a single point and then a triangle over an axis of symmetry. This part of the lesson proceeds in a manner similar to the previous part. It is worth noting that reflections can be difficult to draw using a paper-and-pencil approach. The reflection of a point over an axis is always performed perpendicular to that axis. This means that the axis of symmetry can be thought of as the perpendicular bisector of a segment whose endpoints are a pair of corresponding preimage and image points.

- Begin this part of the lesson by asking students to explain what a reflection is and what the necessary components of a reflection are.

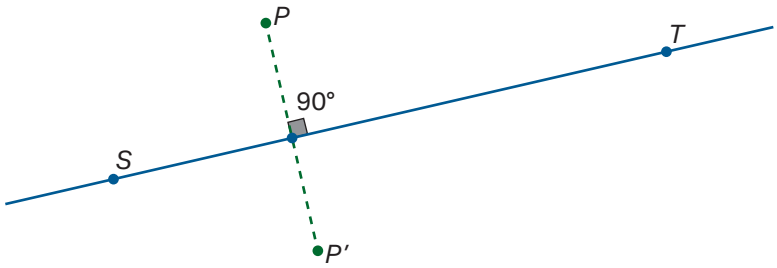
  A reflection is a transformation in which a figure is flipped. A reflection requires an axis of symmetry.

- Let students know that they will learn how to reflect a single point and then how to reflect a triangle.
- Make sure students using the paper-and-pencil approach have paper, compasses, protractors, and straightedges. If students are using a dynamic geometry approach, have them open new pages or clear their screens.

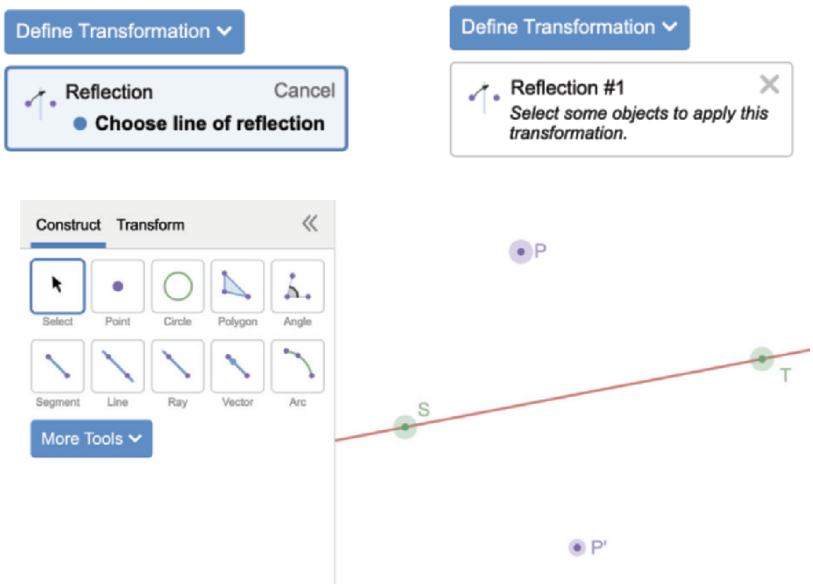
- Let students know that their task is:

  Reflect point $P$ over line $\overleftrightarrow{ST}$ and label the reflected point $P'$.

  You could display the task in a central location in the room to allow students to reference it as needed.
- Allow students some time to work on the task. Some students will find it challenging to complete the task without more explicit instruction. If you have not already paired students, you can form pairs of students who would benefit from shared exploration to figure out how to accomplish the task. For students who need some additional guidance, you can suggest folding the paper along the axis of symmetry. Students may want to use patty paper for this transformation.
- Students using a pencil-and-paper approach will have to choose and label a point $P$ to be reflected and a line $\overleftrightarrow{ST}$ to serve as the axis of symmetry. Students should use a compass and the perpendicular bisector construction to locate point $P'$, an endpoint of a segment bisected by the axis of symmetry. Because reflections preserve the distance between pairs of points, the distance between point $P$ and line $\overleftrightarrow{ST}$ should be equal to the distance between point $P'$ and line $\overleftrightarrow{ST}$. It would be possible to use a protractor to measure a 90° angle and a ruler to measure the distance from $P$ to $\overleftrightarrow{ST}$, but this imprecise method will likely introduce some error.




- Students using dynamic geometry software will have to choose and label a point $P$ to be reflected and a line $\overleftrightarrow{ST}$ to serve as the axis of symmetry. Then they should navigate to the "Transform" tab, pull down the "Define Transformation" menu, and select "Reflection." Next, the program will prompt them to choose the axis of symmetry, followed by the objects on which to apply the reflection. Once students choose line $\overleftrightarrow{ST}$ and point $P$, the program will show an "APPLY" button, which will perform the transformation when clicked. The program does not automatically label the transformed point. Students will have to select the "Construct" tab and label the point as usual.

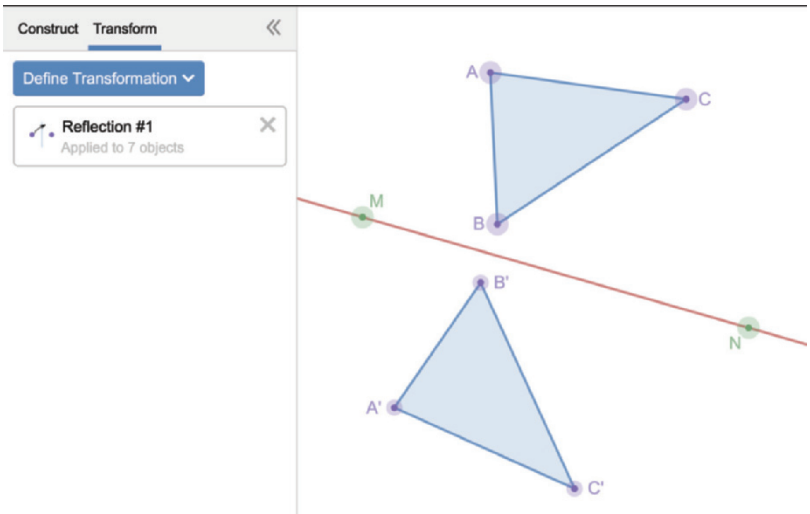



- Ask students to define the transformation and write a statement using function notation that relates the preimage and image.

  The transformation $T$ is the reflection across line $\overleftrightarrow{ST}$. The relationship between $P$ and $P'$ can be expressed as $T(P) = P'$.

- Once students have completed the reflection of a single point, give them another task. Explain that their next task is:

  Reflect triangle $\Delta ABC$ across line $\overleftrightarrow{MN}$ and label the reflected triangle as $\Delta A'B'C'$.



- If you choose, you can repeat the reflection exercise with a more complex shape and a different axis of reflection. It is not necessary for students to master reflecting a figure yet, but they should become familiar with defining and using the necessary components of this transformation.

## PART 3: PERFORMING A TRANSLATION

In this part of the lesson, students experiment with translating a single point, a line segment, and a triangle. This part of the lesson proceeds in a manner similar to the previous two parts. Translations can be challenging in the synthetic plane, because the directions horizontal and vertical do not have meaning without axes for reference. To avoid that confusion, synthetic translations are often defined by a point that is a specified distance and direction away from a point in the preimage. The image of the translation is the result of translating all points in the preimage by the defined distance in the defined direction.

- Begin this part of the lesson by asking students to explain what a translation is and what the necessary components of a translation are.

  A translation is a transformation in which a figure is slid horizontally and/or vertically without any rotation. A translation requires a distance and direction.

- Let students know that they will learn how to perform a translation of a single point, of a segment, and then of a triangle.
- Make sure students using a paper-and-pencil approach have paper, compasses, protractors, and straightedges. If students are using dynamic geometry software, have them open new pages or clear their screens.
- Let students know that their task is:

  > Translate point $P$ to a point $P'$ that is a distance and direction that they choose from $P$.

  You could display the task in a central location in the room to allow students to reference it as needed.

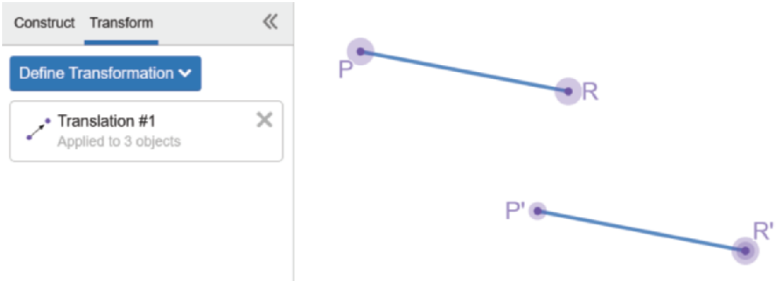
- Allow students some time to work on the task. Because this task is more open-ended than the previous two, students may wonder if they have missed some part of the instruction. Some students may be confused by the openness of the task. You may have to assure students during their work that they are correctly performing the translation.
- Students using a pencil-and-paper approach will need to choose and label a point $P$ to be translated and the point $P'$ to which it is translated.

- Students using dynamic geometry software will have to choose and label a point $P$ to be translated. Then they should navigate to the "Transform" tab, pull down the "Define Transformation" menu, and select "Translation." The program will prompt students to choose a "point (from)" and a "point (to)." Next, the program will show an "APPLY" button that will perform the transformation when clicked. The program does not automatically label the transformed point. Students will have to select the "Construct" tab and then label the point as usual.


- Ask students to define the transformation and write a statement using function notation that relates the preimage and image.

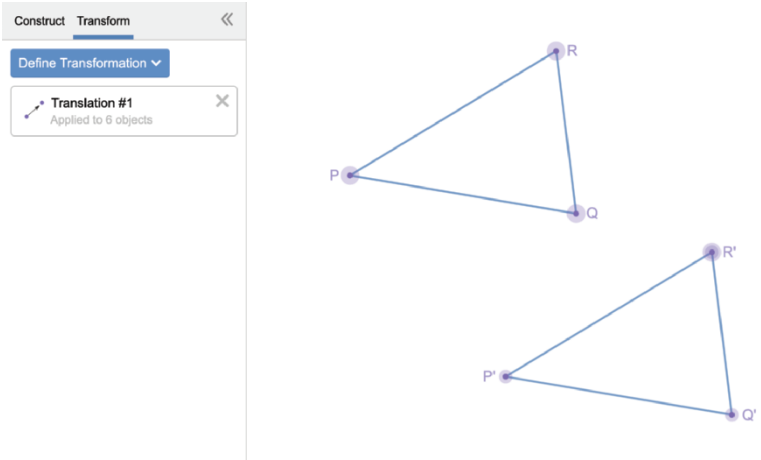
  The transformation $T$ is the translation of $P$ to $P'$ and can be expressed as $T(P) = P'$.

- Once students have defined point $P'$ as the distance and direction of the translation, ask them to add a point $R$ to use with point $P$ as the endpoints of a segment $\overline{PR}$. Then have students translate the segment to create $\overline{P'R'}$. The result should be two congruent segments as shown.




- Once students have completed the translation on a segment, give them another task. Explain that their next task is:

  Choose a point $Q$ and then translate triangle $\Delta PQR$ to $\Delta P'Q'R'$.



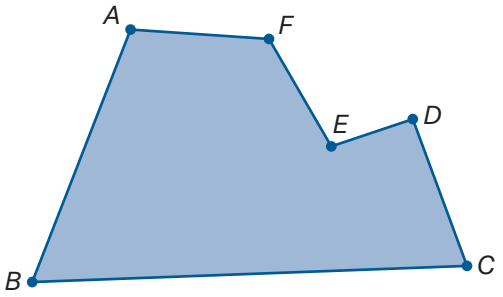

- If you choose, you can repeat the translation exercise with a more complex figure. It is not necessary that students master translating a figure yet, but they should become familiar with defining and using the necessary components of this transformation.

UNIT 3

## ASSESS AND REFLECT ON THE LESSON

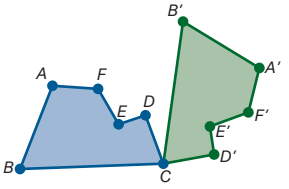
### FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

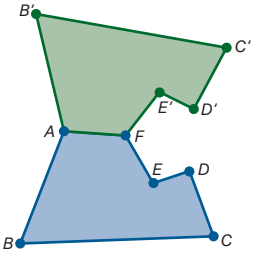



Use the polygon above to perform each of the following transformations.

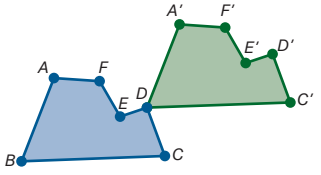
(a) Rotate hexagon *ABCDEF* about point *C* by 100° clockwise.




(b) Reflect hexagon *ABCDEF* over the line containing segment $\overline{AF}$.




(c) Translate hexagon *ABCDEF* so *B′* coincides with *D*.



LESSON 3.3

UNIT 3

# Rigid Motions in the Coordinate Plane

## OVERVIEW

### LESSON DESCRIPTION

**Part 1: Investigating Rigid Motions with Desmos**

Students investigate transformations in the coordinate plane through an interactive digital activity.

**Part 2: Exploring How Rigid Motions Affect Coordinates**

Students work in small groups to perform rigid motions in the coordinate plane and try to find a formula that describes how the coordinates change under a given transformation.

**Part 3: Combining Rigid Motions**

Students use what they know about rigid motions to determine sequences of rigid motions that map a preimage to an image.

### AREAS OF FOCUS

- Engagement in Mathematical Argumentation
- Connections Among Multiple Representations

### SUGGESTED TIMING

~90 minutes

### LESSON SEQUENCE

- This lesson is part of a lesson sequence (~315 minutes) that includes Lessons 3.1 through 3.5.

### HANDOUTS

**Lesson**

- 3.3.A: Exploring Vertical Translations
- 3.3.B: Exploring Horizontal Translations
- 3.3.C: Exploring Reflections over the $x$-Axis
- 3.3.D: Exploring Reflections over the $y$-Axis
- 3.3.E: Exploring Reflections over the Line $y = x$

UNIT 3

## CONTENT FOCUS

This lesson provides a transition from transformations without coordinates to transformations with coordinates. For some transformations, like translations and reflections, the coordinate plane allows you to specify distances and axes of symmetry. For rotations, the coordinate plane can present a challenge, because the effect of a rotation by most angle measures on one or more points is difficult to express precisely using coordinates. A goal of this lesson is for students to understand that transformations can be expressed using function notation and algebraic formulas, although some transformations are more difficult to write this way than others.

- 3.3.F: Exploring 90° Counterclockwise Rotations about the Origin
- 3.3.G: Exploring 180° Counterclockwise Rotations about the Origin
- 3.3.H: Exploring 90° Clockwise Rotations about the Origin
- 3.3.I: Sequences of Transformations

## MATERIALS

- access to **Desmos.com**

## COURSE FRAMEWORK CONNECTIONS

| Enduring Understandings | |
|---|---|
| ▪ Transformations are functions that can affect the measurements of a geometric figure.<br>▪ Congruent figures have equal corresponding angle measures and equal distances between corresponding pairs of points. | |
| **Learning Objectives** | **Essential Knowledge** |
| **3.1.1** Perform transformations on points in a plane. | **3.1.1a** Transformations describe motions in the plane. Analyzing these transformations indicates if and how these motions affect lengths and angle measures of figures. Congruence and similarity are defined in terms of measurements that are preserved by transformations.<br>**3.1.1b** A transformation is a function whose inputs and outputs are points in the plane. A set of all input points of a transformation is called a preimage; a set of all output points of the preimage is called an image. |

| Learning Objectives | Essential Knowledge |
|---|---|
| | **3.1.1c** A rigid motion transformation preserves both the distance between pairs of points and the angle measures. A similarity transformation preserves angle measures but not necessarily distances between pairs of points. |
| **3.1.2** Express transformations using function notation. | **3.1.2a** Given a transformation $T$ and two points, $A$ and $B$, the notation $T(A) = B$ means that the image of point $A$ under transformation $T$ is point $B$. The transformation is said to map point $A$ to point $B$.<br>**3.1.2b** Algebra can be used to express how a transformation affects the $x$- and $y$-coordinates of points. All transformations can be represented using function notation, but some transformations are difficult to define as algebraic expressions. |
| **3.1.3** Prove that a rigid motion transformation maps an object to a congruent object. | **3.1.3a** A rigid motion transformation is a transformation that preserves distances between pairs of points as well as angle measures.<br>**3.1.3b** A translation is a transformation that maps each point in the plane to an image that is a specified distance in a specified direction from the preimage.<br>**3.1.3c** A reflection is a transformation that maps each point in the plane to its mirror image across a line called the axis of symmetry.<br>**3.1.3d** A rotation is a transformation that maps each point in the plane to an image that is turned by a specified angle about a fixed point called the center of rotation. |

| Learning Objectives | Essential Knowledge |
|---|---|
| **3.1.4** Solve problems involving rigid motion transformations. | **3.1.4a** Applying one or more translations, rotations, and reflections maps an object to a congruent object.<br>**3.1.4b** Any transformation that preserves distance between points and angle measures can be written as a sequence of translations, reflections, and/or rotations.<br>**3.1.4c** If two figures are congruent, there must exist a sequence of one or more rigid motion transformations that maps one figure to the other. |

## FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

> Charles and Layla are discussing sequences of transformations. Charles says that the order in which you perform of a sequence of transformations never matters. Layla says that for many specific transformations, the order in which you perform the transformations does matter. Who is correct? Explain your answer.

## PART 1: INVESTIGATING RIGID MOTIONS WITH DESMOS

The lesson begins with a Desmos classroom activity. In this activity, students use their understanding of translations, reflections, and rotations to map a preimage to a desired image. The activity uses an unmarked coordinate plane to help students transition from performing transformations without coordinates to performing them with coordinates. This Desmos activity will deepen students' understanding of transformations and help prepare them for work in the coordinate plane.

- Before teaching the lesson, you will have to log in to a teacher account (or create a free account) at **teacher.desmos.com** and select the activity "Transformation Golf: Rigid Motion." The activity can be found in the "Transformation Bundle."
- Be sure to work through the activity before using it in class so you can anticipate where students might struggle. The activity has several teacher tips for implementation and includes suggested answers for each slide.
- Let students know that they will be working on a digital activity through Desmos. Pair the students so that they can work on the activity with a partner. This activity has a high potential to spark academic conversations between students. They can work on their mathematical argumentation throughout the activity.
- Each pair of students will need access to a laptop or tablet. The activity does not work well on a mobile device.
- Be sure to create a class code for students. You can display the code in a central location for all students to reference. They will need the code to access the activity once they navigate to **student.desmos.com**.
- While students are working, circulate around the room and help answer any questions students have. You can use the teacher dashboard to monitor student progress and highlight student responses that you would like to address as a class or with particular student pairs. Additional support for using Desmos classroom activities can be found at **learn.desmos.com/activities**.

**Meeting Learners' Needs**

You might decide to begin the lesson by having students describe translations, reflections, and rotations and recall the necessary components of each type of transformation. This reminder will help students focus their attention and prepare them for the activity.

- To bring the activity to a close, you should address the questions on slides 16 and 17.

In a sequence of translations, reflections, and rotations, what will NEVER change from the pre-image to the image?

*(Select all that apply.)*

- ☐ The lengths of the line segments.
- ☐ The position of the object.
- ☐ The measures of the angles.
- ☐ The orientation of the object.

Which of these statements is true?

*(Select all that apply.)*

- ☐ Any translation can be replaced by two reflections.
- ☐ Any translation can be replaced by two rotations.
- ☐ Any rotation can be replaced by a reflection.
- ☐ Any reflection can be replaced by a rotation followed by a translation.

- Slide 16—In a sequence of translations, reflections, and rotations, what will NEVER change from the preimage to the image?

  The lengths of the line segments and the measures of the angles.

- Why will the lengths of the line segments and the measures of the angles never change?

  Any sequence of translations, reflections, and/or rotations will map a preimage to a congruent image. Congruent figures have equal corresponding side lengths and angle measures.

- Slide 17—Which of these statements is true? Why or why not?

  "Any translation can be replaced by two reflections" is true because one reflection can map any vertex of the preimage to the corresponding vertex of the image, and a second reflection can map the remaining points of the preimage to their corresponding points of the image. "Any translation can be replaced by two rotations" is true because a 180° rotation can be used to map any vertex of the preimage to the corresponding vertex of the image, and then a second 180° rotation can be used to map the remaining points of the preimage to their corresponding points of the image. "Any rotation can be replaced by a reflection" is not true in general. As a counterexample, there is no reflection of the L shape from the activity that will produce the same image as a 180° rotation. "Any reflection can be replaced by a rotation followed by a translation" is not true in general. As a counterexample, there is no sequence of rotations and translations that will flip the orientation of the L shape from the activity like a reflection would.

**Guiding Student Thinking**

Students may struggle to determine whether or not the statements from slide 17 are true. It is not essential for them to determine the truth of each statement at this point in the lesson. In Part 3, students will use sequences of transformations to map a preimage to an image. After Part 3, they can revisit these statements and then make a final decision.

## PART 2: EXPLORING HOW RIGID MOTIONS AFFECT COORDINATES

In this part of the lesson, students investigate a few specific but important transformations. The activity takes the form of a jigsaw, in which students work in pairs or small groups to determine the effect of a specified transformation on the coordinates of the vertices of a polygon. Then the groups who have explored different transformations will join together to share their observations. The activity culminates in the class generalizing their results and developing formulas for these important transformations.

- There are eight transformations for students to explore: vertical translation, horizontal translation, reflection over the $x$-axis, reflection over the $y$-axis, reflection over the line $y = x$, rotation about the origin by 90° counterclockwise, rotation about the origin by 180° counterclockwise, and rotation about the origin by 90° clockwise. Each of these transformations is addressed in its own handout.
- Have students form pairs or small groups of three. Distribute a different handout to each group. Each handout has several examples of a single transformation. The task for students is to determine the effect of the transformation on the coordinates of the vertices (and all other points) of the polygon.
- You can use a table like the one shown to organize and rearrange students in groups as they change throughout the activity.

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| Student 1 | | | | | | | | |
| Student 2 | | | | | | | | |
| Student 3 | | | | | | | | |
| Student 4 | | | | | | | | |

- Once each group has determined the effect of their transformation on the coordinates of the points of the polygon, they can join with another group that has a transformation of the same family. That is, groups with translations (3.3.A and 3.3.B) can join together, groups with reflections (3.3.C, 3.3.D, and 3.3.E) can join together, and groups with rotations (3.3.F, 3.3.G, and 3.3.H) can join together.

**Classroom Ideas**
If you have a smaller class, you might need each group to explore multiple transformations.

- Have the students in these larger groups share their transformations and the effects of the transformations on the coordinates of points in the polygon. Encourage students to identify similarities and differences between the transformations. Within each group, every student should develop some familiarity with the description of the effect of transformations of the same family on the coordinates of the polygon.

**Guiding Student Thinking**

At this point in the lesson, students may feel more comfortable describing a transformation's effect on the coordinates verbally than writing it as an algebraic statement. For example, students might describe a reflection over the $x$-axis as "keeping the $x$-value the same and changing the sign of the $y$-value." This is an important step toward understanding the utility of algebraic notation. Students often do not recognize the power of algebraic notation to express long phrases using a few symbols. After writing the full description down or saying it out loud several times, students might find that they prefer the more efficient way to express the formula.

- Now have students organize themselves into new groups in which each group has at least one representative from each transformation family. You can use the rows from the table on the previous page to quickly assign groups. Students should share the effect of their transformation on the coordinates of the vertices of the polygon. Encourage students to identify similarities and differences among the transformations.
- Each group should create a summary of the effects of each type of transformation. They could make a small poster or write a summary on a personal white board.
- Bring the whole class together, and have volunteers share their descriptions for the transformations. Students should have concluded the following:

| Handout | Transformation | Verbal Description | Algebraic Description in Function Notation |
|---|---|---|---|
| 3.3.A | vertical translation | Keep the $x$-coordinate the same; add or subtract the vertical change to the $y$-coordinate. | $T(x,y)=(x,y+k)$ |
| 3.3.B | horizontal translation | Keep the $y$-coordinate the same; add or subtract the horizontal change to the $x$-coordinate. | $T(x,y)=(x+h,y)$ |

| Handout | Transformation | Verbal Description | Algebraic Description in Function Notation |
|---|---|---|---|
| 3.3.C | reflection over the $x$-axis | Keep the $x$-coordinate the same; change the sign of the $y$-coordinate. | $T(x, y) = (x, -y)$ |
| 3.3.D | reflection over the $y$-axis | Keep the $y$-coordinate the same; change the sign of the $x$-coordinate. | $T(x, y) = (-x, y)$ |
| 3.3.E | reflection over the line $y = x$ | Switch the values of the $x$- and $y$-coordinates. | $T(x, y) = (y, x)$ |
| 3.3.F | rotation about the origin by 90° counterclockwise | Change the sign of the $y$-coordinate, and then switch the values of the $x$- and $y$-coordinates. | $T(x, y) = (-y, x)$ |
| 3.3.G | rotation about the origin by 180° counterclockwise | Change the signs of the $x$- and $y$-coordinates. | $T(x, y) = (-x, -y)$ |
| 3.3.H | rotation about the origin by 90° clockwise | Change the sign of the $x$-coordinate, and then switch the values of the $x$- and $y$-coordinates. | $T(x, y) = (y, -x)$ |

- Students may need help writing the algebraic expressions for the effect of the transformations using function notation. Encourage students to try to rewrite their verbal description algebraically. If it is not correct, help them see why it does not result in the transformation they expected. A dynamic graph could help students see the effect of their algebraic mapping quickly.

**Guiding Student Thinking**

Function notation is only one way for students to express the algebraic description of the effect of the transformation. Some students might use arrow notation, such as $(x, y) \rightarrow (-x, -y)$, to show the input and output of the transformation. You can validate the notation that students create, provided that it correctly describes the effect of the transformation. You can show students how to translate their own notation into conventional function notation.

- This is a good opportunity to reinforce that transformations are functions. Just like the functions students learned in Algebra 1, transformations have inputs and outputs. In Algebra 1, an input was a single number, and an output was another number. For transformations, the input is a point, and the output is another point.

## PART 3: COMBINING RIGID MOTIONS

In this part of the lesson, students use what they've learned in Parts 1 and 2 to determine a sequence of transformations that maps a preimage to an image. There are two main goals for Part 3. First, students learn that there is usually more than one way to order transformations to map a preimage to a given image. Second, students recognize that a few well-chosen transformations can be ordered to achieve a wide range of complex transformations.

- Distribute **Handout 3.3.I: Sequences of Transformations**. Students can work independently or in pairs on the problems. The task is for students to determine at least two different sequences of transformations that map each preimage to the given image.
- If students are working independently, have them pair up once they have made some progress on the problems. If students are already working in pairs, have two pairs form a small group once they have made some progress on the problems. In these groups, students should share their transformations with each other and take time to critically examine each transformation and determine if it successfully maps the preimage to the image.
- Encourage students to find as many transformations as possible that will map each preimage to its image. You can expect that students will want to write down the transformations as descriptions, such as, "Do a rotation of 90°, and then translate." Be sure that students are as specific as possible in their descriptions, such as, "First perform a rotation of 90° counterclockwise about the origin, and then translate 2 units down and 5 units to the right."

**Meeting Learners' Needs**

Some students may wonder if it is possible to rotate a polygon by an angle other than 90° or 180°. For students who would welcome a challenge, you can ask them to do some internet research to determine if it is possible. While a polygon can be rotated by any angle, the mathematics needed to determine the coordinates of the resulting vertices is beyond the scope of this course. In general, performing a rotation of a geometric figure in the coordinate plane by an arbitrary angle involves trigonometry and matrix algebra. Students interested in pursing a STEM field may encounter rotation matrices in a college-level Linear Algebra course.

- Try to get students to write each of their transformations as an algebraic expression in function notation. Repeated use of function notation will help students become more familiar with it.
- To debrief the lesson, provide some time for student groups to present to the class at least one sequence of rigid motions that maps a preimage to its image. Other students should have an opportunity to critique each mapping sequence. Allocate time for students who have different solutions to share them. It is important that students see multiple solution pathways from a preimage to an image.

**Meeting Learners' Needs**
You can challenge students who finish the assignment quickly to write each of their transformation sequences as a single algebraic expression in function notation that maps the preimage coordinates to the image coordinates. This assignment primes students to think about function composition, a major concept they will explore in Algebra 2.

## ASSESS AND REFLECT ON THE LESSON

### FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

Charles and Layla are discussing sequences of transformations. Charles says that the order in which you perform of a sequence of transformations never matters. Layla says that for many specific transformations, the order in which you perform the transformations does matter. Who is correct? Explain your answer.

Layla is correct. Some transformations must be performed in the order specified. For example, translating a polygon vertically by 2 units and then reflecting it over the $x$-axis will map to a different image than reflecting the same preimage over the $x$-axis and then translating it vertically by 2 units.

**Guiding Student Thinking**

If students struggle with the Formative Assessment Goal, you can encourage them to generate examples of transformation sequences to test the differing claims made by Charles and Layla.

### HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

#### Handout 3.3.I: Sequences of Transformations

1. One possible sequence of transformations is a reflection over the $y$-axis followed by a horizontal translation of 5 units left and a vertical translation of 3 units down.
2. One possible sequence of transformations is a reflection over the line $y = x$ followed by a horizontal translation of 5 units left and a vertical translation of 4 units up.
3. One possible sequence of transformations is rotation about the origin by 180°, then a reflection over the line $y = x$, followed by a vertical translation of 1 unit down and a horizontal translation of 1 unit left.

LESSON 3.4

UNIT 3

# Dilations

## OVERVIEW

### LESSON DESCRIPTION

**Part 1: Dilating a Figure from a Point on the Figure**

Students investigate dilations of a geometric figure in the synthetic plane. In this part of the lesson, the center of dilation is a point on the figure, such as a vertex of a triangle. As in Lesson 3.2, this part of the lesson can be explored using Desmos Geometry or a paper-and-pencil approach.

**Part 2: Dilating a Figure from a Point not on the Figure**

Students investigate dilations of a geometric figure in the synthetic plane. In this part of the lesson, the center of dilation is a point not on the figure, such as a point in the interior of a triangle. Again, this part of the lesson can be explored using Desmos Geometry or a paper-and-pencil approach.

**Part 3: Dilating a Figure in the Coordinate Plane**

Students explore dilations of a geometric figure in the analytic plane. As in Lesson 3.3, students examine the effect of a transformation on the coordinates of the vertices of a polygon. Students only work with dilations where the center of dilation is the origin.

### AREAS OF FOCUS

- Engagement in Mathematical Argumentation
- Connections Among Multiple Representations

### SUGGESTED TIMING

~90 minutes

### LESSON SEQUENCE

- This lesson is part of a lesson sequence (~315 minutes) that includes Lessons 3.1 through 3.5.

### HANDOUTS

**Lesson**

- 3.4: Exploring Dilations

### MATERIALS

- unlined paper
- rulers with millimeter markings

UNIT 3

## CONTENT FOCUS

A dilation is a transformation that scales the distances between points but preserves angle measures. Because the angle measures remain fixed but distances between points are scaled, a dilation is not a rigid motion transformation, and the image under a dilation is similar, not congruent, to the preimage. Students might come to the course thinking of a dilation as only a scaling of a figure to be larger or smaller. Mathematically, a dilation, as with a rotation, is performed with reference to a point. For a dilation, this point is called the center of dilation. The center of dilation can be a point inside, outside, or on the figure being dilated. A dilation scales the distances that the points of the preimage are from the center of dilation by a value called the scale factor. This means, for example, that each point of the image of a dilation with a scale factor of 2 is twice as far from the center of dilation as the corresponding preimage point is. Students should come to understand that the image of a dilation is larger or smaller than the preimage and is farther away from or closer to the center of dilation than the preimage, yet it has the same corresponding angle measures as the preimage.

## COURSE FRAMEWORK CONNECTIONS

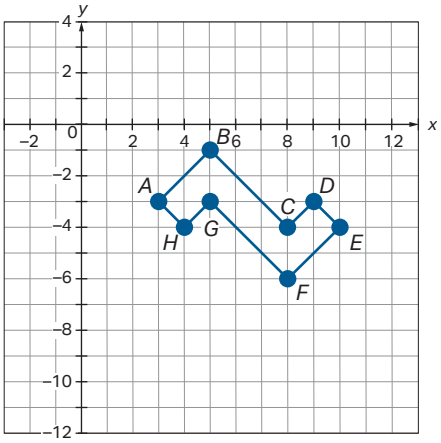
| Enduring Understandings | |
|---|---|
| ▪ Transformations are functions that can affect the measurements of a geometric figure.<br>▪ Similar figures have equal corresponding angle measurements, and the distances between corresponding pairs of points are proportional. | |
| **Learning Objectives** | **Essential Knowledge** |
| **3.1.5** Prove that a similarity transformation maps an object to a similar object. | **3.1.5a** A similarity transformation is a sequence of a dilation and/or one or more rigid motion transformations.<br>**3.1.5b** A dilation from a fixed point, called the center, with a scale factor $k$ is a transformation that maps each point in the plane to an image whose distance from the center is $k$ times the distance between the center and the preimage, in the same direction as the preimage.<br>**3.1.5c** Dilations of figures do not affect the angle measures of a figure. |

| Learning Objectives | Essential Knowledge |
|---|---|
| **3.1.6** Solve problems involving similarity transformations. | **3.1.6a** Dilating the plane by a scale factor $k$ with center (0, 0) will scale each coordinate by $k$.<br>**3.1.6b** A dilation maps a line not passing through the center of the dilation to a parallel line and maps a line passing through the center of dilation to itself.<br>**3.1.6c** The scale factor of a dilation can be determined by dividing a length from the image by its corresponding length in the preimage. |

**FORMATIVE ASSESSMENT GOAL**

This lesson should prepare students to complete the following formative assessment activity.

Use the polygon below to answer the questions.




(a) Create a polygon *JKLMNOPQ* that is a dilation of polygon *ABCDEFGH* with a scale factor of 3 about the origin.

(b) Create a polygon *STUVWXYZ* that is a dilation of polygon *ABCDEFGH* with a scale factor of $\frac{1}{3}$ about the origin.

(c) Are polygons *JKLMNOPQ* and *STUVWXYZ* similar to each other? If they are, what is the scale factor by which one can be dilated to coincide with the other?

UNIT 3

## PART 1: DILATING A FIGURE FROM A POINT ON THE FIGURE

In this part of the lesson, students explore dilations without referencing the coordinate plane. Dilations differ from translations, rotations, and reflections in that the image is similar but not congruent to the preimage. The distance from the center of dilation to each point in the image is scaled up or down from its distance to the corresponding point in the preimage.

- To begin the lesson, you can ask students to think about everyday examples of dilations.
  - What are some examples of situations in which you might scale up or scale down an image of something?

    I might zoom in on or out from a photograph or a map, or use a diagram in an instructional manual to build furniture or a model kit.
  - When you zoom in on or zoom out from a picture or a map, what changes and what stays the same? Is the picture or map congruent to the original picture or map after you zoom in or zoom out?

    The shape is the same, but the size changes.
  - What mathematical term describes the relationship between the shapes before and after zooming?

    The shapes before and after zooming are not congruent. The shapes would be best described as similar.
  - What are some other examples of scaling up or down that you have observed?

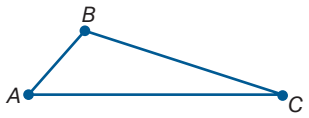
    A projection on a screen, like in a classroom or a movie theatre, is a scaled up version of a still or moving image.
- You can tell students that the mathematical term for scaling up or down is *dilation*. This is a new transformation that they will learn about in this lesson. You can help students make a connection to prior lessons by having them think about right triangle trigonometry. Similar triangles are dilations of each other.
- Let students know that they will learn how to perform a dilation on a figure.

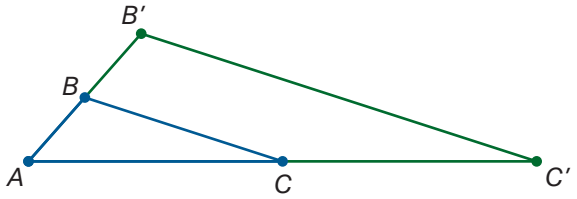
**Meeting Learners' Needs**
You can help students understand dilations by identifying maps and blueprints as examples of dilations. Surveyors "shrink" large landscapes to fit on laptop screens by keeping the shapes the same but scaling down all other features by the same scale factor. In the resulting image, the relative sizes of the figures are consistent with each other.

- If you are using a paper-and-pencil approach, distribute paper and rulers to students. Some students might prefer to use compasses if you have them readily available. If you are using dynamic geometry, have students open the program. All of the screenshots in this lesson are taken from the Desmos Geometry tool (**desmos.com/geometry**).

**Classroom Ideas**
You could pair students so one is using a paper-and-pencil approach while the other is using a dynamic geometry program. Then, you can have them compare and contrast their approaches.

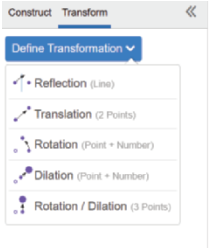
- Let students know that their task is to dilate triangle $\Delta ABC$ by a factor of 2 from vertex $A$. You could display the task in a central location in the room to help students reference it.
- Allow students some time to work on the task. Some students will find it challenging to do this without more explicit instruction. You can pair up some students, if they are not already paired, so more than one person is trying to figure out how to complete the task.
- Students using a pencil-and-paper approach will have to create the preimage triangle and label its vertices. From point $A$ they will have to determine the lengths of sides $\overline{AB}$ and $\overline{AC}$. Then they will have to determine the location of a point, $B'$, along $\overline{AB}$ so that $\overline{AB'}$ is twice the length of $\overline{AB}$. Then they will repeat the process to determine a point, $C'$, along $\overline{AC}$ so that $\overline{AC'}$ is twice the length of $\overline{AC}$. Then they can construct segment $\overline{B'C'}$. Finally, they should verify that $m\overline{B'C'} = 2 \bullet m\overline{BC}$.

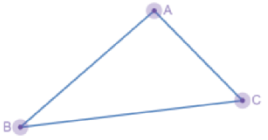


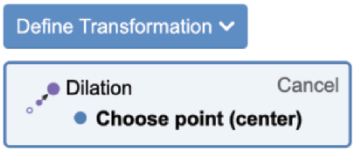



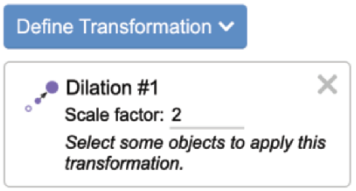

- Students using dynamic geometry software will have to create triangle $ABC$ and label its vertices. Then they should navigate to the "Transform" tab, pull down the "Define Transformation" menu, and select "Dilation." The program will then prompt them to choose the center of dilation. After students choose the center, they can adjust the scale factor. The program will prompt students to select the

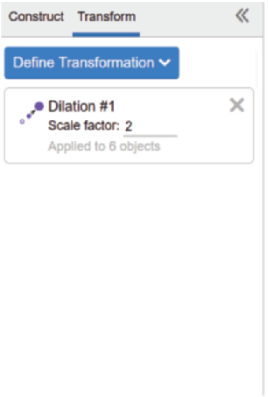
objects on which to apply the transformation. Once students choose $\Delta ABC$, the program will show an "APPLY" button that will perform the transformation when clicked. The program does not automatically label the point. Students will have to select the "Construct" tab and then label the points as usual.

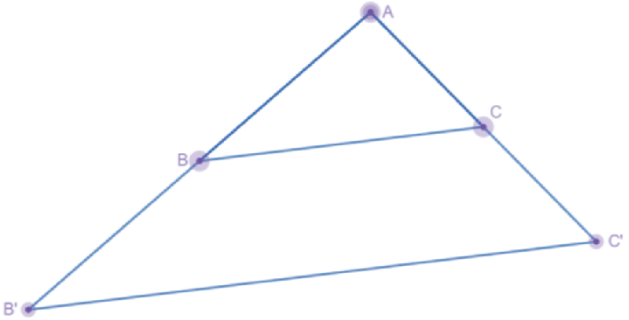


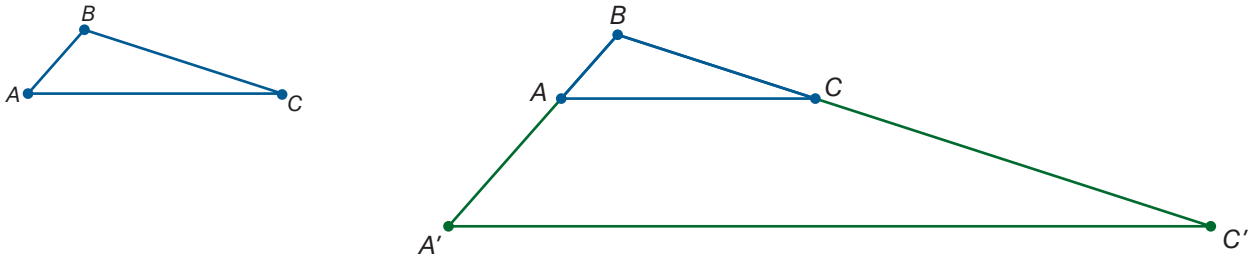



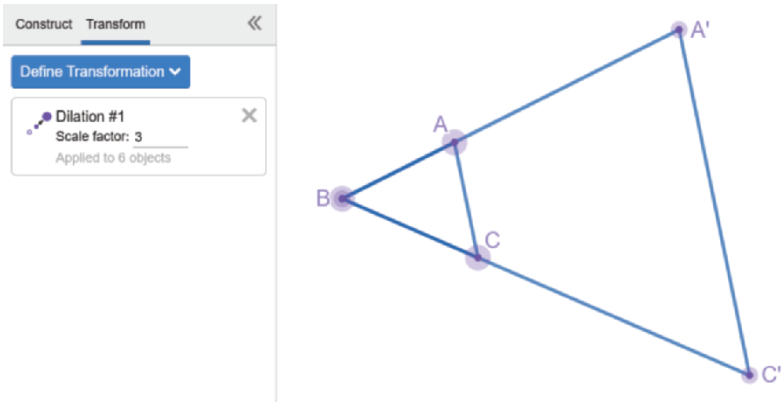




- Encourage students to use function notation to represent the dilation. If the dilation is denoted by $D$ and $P$ is a point in the plane, then the transformation $D$ takes the point $P$ to a point $P'$. This can be written in function notation as $D(P) = P'$. The transformation $D$ is a dilation by a scale factor of 2 whose center of dilation is point $A$.
- Now you can have students determine how they would dilate triangle $ABC$ using a scale factor of 3 from a different vertex, such as $B$.

- As before, students using a pencil-and-paper approach should determine the lengths of sides $\overline{BA}$ and $\overline{BC}$. Then they will have to determine a point, $A'$, along $\overline{BA}$ so that $\overline{BA'}$ is three times the length of $\overline{BA}$. Then they must repeat the process to determine a point, $C'$, along $\overline{BC}$ so that $\overline{BC'}$ is three times the length of $\overline{BC}$. They can then construct segment $\overline{A'C'}$. Finally, they should verify that $m\overline{A'C'} = 3 \bullet m\overline{AC}$.




- Students using dynamic geometry software can undo the original dilation by clicking the "X" on upper right side of the dilation. Then they can define a new dilation by navigating to the "Transform" tab, pulling down the "Define Transformation" menu, and selecting "Dilation." The program will then prompt them to choose the center of dilation. After students choose the center, they can adjust the scale factor. The program will prompt students to select the objects on which to apply the transformation. Once students choose $\Delta ABC$, the program will show an "APPLY" button that will perform the transformation when clicked. As before, the program does not automatically label the point. Students must select the "Construct" tab and then label the points as usual.



- Before giving students another task, ask them to think about why the dilation does not seem to affect point $B$.
  - In the dilation, two vertices of the triangles have distinct images, but the other vertex doesn't seem to be affected. Why does it look like the center of dilation is not affected by the dilation?

    Dilations scale the distances between a center of dilation and all other points. Because one vertex of the triangle is at the center, its distance from itself is zero. Zero scaled by any factor will always be zero. So the vertex that is the center of dilation will have the same location in the preimage and image.
- If you choose, you can challenge students to dilate a triangle in which the center of dilation is a nonvertex point on the triangle. The processes for both a paper-and-pencil approach and a dynamic geometry approach will be similar to dilating from a vertex.

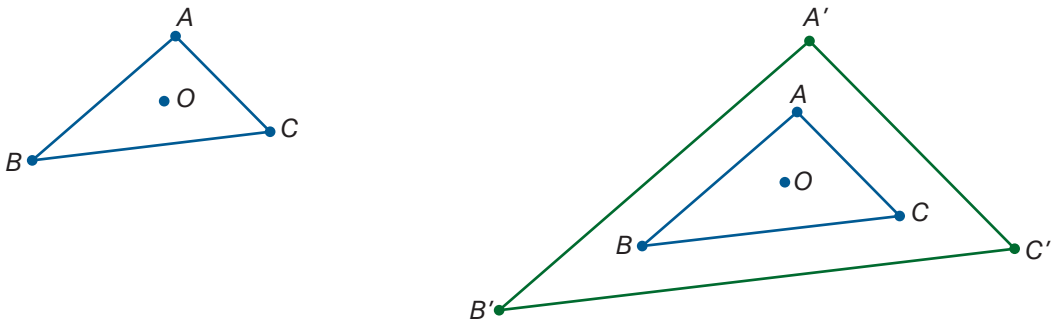
## PART 2: DILATING A FIGURE FROM A POINT NOT ON THE FIGURE

In both of the dilation tasks for Part 1, students defined a center of dilation that was a point on the triangle. However, the center of dilation does not have to be a point on a figure; it can be any point in the plane. In this part of the lesson, students will explore dilations in which the center is not on the figure.

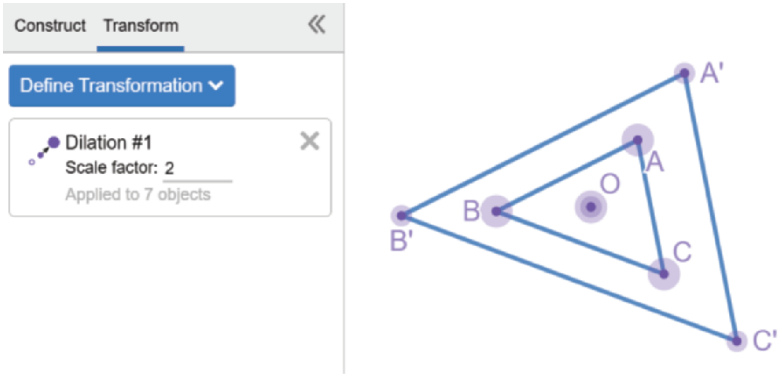
- You can start this part of the lesson by posing an open-ended question:
  - If you want to dilate a triangle, does the center of dilation have to be on the triangle? Can it be inside the triangle? Outside the triangle?
- You can let students have some time to think about possible locations for a center of dilation. They do not have to answer the question at this point. Part of doing mathematics is asking a question and then carrying out an investigation to search for an answer. You can let students know that their task is to dilate triangle $ABC$ by a scale factor of 2 using a point, $O$, that is inside the triangle as the center of dilation.
- Students using a pencil-and-paper approach should construct triangle $ABC$ and a center of dilation inside the triangle. They must determine the distances between points $O$ and $A$, points $O$ and $B$, and points $O$ and $C$. Then they will have to determine a point, $A'$, along $\overrightarrow{OA}$ so that the distance between points $O$ and $A'$ is twice the distance between points $O$ and $A$. Then they should repeat the process to determine a point, $B'$, along

**Meeting Learners' Needs**
Some students may benefit from drawing segments $\overline{OA}$, $\overline{OB}$, and $\overline{OC}$ as well as rays $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$. This will help them make sure the dilations are emanating from point $O$.

$\overline{OB}$ so that the distance between points $O$ and $B'$ is twice the distance between points $O$ and $B$. Then they should repeat the process one more time to determine a point, $C'$, along $\overline{OC}$ so that the distance between points $O$ and $C'$ is twice the distance between points $O$ and $C$. They should construct segments $\overline{A'B'}$, $\overline{B'C'}$, and $\overline{A'C'}$ to complete the dilated triangle. Finally, they should verify that $m\overline{A'B'} = 2 \bullet m\overline{AB}$, $m\overline{B'C'} = 2 \bullet m\overline{BC}$, and $m\overline{A'C'} = 2 \bullet m\overline{AC}$.

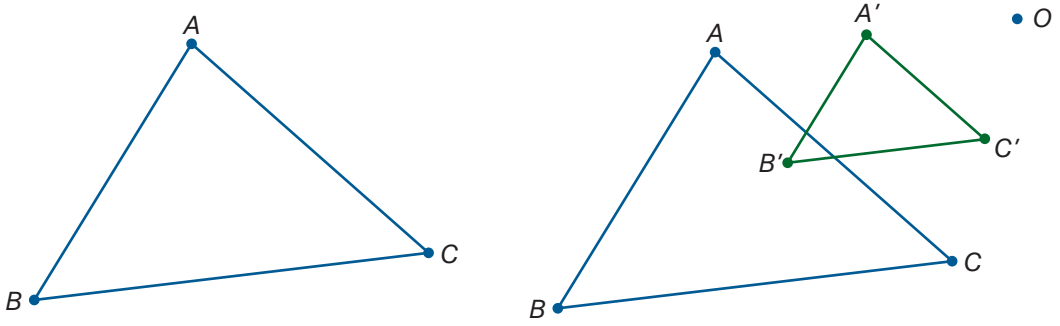



- Students using dynamic geometry software will have to create triangle $ABC$, label its vertices, and plot and label a point $O$ in the interior of the triangle. Then they should navigate to the "Transform" tab, pull down the "Define Transformation" menu, and select "Dilation." The program will then prompt them to choose the center. After students choose the center, they can adjust the scale factor. The program will tell students to select the objects on which to apply the transformation. Once students choose $\Delta ABC$, the program will show an "APPLY" button that will perform the transformation when clicked.

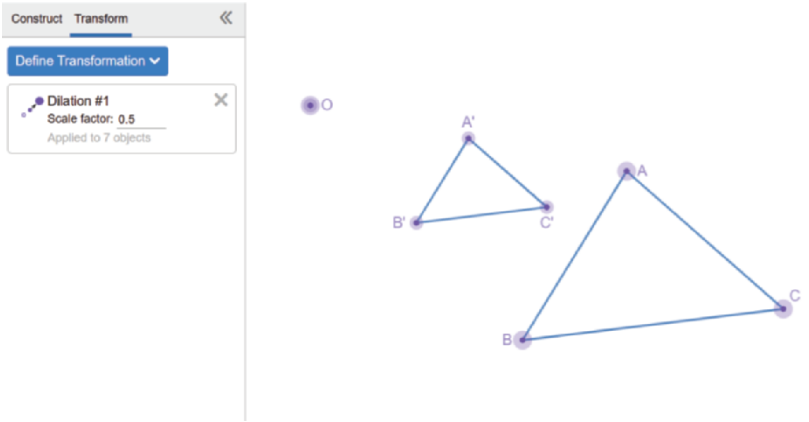



- If you think students would benefit from an additional example, you can challenge them to dilate a different polygon, such as a quadrilateral, by a different scale factor using a different point in its interior as the center of dilation. The procedure will be similar to dilating a triangle.

- Now you can let students know that they will dilate a triangle using a point outside the triangle. Their task is to dilate triangle $ABC$ by a scale factor of $\frac{1}{2}$ using a new center of rotation, again labeled $O$, that is outside the triangle.
- Students using a pencil-and-paper approach should construct a triangle $ABC$ with a center of dilation, $O$, outside the triangle. They should determine the distance between points $O$ and $A$, points $O$ and $B$, and points $O$ and $C$. Then they will have to determine a point, $A'$, along $\overrightarrow{OA}$ so that the distance between points $O$ and $A'$ is half the distance between points $O$ and $A$. Then they should repeat the process to determine a point, $B'$, along $\overrightarrow{OB}$ so that the distance between points $O$ and $B'$ is half the distance between points $O$ and $B$, and a point, $C'$, along $\overrightarrow{OC}$ so that the distance between points $O$ and $C'$ is half the distance between points $O$ and $C$. They should construct segments $\overline{A'B'}$, $\overline{B'C'}$, and $\overline{A'C'}$ to complete the dilated triangle. Finally, they should verify that $m\overline{A'B'} = \frac{1}{2} \bullet m\overline{AB}$, $m\overline{B'C'} = \frac{1}{2} \bullet m\overline{BC}$, and $m\overline{A'C'} = \frac{1}{2} \bullet m\overline{AC}$.




- Students using dynamic geometry software will have to create a triangle $ABC$, label its vertices, and create and label a point $O$ that is outside the triangle. Then they should navigate to the "Transform" tab, pull down the "Define Transformation" menu, and select "Dilation." The program will then prompt them to choose the center of dilation. After students choose the center, they can adjust the scale factor. The program will ask students to select the objects on which to apply the transformation. Once students choose $\Delta ABC$, the program will show an "APPLY" button that will perform the transformation when clicked. As before, students will have to select the "Construct" tab and then label the points as usual.



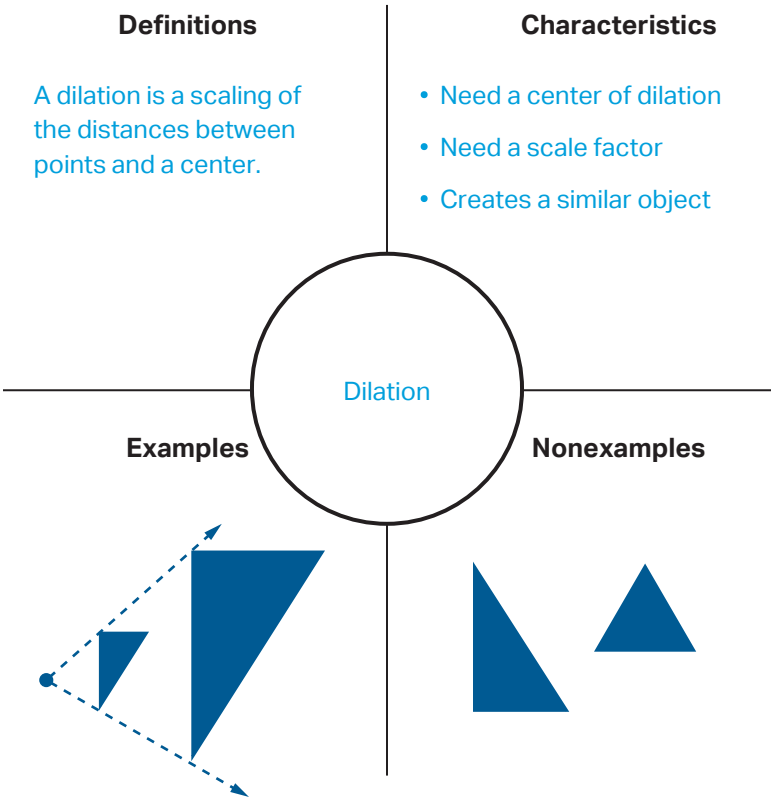

- You can debrief this part of the lesson by returning to the open-ended question from the beginning of the section and asking students to reflect on what happens to the distance between points:
  - If you want to dilate a geometric figure, does the center of dilation have to be on the figure? Can it be inside the figure? Outside the figure?

    You can dilate a figure from any point in the plane, whether it is on the figure, inside the figure, or outside the figure.
  - When you perform a dilation, is the image congruent to the preimage? Why or why not?

    For any scale factor other than 1, the image of a dilation is not congruent to the preimage. The image is similar to the preimage because the distances between pairs of points of the figure are scaled by the same amount, but the angle measures are not changed. This means the figure has the same shape but is a different size.
  - Suppose a dilation has a scale factor of $k$ and the center of dilation is point $P$. You perform the dilation on point $A$, and its image is $A'$. What can you conclude about the distance between points $P$ and $A$ and the distance between points $P$ and $A'$?

    The distance between points $P$ and $A'$ is $k$ times the distance between points $P$ and $A$.

- To end this part of the lesson, students should create a vocabulary organizer (**Handout 3.1.C: Vocabulary Graphic Organizer**) for dilation that is similar to the ones they made for translations, reflections, and rotations. If you have a word wall in your classroom, you can post your own version of the graphic organizer.




## PART 3: DILATING A FIGURE IN THE COORDINATE PLANE

In this part of the lesson, students investigate the effects that a dilation has on the coordinates of a polygon. To keep the algebraic expressions relatively straightforward, the center of dilation for this activity will be the origin. While it is possible to use any point in the plane as the center of dilation, the mathematics required is beyond the scope of the course.

- You can have students work individually or in pairs. Distribute a copy of **Handout 3.4: Exploring Dilations** to each student. The task for students is to determine the effect of the transformation on the coordinates of the vertices (and all other points) of each polygon. You should let students know that all the dilations on the handout use the origin as the center of dilation.

- Once students develop a conjecture about the general effect of a dilation from the origin on the coordinates of the points of the polygon, they can form small groups to verify that other students made the same conjecture.
- Each group should create a summary of the effects of a dilation. They could make a small poster or write them on a personal whiteboard.
- Bring the class back together as a whole and have volunteers share out the descriptions they came up with. Students should have concluded that a dilation of a figure about the origin multiplies the $x$- and $y$-coordinates of each point in the preimage by the scale factor to produce the coordinates of each corresponding point of the image. This can be expressed algebraically in function notation as $D(x, y) = (kx, ky)$, where $k$ is the scale factor.
- You can debrief this part of the lesson by asking students some clarifying questions about dilations.
  - Suppose a dilation has a scale factor of $k$, and points on the image are farther away from the center of dilation than points on the preimage. What can you conclude about the value of the scale factor? How do you know?

    The scale factor would be greater than 1. If each point in the image is farther away from the center of dilation than the corresponding point in the preimage, then the image will be "larger" than the preimage. That means the distances between the center of dilation and the points of the preimage must have been scaled up, or multiplied by a value greater than 1.
  - Suppose a dilation has a scale factor of $k$ and points on the image are closer to the center of dilation than points on the preimage. What can you conclude about the value of the scale factor? How do you know?

    The scale factor would be less than 1. If each point in the image is closer to the center of dilation than the corresponding point in the preimage, then the image will be "smaller" than the preimage. That means the distances between the center of dilation and the points of the preimage must have been scaled down, or multiplied by a value less than 1.
  - Suppose a dilation has a scale factor of 1. What do you think you could conclude about the relationship between the image and the preimage? Is the image larger or smaller than the preimage?

    If the scale factor is 1, then the image is congruent to the preimage, because the distances between the points of the preimage and the center have not changed. The image is the same size as the preimage.

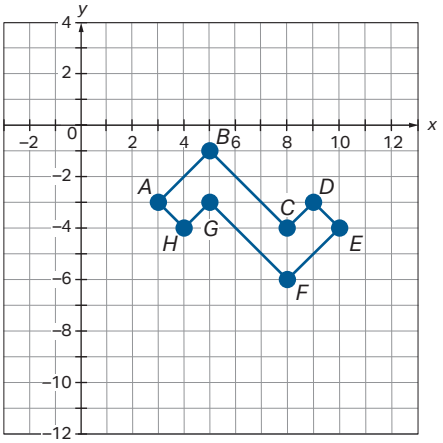
UNIT 3

## ASSESS AND REFLECT ON THE LESSON

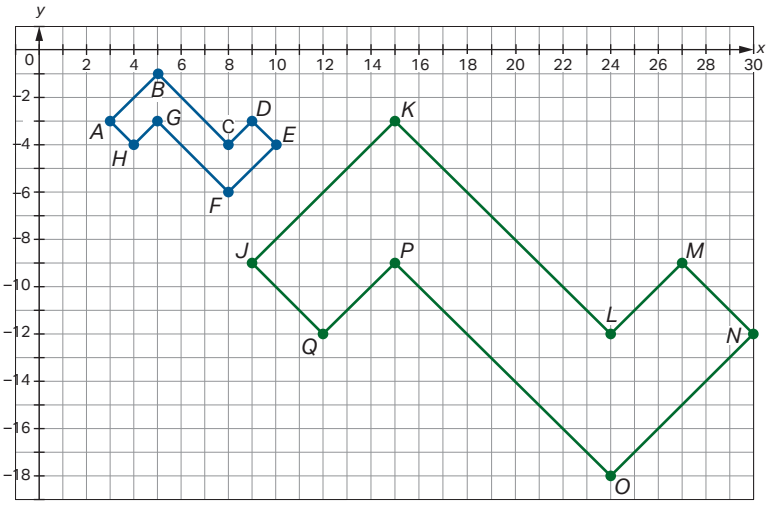
### FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

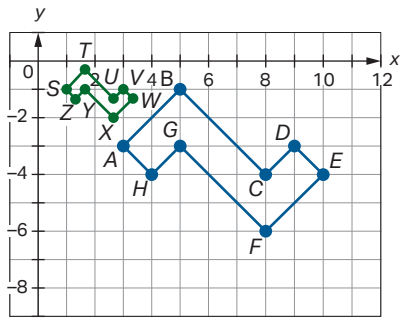
Use the polygon below to answer the questions.




(a) Create a polygon *JKLMNOPQ* that is a dilation of polygon *ABCDEFGH* with a scale factor of 3 about the origin.



(b) Create a polygon *STUVWXYZ* that is a dilation of polygon *ABCDEFGH* with a scale factor of $\frac{1}{3}$ about the origin.




(c) Are polygons *JKLMNOPQ* and *STUVWXYZ* similar to each other? If they are, what is the scale factor by which one can be dilated to coincide with the other?

The polygons are similar. Their angles and shapes are the same, but their sizes are different. *JKLMNOPQ* is 3 times the size of *ABCDEFGH*, and *STUVWXYZ* is $\frac{1}{3}$ the size of *ABCDEFGH*. So, *STUVWXYZ* is a dilation of *JKLMNOPQ* about the origin with a scale factor of $\frac{1}{9}$. Alternatively, *JKLMNOPQ* is a dilation of *STUVWXYZ* about the origin with a scale factor of 9.

## HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

### Handout 3.4: Exploring Dilations

1. Preimage: $A(1, 2)$, $B(2, 6)$, $C(7, 3)$; image: $A'(2, 4)$, $B'(4, 12)$, $C'(14, 6)$
2. Preimage: $A(-2, 1)$, $B(-7, 2)$, $C(-4, 5)$, $D(-1, 3)$; image: $A'(-6, 3)$, $B'(-21, 6)$, $C'(-12, 15)$, $D'(-3, 9)$
3. Preimage: $A(0, 6)$, $B(-4, 0)$, $C(5, 2)$, $D(3, -1)$; image: $A'(0, 3)$, $B'(-2, 0)$, $C'(2.5, 1)$, $D'(1.5, -0.5)$
4. Preimage: $A(-2, -3)$, $B(-5, 4)$, $C(4, 8)$, $D(6, 4)$, $E(3, 5)$; image: $A'(-3, -4.5)$, $B'(-7.5, 6)$, $C'(6, 12)$, $D'(9, 6)$, $E'(4.5, 7.5)$

UNIT 3

LESSON 3.5

# Defining Congruence and Similarity Through Transformation

## OVERVIEW

### LESSON DESCRIPTION

**Part 1: Defining Congruence**

Students formally define congruence through rigid motion transformations.

**Part 2: Defining Similarity**

Students formally define similarity through dilations.

### CONTENT FOCUS

This lesson serves to formalize the definitions of congruence and similarity using transformations. Transformation-based definitions of congruence and similarity are extremely useful because they can be applied to nonpolygonal geometric figures, including those that lack side lengths and angles, such as circles or other figures with curves.

### AREA OF FOCUS

- Engagement in Mathematical Argumentation

### SUGGESTED TIMING

~45 minutes

### LESSON SEQUENCE

- This lesson is part of a lesson sequence (~315 minutes) that includes Lessons 3.1 through 3.5.

### HANDOUTS

**Lesson**

- 3.5.A: Geometric Blobs
- 3.5.B: Are the Figures Congruent?
- 3.5.C: Are the Figures Similar?

### MATERIALS

- patty paper
- ruler
- compass
- graph paper

## COURSE FRAMEWORK CONNECTIONS

| Enduring Understandings | |
|---|---|
| ▪ Transformations are functions that can affect the measurements of a geometric figure.<br>▪ Congruent figures have equal corresponding angle measures and equal distances between corresponding pairs of points.<br>▪ Similar figures have equal corresponding angle measurements, and the distances between corresponding pairs of points are proportional. | |
| **Learning Objectives** | **Essential Knowledge** |
| **3.1.1** Perform transformations on points in a plane. | **3.1.1a** Transformations describe motions in the plane. Analyzing these transformations indicates if and how these motions affect lengths and angle measures of figures. Congruence and similarity are defined in terms of measurements that are preserved by transformations.<br>**3.1.1b** A transformation is a function whose inputs and outputs are points in the plane. A set of all input points of a transformation is called a preimage; a set of all output points of the preimage is called an image.<br>**3.1.1c** A rigid motion transformation preserves both the distance between pairs of points and the angle measures. A similarity transformation preserves angle measures but not necessarily distances between pairs of points. |

| Learning Objectives | Essential Knowledge |
|---|---|
| **3.1.3** Prove that a rigid motion transformation maps an object to a congruent object. | **3.1.3a** A rigid motion transformation is a transformation that preserves distances between pairs of points as well as angle measures.<br>**3.1.3b** A translation is a transformation that maps each point in the plane to an image that is a specified distance in a specified direction from the preimage.<br>**3.1.3c** A reflection is a transformation that maps each point in the plane to its mirror image across a line called the axis of symmetry.<br>**3.1.3d** A rotation is a transformation that maps each point in the plane to an image that is turned by a specified angle about a fixed point called the center of rotation. |
| **3.1.4** Solve problems involving rigid motion transformations. | **3.1.4a** Applying one or more translations, rotations, and reflections maps an object to a congruent object.<br>**3.1.4b** Any transformation that preserves distance between points and angle measures can be written as a sequence of translations, reflections, and/or rotations.<br>**3.1.4c** If two figures are congruent, there must exist a sequence of one or more rigid motion transformations that maps one figure to the other. |
| **3.1.5** Prove that a similarity transformation maps an object to a similar object. | **3.1.5a** A similarity transformation is a sequence of a dilation and/or one or more rigid motion transformations.<br>**3.1.5b** A dilation from a fixed point, called the center, with a scale factor $k$ is a transformation that maps each point in the plane to an image whose distance from the center is $k$ times the distance between the center and the preimage, in the same direction as the preimage.<br>**3.1.5c** Dilations of figures do not affect the angle measures of a figure. |

| Learning Objectives | Essential Knowledge |
|---|---|
| **3.1.6** Solve problems involving similarity transformations. | **3.1.6a** Dilating the plane by a scale factor $k$ with center (0, 0) will scale each coordinate by $k$.<br>**3.1.6c** The scale factor of a dilation can be determined by dividing a length from the image by its corresponding length in the preimage.<br>**3.1.6d** The perimeter of the image of a figure is the perimeter of the preimage scaled by the same scale factor as the dilation. |

**FORMATIVE ASSESSMENT GOAL**

This lesson should prepare students to complete the following formative assessment activity.

1. Is it possible for two congruent polygons to have corresponding sides with different lengths? Explain why or why not using transformations.
2. Is it possible for two similar polygons to have corresponding angles with different measures? Explain why or why not using transformations.

## PART 1: DEFINING CONGRUENCE

This part of the lesson formally defines congruence through rigid motion transformations. Some students will recognize that translations, rotations, and reflections map a preimage to a congruent image because these transformations preserve both the distances between pairs of points and the angle measures of a figure.

- Begin the lesson by asking students to think about how they would know if two shapes are congruent. Give them some time to brainstorm, and record their suggestions. Students might suggest determining if corresponding sides have the same lengths or if corresponding angles have the same measures. Students might suggest trying to transform one figure to coincide with the other, without using dilations.
- Let students know that they will engage in an activity in which they have to determine if and why two figures are congruent. Assign student pairs and distribute a copy of **Handout 3.5.A: Geometric Blobs** to each pair. You could display a copy in a central location in the room for students to reference as needed.

**Instructional Rationale**

This part of the lesson focuses students' attention on geometric figures that do not have measurable angles or line segments. It will help them recognize that transformation-based definitions of congruence and similarity are more generalizable and useful than definitions based on measurements.

- As you circulate around the room while students work, look and listen for the justifications they use. Students may state that the figures have the same size and shape. Encourage students to explain how, exactly, they know the figures have the same size and shape. Students may need some help recognizing that congruent figures can be moved so all corresponding points that make up the figures coincide.
- Once students have finished matching figures and developing explanations for their choices, take some time to debrief the activity. It is important that students state that they know the figures are congruent because one can be moved so all corresponding points that make up the figures coincide.

  The matches are A-G, B-C-L, D-K, E-F-I, and H-J.

- Provide two important definitions for students:

  *A rigid motion transformation is a sequence of translations, rotations, and/or reflections.*

*Two figures are congruent if and only if a rigid motion transformation can map one figure so it coincides with the other.*

**Guiding Student Thinking**

Students may interpret the word *sequence* in the definition of a rigid motion transformation to mean that it must include more than one transformation. When we use the word sequence in everyday speech, we usually mean several things in succession. Mathematically, a *sequence* is an ordered set of mathematical objects. So, a single transformation could be thought of as a sequence, because a transformation is a mathematical object and there is an order to performing it; you perform it first. That it is the only transformation to perform does not matter to the mathematical definition of sequence.

- In the next portion of the lesson, students will explore some triangles and determine if they are congruent. You can ask students some questions like the ones below to help them recall prior knowledge about congruent segments and angles.
  - How can we tell if two line segments are congruent?

    Two line segments are congruent if they have the same length. Two line segments are congruent if one can be translated, rotated, and/or reflected so it coincides with the other.
  - How can we tell if two angles are congruent?

    Two angles are congruent if they have the same measure. Two angles are congruent if one can be translated, rotated, and/or reflected so it coincides with the other.
  - How do you think we could figure out whether or not two triangles are congruent?

    Two triangles are congruent if one can be translated, rotated, and/or reflected so it coincides with the other. This means that the corresponding line segments have the same length and the corresponding angles have the same measure.
- Allow students some time to brainstorm and generate ideas to determine if two triangles are congruent.
- To test their conjectures about how to determine if two triangles are congruent, have students work through the examples on **Handout 3.5.B: Are the Figures Congruent?** You can decide if students should work individually or with a partner. Students should have access to rulers, protractors, compasses, and patty paper.

- Once students have had some time to work, have them share what they decided about each pair of triangles. For this activity, it is not enough that students just determine if the triangles are congruent. They should be able to justify why the triangles are or are not congruent.

**Instructional Rationale**

In this part of the activity, highlight for students that there are two equivalent methods for determining congruence: determining if a rigid motion transformation maps one figure to the other and comparing the measurements of all corresponding parts. It is important that students explain that all six parts—the three corresponding angles and the three corresponding sides—of the triangles are congruent. This will motivate a preference for using rigid transformations for their future explorations of triangle congruence, because justifying that all six parts are congruent is comparatively excessive and unnecessary. Proving that certain combinations of those parts are congruent is all that is necessary.

- Solutions for the handout are provided here:

  (1) $\Delta ABC \cong \Delta XYZ$, because one of the triangles can be transformed without a dilation to coincide with the other. That means the corresponding side lengths and corresponding angle measures are equal, so the triangles are congruent.
  (2) $\Delta RTS \ncong \Delta OQP$, because the lengths of their corresponding sides are not equal. Also, measuring any pair of corresponding angles in the triangles indicates the angles are not congruent. For example $m\angle P \neq m\angle S$, and if one set of corresponding angles is not congruent, then the triangles cannot be congruent.
  (3) $\Delta KMC \cong \Delta RDL$, because one of the triangles can be transformed without a dilation to coincide with the other. Also, the corresponding side lengths and angle measures are equal, so the triangles are congruent.
  (4) $\Delta ABD \ncong \Delta EFH$, because the lengths of their corresponding sides are not equal. The corresponding angles are congruent, which means the triangles are similar but not congruent.

## PART 2: DEFINING SIMILARITY

In Unit 2, students explored similar right triangles. In this part of the lesson, they develop a definition of similarity in terms of rigid motion transformations and dilations that can be applied to figures beyond only right triangles.

- Begin by asking students some questions like the ones below to have them recall prior knowledge about similarity.

- How could we tell if two right triangles were similar?

  Two right triangles were similar if the triangles had congruent corresponding angles and corresponding side lengths were in proportion.

- What transformations have we explored that affect the size of a figure but not its shape?

  Dilations will scale the distances between points and the center of dilation but not change any angle measures. Therefore the shape of the image is the same as the preimage, but the size could be different.

- Let students know that their task will be to analyze pairs of figures to determine if and why they are similar. Assign student pairs and distribute a copy of **Handout 3.5.C: Are the Figures Similar?** to each pair. You could display a copy in a central location in the room.
- As you circulate around the room while students work, look and listen for the justifications they use. Students may state that the figures have the same shape but are different sizes. Encourage students to explain how, exactly, they know the figures are the same shape but are different sizes. Students may need some help recognizing that the similar figures can be moved so all corresponding points that make up the figure would coincide after a dilation of one of the figures.
- Solutions for the handout are provided here:

  (1) $\Delta ABD \sim \Delta EFH$, because the lengths of their corresponding sides are in proportion and their corresponding angles are congruent. Each side of $\Delta EFH$ is 1.5 times larger than the corresponding side of $\Delta ABD$.
  (2) $\Delta ABC \sim \Delta JKL$, because the lengths of their corresponding sides are in proportion and their corresponding angles are congruent. Each side of $\Delta JKL$ is 1.5 times larger than the corresponding side of $\Delta ABC$.
  (3) Quadrilateral *ABCD* is similar to quadrilateral *JKLM*, because one of the quadrilaterals can be reflected so it coincides with the other after a dilation. Each side of *JKLM* is $\frac{1}{2}$ the size of the corresponding side of *ABCD*.
  (4) The pentagons are not similar, because all corresponding angles are not congruent and therefore do not have the same shape.

- Provide two important definitions for students:

  *A similarity transformation is a sequence of translations, rotations, reflections, and/or dilations.*

  *Two figures are similar if and only if a similarity transformation can map one figure so it coincides with the other.*

## ASSESS AND REFLECT ON THE LESSON

### FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

1. Is it possible for two congruent polygons to have corresponding sides with different lengths? Explain why or why not using transformations.

   It is not possible for two congruent polygons to have corresponding sides with different lengths, because rigid motions do not affect the distances between pairs of points. If the polygons are congruent, then corresponding sides must have lengths with equal measures.

2. Is it possible for two similar polygons to have corresponding angles with different measures? Explain why or why not using transformations.

   It is not possible for two similar polygons to have corresponding angles with different measures, because neither dilations nor rigid motions affect angle measures. If the polygons are similar, then corresponding angles must have equal measures.

### HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

**Handout 3.5.A: Geometric Blobs**

See lesson for solutions.

**Handout 3.5.B: Are the Figures Congruent?**

See lesson for solutions.

**Handout 3.5.C: Are the Figures Similar?**

See lesson for solutions.

UNIT 3

PRACTICE PERFORMANCE TASK

# Transformations in the Coordinate Plane

## OVERVIEW

### DESCRIPTION

In this practice performance task, students analyze several transformations in the coordinate plane.

### CONTENT FOCUS

Earlier in this unit, students performed translations, rotations about the origin, reflections across the $x$- and $y$-axes, and dilations about the origin. This practice performance task is designed to extend students' understandings of these basic transformations by investigating previously unexplored axes of reflection, angles of rotation, and centers of dilation.

### AREAS OF FOCUS

- Engagement in Mathematical Argumentation
- Connections Among Multiple Representations

### SUGGESTED TIMING

~45 minutes

### HANDOUTS

Unit 3 Practice Performance Task: Transformations in the Coordinate Plane

### MATERIALS

- scientific calculator
- graphing utility
- graph paper
- straightedge
- compass
- patty paper
- ruler
- protractor
- dynamic geometry software

UNIT 3

## COURSE FRAMEWORK CONNECTIONS

| Enduring Understandings | |
|---|---|
| ▪ Transformations are functions that can affect the measurements of a geometric figure.<br>▪ Congruent figures have equal corresponding angle measures and equal distances between corresponding pairs of points.<br>▪ Similar figures have equal corresponding angle measurements, and the distances between corresponding pairs of points are proportional. | |
| **Learning Objectives** | **Essential Knowledge** |
| **3.1.1** Perform transformations on points in a plane. | **3.1.1a** Transformations describe motions in the plane. Analyzing these transformations indicates if and how these motions affect lengths and angle measures of figures. Congruence and similarity are defined in terms of measurements that are preserved by transformations.<br>**3.1.1b** A transformation is a function whose inputs and outputs are points in the plane. A set of all input points of a transformation is called a preimage; a set of all output points of the preimage is called an image.<br>**3.1.1c** A rigid motion transformation preserves both the distance between pairs of points and the angle measures. A similarity transformation preserves angle measures but not necessarily distances between pairs of points. |
| **3.1.3** Prove that a rigid motion transformation maps an object to a congruent object. | **3.1.3a** A rigid motion transformation is a transformation that preserves distances between pairs of points as well as angle measures.<br>**3.1.3b** A translation is a transformation that maps each point in the plane to an image that is a specified distance in a specified direction from the preimage. |

| Learning Objectives | Essential Knowledge |
|---|---|
| | **3.1.3c** A reflection is a transformation that maps each point in the plane to its mirror image across a line called the axis of symmetry.<br>**3.1.3d** A rotation is a transformation that maps each point in the plane to an image that is turned by a specified angle about a fixed point called the center of rotation. |
| **3.1.4** Solve problems involving rigid motion transformations. | **3.1.4a** Applying one or more translations, rotations, and reflections maps an object to a congruent object.<br>**3.1.4b** Any transformation that preserves distance between points and angle measures can be written as a sequence of translations, reflections, and/or rotations.<br>**3.1.4c** If two figures are congruent, there must exist a sequence of one or more rigid motion transformations that maps one figure to the other. |
| **3.1.5** Prove that a similarity transformation maps an object to a similar object. | **3.1.5a** A similarity transformation is a sequence of a dilation and/or one or more rigid motion transformations.<br>**3.1.5b** A dilation from a fixed point, called the center, with a scale factor $k$ is a transformation that maps each point in the plane to an image whose distance from the center is $k$ times the distance between the center and the preimage, in the same direction as the preimage.<br>**3.1.5c** Dilations of figures do not affect the angle measures of a figure. |
| **3.1.6** Solve problems involving similarity transformations. | **3.1.6a** Dilating the plane by a scale factor $k$ with center (0, 0) will scale each coordinate by $k$.<br>**3.1.6c** The scale factor of a dilation can be determined by dividing a length from the image by its corresponding length in the preimage. |

## SUPPORTING STUDENTS

### BEFORE THE TASK

In this practice performance task, students are expected to use their understanding of transformations in the coordinate plane to identify some of the necessary characteristics of each transformation.

- To prepare students to engage in the task, you could begin the class by posing some questions to them. These could be part of a warmup or could be asked verbally.
  - What is a rotation? What information do you need to perform a rotation?

    A rotation is when a point or figure is turned about a center. To perform a rotation, you need an angle measure, a direction, and a center of rotation.
  - What is a reflection? What information do you need to perform a reflection?

    A reflection is when a point or figure is flipped over an axis. To perform a reflection, you need an axis of symmetry.
  - What is a translation? What information do you need to perform a translation?

    A translation is when a point or figure is slid in the plane without turning or flipping. To perform a translation, you need a direction and a distance.
  - What is a dilation? What information do you need to perform a dilation?

    A dilation is when a point or figure is scaled up or down about a center. To perform a dilation, you need a scale factor and a center of dilation.

### DURING THE TASK

The goal of this task is for students to extend their understanding about transformations in the plane. The transformations in the task use centers or axes that may be unfamiliar to students. Because the mathematics involved is necessarily more challenging than students' previous experience with transformations, you could choose to implement the task as a small group experience.

- Students could work in pairs or groups of three to complete the task. There is ample work and enough potential discussion areas for up to, but not more than, three students in each group. In larger groups, some students may not have an opportunity to meaningfully engage in the task.
- You could chunk the task into four parts and have students complete one part at a time. Students could check their solutions with you or the scoring guidelines before moving on to the next part. During the check, spend a few moments discussing the solution with the student. Focus on what changes, if any, students could make to their solution to craft a more complete response the next time they engage in a performance task.

During the task, students might ask about which tools they can use to answer the questions. Some students will be very comfortable with the coordinate plane as presented, but others might want to draw graphs of the preimage and images on their own coordinate planes. Some students will want to make direct measurements with a ruler or protractor. Some students may prefer indirect measurements using the classical construction tools of a straightedge and compass. Still other students may want to use a graphing utility to digitally construct their own versions of the transformations. You will have to exercise your professional judgment about which tools students should and should not use. The implementation of this task should be adapted to your particular classroom and available materials.

### AFTER THE TASK

Whether you decide to have students score their own solutions, have them score their classmates' solutions, or score the solutions yourself, the results of the practice performance task should be used to inform instruction.

Students should understand that converting their score into a percent does not provide a good measure of how they performed on the task. You can use the suggested scoring conversion guide that follows the scoring guidelines to discuss their performance.

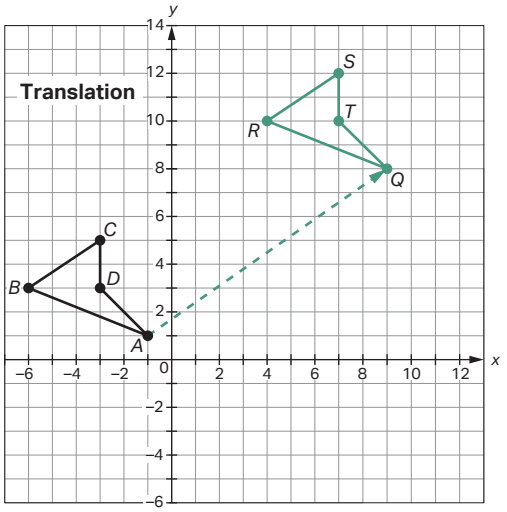
UNIT 3

## SCORING GUIDELINES

There are 12 possible points for this performance task.

**Student Stimulus and Part (a)**

(a) Which quadrilateral could be the image of *ABCD* under a translation? Describe the components of the translation that maps *ABCD* to this image.

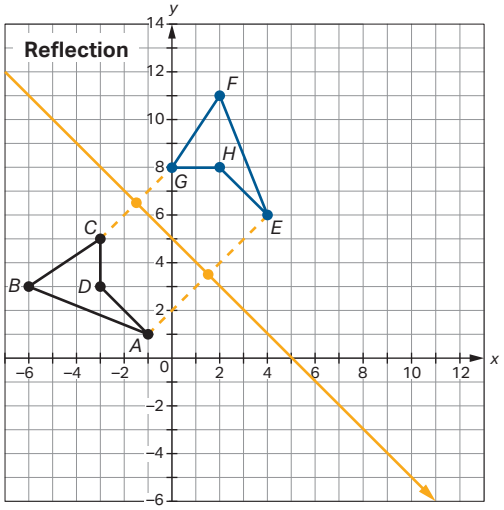
| Sample Solutions | Points Possible |
|---|---|
| Quadrilateral *QRST* could be the image of quadrilateral *ABCD* under a translation.<br>  <br><br>The segment joining any two corresponding points of the preimage and image can be used to determine the direction and distance of the translation.<br>There are two possible descriptions of the translation that maps *ABCD* to *QRST*.<br>Description 1:<br>The distance is the square root of 149 units along a line with slope of 0.7.<br>Description 2:<br>Translate all points of *ABCD* vertically by 7 units and horizontally by 10 units. | **3 points maximum**<br>1 point for identifying quadrilateral *QRST* as the image of *ABCD* under a translation<br>1 point for identifying the distance of the translation<br>1 point for describing the direction of the translation<br>*Scoring note*: If students state the horizontal and vertical components of the translation, they should receive 1 point for each correct component of the distance instead of separate points for distance and direction. |

| Targeted Feedback for Student Responses |
|---|
| If students struggle with part (a), it could be because they do not understand how to separate the components of a translation. Translations in the coordinate plane can be described using the horizontal and vertical components of the transformation. |

TEACHER NOTES AND REFLECTIONS

## Student Stimulus and Part (b)

(b) Which quadrilateral could be the image of *ABCD* under a reflection? Describe the components of the reflection that maps *ABCD* to this image.

| Sample Solutions | Points Possible |
|---|---|
| Quadrilateral *EFGH* could be the image of quadrilateral *ABCD* under a reflection.   | **3 points maximum**<br>1 point for identifying quadrilateral *EFGH* as the image of *ABCD* under a reflection<br>1 point for identifying the axis of symmetry<br>1 point for writing an equation for the axis of symmetry |

*continues*

| | |
|---|---|
| The perpendicular bisector of a segment joining any two corresponding points is the axis of symmetry. An equation for the axis of symmetry is $y = -1x + 5$. Reflecting all points that make up *ABCD* across the line $y = -1x + 5$ will yield *EFGH*. | |

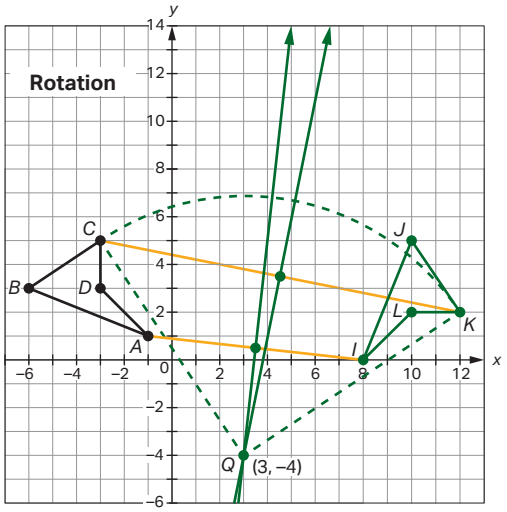
**Targeted Feedback for Student Responses**

Students may experience some difficulty determining where the axis of symmetry is located. You can suggest that they fold the graph so all corresponding vertices coincide.

TEACHER NOTES AND REFLECTIONS

### Student Stimulus and Part (c)

(c) Which quadrilateral could be the image of *ABCD* under a rotation? Describe the components of the rotation that maps *ABCD* to this image.

| Sample Solutions | Points Possible |
|---|---|
| Quadrilateral *IJKL* could be the image of quadrilateral *ABCD* under a rotation.<br>  <br><br>The perpendicular bisectors of any two segments whose endpoints are corresponding points of the preimage and image will intersect at the center of rotation, indicated here by point *Q*.<br>Rotating all points in *ABCD* around the point *Q* (3, –4) at an angle of 90° clockwise will yield *IJKL*.<br>Alternatively, students may identify that *ABCD* can be rotated about any other specified point by an angle of 90° clockwise and then translated to coincide with *IJKL*. | **3 points maximum**<br>1 point for identifying quadrilateral *IJKL* as the image of *ABCD* under a rotation<br>1 point for identifying the center of rotation<br>1 point for identifying the angle measure of the rotation<br>*Scoring note*: If students identify a rotation about a specific point by 90° and then translate the result, they should receive 1 point for correctly identifying the rotation components and 1 point for correctly identifying the translation components. |

**Targeted Feedback for Student Responses**

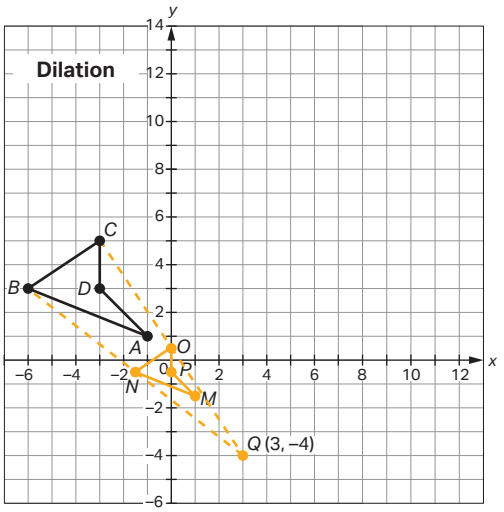
Students may need support to determine the location of the center of rotation. You can suggest that they try to rotate the figure about a center of their choosing, like the origin or a vertex of the figure, and then translate it to coincide with the image.

UNIT 3

TEACHER NOTES AND REFLECTIONS

## Student Stimulus and Part (d)

(d) Which quadrilateral could be the image of *ABCD* under a dilation? Describe the components of the dilation that maps *ABCD* to this image.

| Sample Solutions | Points Possible |
|---|---|
| Quadrilateral *MNOP* could be the image of quadrilateral *ABCD* under a dilation.<br>  <br><br>Any two corresponding points of the preimage and image will lie on a ray whose endpoint is the center of dilation. To find the endpoint of the ray, determine the intersection point of two lines that contain corresponding points of the preimage and image.<br>The center of dilation is (3, –4). The scale factor of the dilation is $\frac{1}{2}$. | **3 points maximum**<br>1 point for identifying quadrilateral *MNOP* as the image of *ABCD* under a dilation<br>1 point for identifying the center of dilation<br>1 point for identifying the scale factor |

**Targeted Feedback for Student Responses**

If students cannot find the center of the dilation, you may need to remind them that any pair of corresponding points of the preimage and image will be collinear with the center. Identifying two lines through such pairs will indicate where the center is located. Students who state that the scale factor is 2, not $\frac{1}{2}$, may have a misunderstanding about how to calculate the scale factor. The scale factor is determined by the ratio:

$$\frac{\text{distance from image point to center}}{\text{distance from preimage point to center}}.$$

TEACHER NOTES AND REFLECTIONS

| Points Received | Appropriate Letter Grade (If Graded) | How Students Should Interpret Their Score |
|---|---|---|
| 11 or 12 points | A | "I know all of this geometry really well." |
| 8 to 10 points | B | "I know all of this geometry well, but I made a few mistakes." |
| 5 to 7 points | C | "I know some of this geometry well, but not all of it." |
| 2 to 4 points | D | "I only know a little bit of this geometry." |
| 0 or 1 point | F | "I don't know much of this geometry at all." |

UNIT 3

LESSON 3.6

# The Equation of a Circle

## OVERVIEW

### LESSON DESCRIPTION

**Part 1: Finding Points on a Circle**
Students use the Pythagorean theorem to determine the coordinates of points on a circle with a radius of 1 unit.

**Part 2: Writing the Equation of a Circle Whose Center Is the Origin**
Students use the Pythagorean theorem in the coordinate plane to write an equation for the coordinates of all points that lie on a circle whose center is the origin.

**Part 3: Writing the General Equation of a Circle**
Students extend their thinking from Part 2 to write a general equation for any circle in the coordinate plane.

### CONTENT FOCUS

In this lesson, students explore how to write an equation for a circle. The mathematics necessary to derive the equation is directly related to the Pythagorean theorem in the coordinate plane and, therefore, to the distance formula. This lesson expects students to understand and justify why the equation for a circle, the Pythagorean theorem, and the distance formula are closely related. This lesson sets up some of the algebraic reasoning necessary for solving systems of equations that involve circles and lines.

### AREAS OF FOCUS

- Connections Among Multiple Representations
- Engagement in Mathematical Argumentation

### SUGGESTED TIMING

~90 minutes

### LESSON SEQUENCE

This lesson is part of a lesson sequence (~180 minutes total) with Lessons 3.7 and 3.8.

### HANDOUTS

**Lesson**

- 3.6.A: Unit Circle with Coordinates
- 3.6.B: Unit Circle Without Coordinates

### MATERIALS

- straightedge/ruler
- calculator

## COURSE FRAMEWORK CONNECTIONS

| Enduring Understandings | |
|---|---|
| ▪ The geometry of a circle is completely determined by its radius. | |
| **Learning Objectives** | **Essential Knowledge** |
| **3.3.1** Determine whether a particular point lies on a given circle. | **3.3.1a** A point in the coordinate plane lies on a circle if its coordinates satisfy the equation of a circle.<br>**3.3.1b** All points that lie on a circle are equidistant from the center of the circle. |
| **3.3.2** Translate between the geometric and algebraic representations of a circle. | **3.3.2a** A circle is the set of all points equidistant from a given point.<br>**3.3.2b** In the coordinate plane, the graph of the equation $(x-h)^2+(y-k)^2=r^2$ is the set of all points located $r$ units from the point $(h,k)$. This is a circle with radius $r$ and center $(h,k)$. |

## FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

(a) Mason says that a point with coordinates $(4,4)$ lies on a circle centered at the origin with a radius of 4 units. Is Mason correct? Explain why or why not.

(b) Susan knows that the coordinates of the center of a circle and a point on the circle are $(-3, 2)$ and $(10, -2)$, but she can't remember which coordinate pair is which. What could be an equation of the circle?

UNIT 3

## PART 1: FINDING POINTS ON A CIRCLE

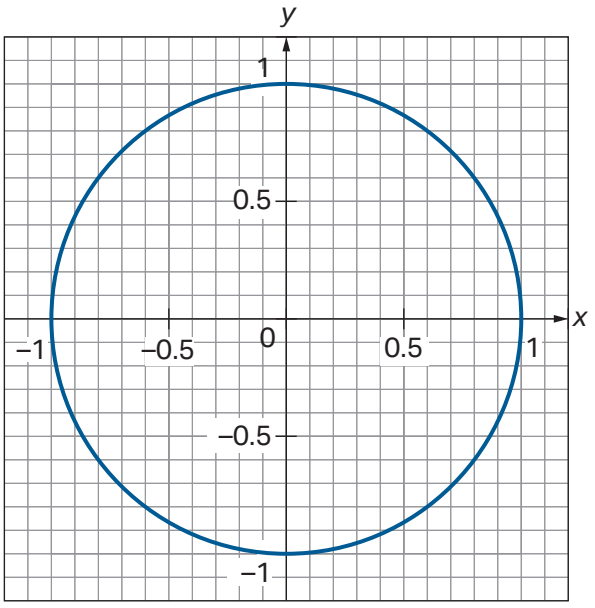
In this part of the lesson, students use their understanding of the Pythagorean theorem in the coordinate plane to explore the graph of a circle. They work to determine the coordinates of points on the circle. Part 1 of the lesson is a critical first step toward understanding how to write the general equation of a circle.

- Begin by asking students some questions to activate their prior knowledge about circles:
  - What are some distinguishing characteristics of a circle? What makes a circle a special kind of geometric figure?

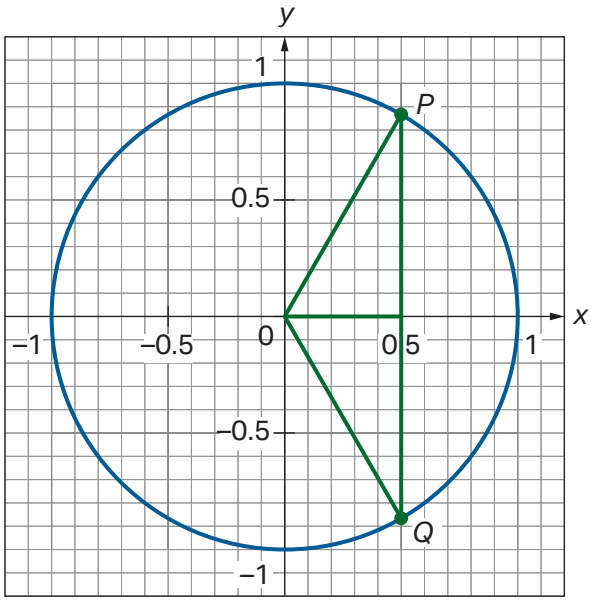
    A circle is the set of all points in a plane equidistant from a given point. All circles are similar to each other.

- Distribute **Handout 3.6.A: Unit Circle with Coordinates** to each student, and also display it for the class to see. Ask students what they notice about the graph of the circle. You can give students a few minutes to think about it. Some students will benefit from writing down their observations. You could have students who sit close to each other share their observations with each other before sharing with the class. Record students' observations in a central location.

  Students may observe that the circle has reflection symmetry across the $y$-axis and across the $x$-axis. Students may observe that the radius of the circle is 1 unit. Students may observe that the circle has two $x$-intercepts and two $y$-intercepts, and that the intercepts in each pair are opposites of each other. Students may observe that the center of the circle is the origin.



- You can let students know that a circle with a radius of 1 unit is called a *unit circle*. In Algebra 2, they will explore the unit circle and its relationship to trigonometry in more detail.
- If students have not yet identified the intercepts, ask them to determine a few points on the circle whose coordinates can be identified just by looking at the graph. They should state that the easiest-to-find coordinates are (1, 0), (0, 1), (–1, 0), and (0, –1).
- Ask students if there are any other points on the circle whose coordinates can be determined by inspection. Students may guess some coordinates, but they should begin to recognize that the coordinates they guess are approximate rather than exact.
- Once students realize that the coordinates of points other than the $x$- and $y$-intercepts cannot be determined by inspection, ask them for ideas about how they might find the exact coordinates. Let students work in pairs to brainstorm what they might do.
- To get students started, you can ask them to determine the $y$-coordinates of the points in the circle where the $x$-coordinate is $\frac{1}{2}$. Some students may need help drawing a right triangle whose hypotenuse is the radius of the circle, whose legs are horizontal and vertical segments, and whose vertical leg has an $x$-coordinate of $\frac{1}{2}$. Other students may need help to figure out that the hypotenuse of the right triangle is the radius of the circle.



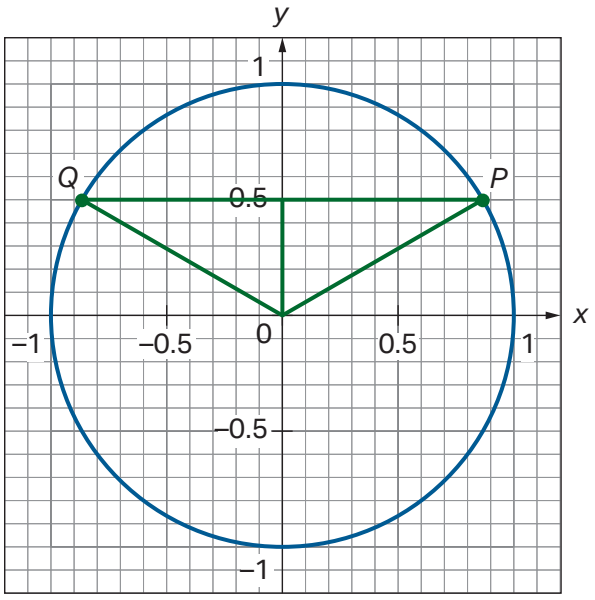

$$\left(\frac{1}{2}\right)^2 + y^2 = 1^2$$
$$y^2 = \frac{3}{4}$$
$$y = \pm\sqrt{\frac{3}{4}}$$
$$y = \pm\frac{\sqrt{3}}{2}$$

The coordinates of the points on the circle where the $x$-coordinate is $\frac{1}{2}$ are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

- To emphasize that this process can be used to find either the $x$- or $y$-coordinate, ask students to find the points where the $y$-coordinate is $\frac{1}{2}$. As before, some students may need help drawing a right triangle whose hypotenuse is the radius of the circle, whose legs are horizontal and vertical segments, and whose horizontal leg has a $y$-coordinate of $\frac{1}{2}$. Other students may need help to figure out that the hypotenuse of the right triangle is the radius of the circle.

**Meeting Learners' Needs**

Encourage students to use fractions instead of decimals. This will help them develop their fraction arithmetic. Some students may need to review how to simplify square roots, and especially square roots of fractions. This is an appropriate point in the lesson to spend some time on that skill.




$$x^2 + \left(\frac{1}{2}\right)^2 = 1^2$$

$$x^2 = \frac{3}{4}$$

$$x = \pm\sqrt{\frac{3}{4}}$$

$$x = \pm\frac{\sqrt{3}}{2}$$

The coordinates of the points on the circle where the $y$-coordinate is $\frac{1}{2}$ are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

- You can continue to repeat this process of indicating a specific $x$- or $y$-coordinate and asking students to find the corresponding $y$- or $x$-coordinate until they recognize that drawing a right triangle and using the Pythagorean theorem will be a useful technique for finding the coordinates of any point on the circle.

**Instructional Rationale**

It is worthwhile to have students do a sufficient number of examples so they acknowledge the utility of the Pythagorean theorem in finding the coordinates of points on the circle. It is critical for students to understand that the repeated process of calculating coordinate pairs using the Pythagorean equation will lead to a general equation for all the points on the circle.

- Students should find the exact value of each coordinate, but they can use a calculator to find a decimal approximation of each exact value to verify the location of the point on the graph.

## PART 2: WRITING THE EQUATION OF A CIRCLE WHOSE CENTER IS THE ORIGIN

In this part of the lesson, students generalize their use of the Pythagorean theorem to generate an equation for a circle whose center is the origin. Students will use the equation of a circle to determine whether a point lies on the circle.

- On **Handout 3.6.B: Unit Circle Without Coordinates**, ask students to choose any point on the circle. Encourage students to choose points that are not in the first quadrant. They should label this point $(x, y)$.

**Meeting Learners' Needs**

As a challenge, you can ask students to determine the coordinates of the point (or points) on the circle where the $x$- and $y$-coordinates are equal. This leads to the equation:

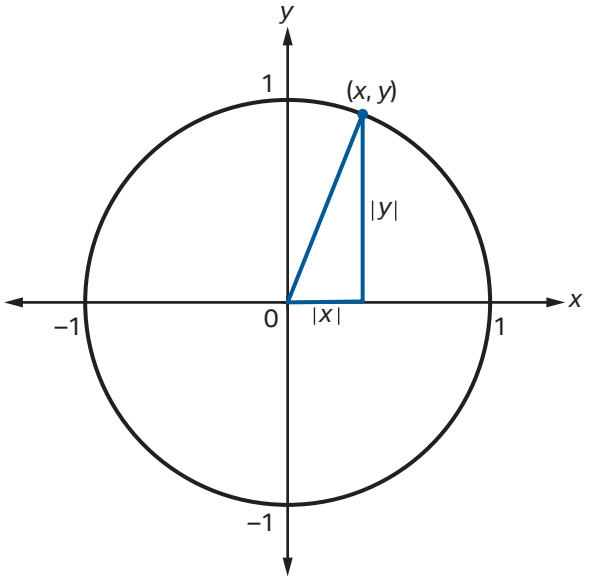
$$a^2 + a^2 = 1^2$$
$$2a^2 = 1$$
$$a^2 = \frac{1}{2}$$
$$a = \pm\sqrt{\frac{1}{2}}$$

Therefore, the coordinates of points whose $x$- and $y$-coordinates are equal are $\left(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right)$ and $\left(-\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}\right)$.

**Guiding Student Thinking**

Students might struggle with the idea of a generic point on the circle. A few students might want to include the appropriate signs for each quadrant. You might have to assure them that they can choose any point and that the signs of the $x$- and $y$-coordinates will not impact their calculations. Students will understand this better after they draw a right triangle whose hypotenuse is the radius of the circle and whose vertices include the origin and a point on the $x$-axis (or the $y$-axis). The critical component of the ensuing activity and debrief is students' coming to understand that every point on the unit circle is a distance of 1 unit from the origin. This characteristic is what makes it possible to write an equation for the circle.

- Ask students to draw a right triangle whose hypotenuse is the radius of the circle from (0, 0) to $(x, y)$. It is customary to draw the horizontal leg so it coincides with the $x$-axis, but this is not strictly necessary.




- Ask students some questions like the one below to get them thinking about how to generalize the process they used in Part 1 of the lesson. Students can work in small groups to answer the questions and then report back to the whole class.
  - What do we know about the radius of the circle? How can we determine the horizontal and vertical lengths of the triangle?

    The radius of the circle is 1 unit. We know the horizontal and vertical lengths of the triangle, because they will be the absolute value of the $x$- and $y$-coordinates, respectively, of the point we chose.

- Ask students to write a Pythagorean equation for the triangle they drew. Students should write the equation $x^2 + y^2 = 1$.

**Guiding Student Thinking**

Some students might have difficulty determining the vertical or horizontal lengths of the triangle because it is oriented in the negative direction horizontally or vertically. Distance is defined to be nonnegative, so when students subtract the $x$- or $y$-coordinates of two points, they should attend to the order of the subtraction so the result is positive. You could let students know that could be slightly less strict about the order of the subtraction if they remember to take the absolute value of the difference. That is, $|a-b| = |b-a|$ for all real numbers $a$ and $b$.

- Have students reflect on the equation for a moment by asking them what a solution to this equation means.

  A solution to this equation is an ordered pair $(x, y)$ that makes the equation true. The equation is true when the sum of the $x$-value squared and the $y$-value squared is equal to 1.

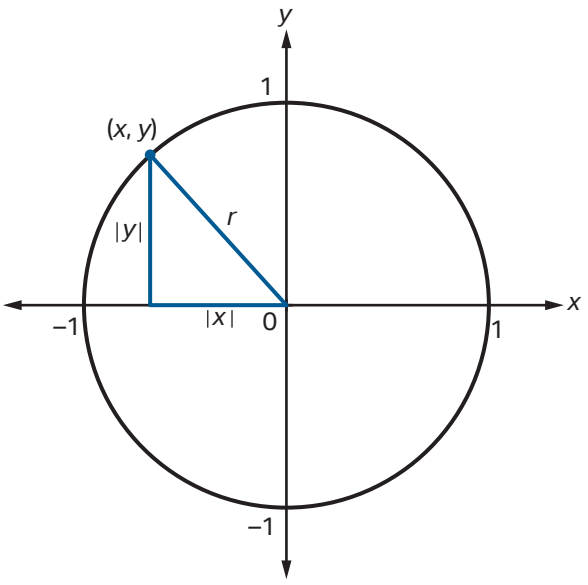
- Next, ask students to investigate how their equation would change if the radius of the circle was a value other than 1. You can assign students different radius values, or you can let them choose their own values.
- To debrief this part of the lesson, ask students to share the radius values they chose and the equations they obtained. To make it a little easier to see a pattern emerge, you can organize the students' radii and equations in a table, as shown:

| Radius | Equation |
|---|---|
| 1 | $x^2 + y^2 = 1$ |
| 2 | $x^2 + y^2 = 4$ |
| 3 | $x^2 + y^2 = 9$ |
| 4 | $x^2 + y^2 = 16$ |
| 5 | $x^2 + y^2 = 25$ |
| … | … |
| $r$ | $x^2 + y^2 = r^2$ |

- Ask students to closely observe the table and write down anything they notice. You might have to prompt students to look at the value of the radius and the constant term in the equation.

  Every equation has an $x^2$-term and a $y^2$-term. Every equation has the radius squared as the constant term.

- You can conclude this part of the lesson by asking students to justify why the radius of the circle would be squared in the equation. It might help students to see a diagram of a circle in the coordinate plane without tick marks on the axes:




**Meeting Learners' Needs**

For students who are struggling to get started, you can suggest that they draw a circle on graph paper or unlined paper; draw a right triangle whose hypotenuse is a radius of the circle, whose legs are horizontal and vertical segments, and that has a point on the circle whose coordinates are $(x, y)$; and repeat the process of writing a Pythagorean equation for the triangle. If students need more prompting, you can suggest some specific $x$- and $y$-values, like in Part 1 of the lesson. Some students will need to repeat the entire solution process with another set of values before seeing the regularity in the reasoning.

Because the radius of the circle is the hypotenuse of a right triangle whose legs are horizontal and vertical segments, it will be squared just like the corresponding term of the Pythagorean equation.

## PART 3: WRITING THE GENERAL EQUATION OF A CIRCLE

In this part of the lesson, students extend the work they did with the unit circle and its equation to finding the general equation of any circle. They adapt their approach to circles that are not centered at the origin. As in the previous two parts of the lesson, students start with some numerical examples to establish a pattern and then generalize the pattern.

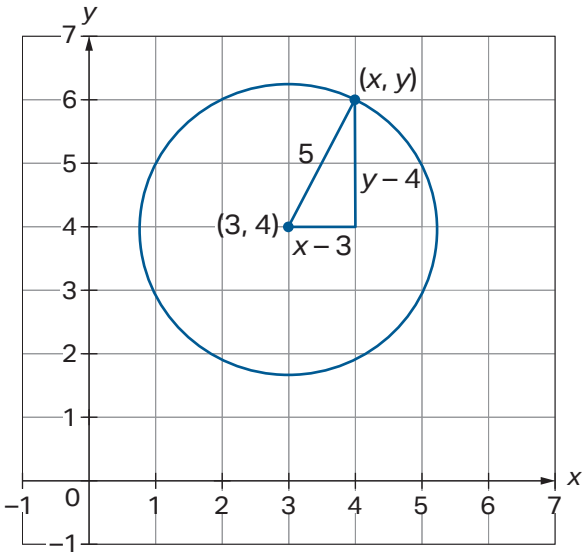
- Begin this part of the lesson by posing a question to the class to motivate the activity:
  - Do you think the center of the circle has to be at the origin? Where else could the center of the circle be located? How do you think the location of the center would affect the equation for the circle?
- Let students ponder the question for a few moments. If students want to volunteer suggestions for how the equation might (or might not) be different, you can record them for the class.
- Pair the students and give each pair some graph paper, an ordered pair for the center of their circle, and a radius. Some students might want to draw a circle using a compass, so you might want to make them available.
- As in the previous parts of the lesson, you can scaffold this investigation. For students who would benefit from a numerical example, specify an $x$- or $y$-coordinate for a point on the circle, and ask students to find the corresponding $y$- or $x$-coordinate. For students who may not need a numerical example to set the pattern, you can ask them to write an equation for the circle. If students need more help to get started, you can point them to the process of identifying a point $(x, y)$ on the circle, drawing a right triangle, and writing a Pythagorean equation with those values. You might have to let students know that they do not need to perform any of the squaring to write the equation in a different form.

**Classroom Ideas**

If you want students to pay attention to the center and not be potentially distracted by the radius, you could assign the same radius to each pair of students.

**Guiding Student Thinking**

Some students will be more comfortable moving toward a generalization if they first establish a pattern with more familiar numbers. Because algebra can be thought of as arithmetic with unknown values, performing the operations with known values will help them understand what is being generalized.




- As you circulate around the room, help students reach the understanding that the horizontal leg length is obtained by subtracting the $x$-coordinates of the center and a point on the circle and vertical leg length is obtained by subtracting the $y$-coordinates of the center and a point on the circle. Once students have written an equation for their circle, have them share their equations. Record the equations in a central location. Some sample centers and radii are shown here:

| Center | Radius | Equation |
|---|---|---|
| $(3, 4)$ | 5 | $(x-3)^2+(y-4)^2=25$ |
| $(-5, 8)$ | 3 | $(x-{}^{-}5)^2+(y-8)^2=9$ |
| $(7, -1)$ | 2 | $(x-7)^2+(y-{}^{-}1)^2=4$ |
| $\left(\frac{1}{2}, \frac{3}{4}\right)$ | 10 | $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{3}{4}\right)^2=100$ |
| … | … | … |
| $(h, k)$ | $r$ | $(x-h)^2+(y-k)^2=r^2$ |

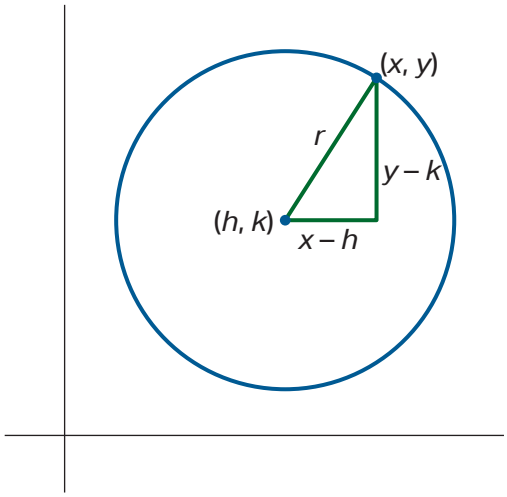
- Ask students to closely observe the table and write down anything they notice.

  Every equation shows the center of the circle in the equation. The $x$-coordinate of the center is subtracted from $x$, and the $y$-coordinate is subtracted from $y$. Every equation has the radius squared as the constant term.

- You can conclude this part of the lesson by asking students to justify why the center of the circle would be subtracted from the $x$- and $y$-variables. It might help students to see a diagram of a circle in the coordinate plane without tick marks on the axes.

**Classroom Ideas**

Students could quickly enter their results in an interactive document or spreadsheet that you create and display. All students would be able to populate it on their own, leaving additional class time available for more debrief discussion.




The endpoints of a radius of a circle are the center of the circle and a point on the circle. A Pythagorean equation will use those two points as the endpoints of the hypotenuse of a right triangle whose legs are horizontal and vertical segments. Subtracting the corresponding $x$- and $y$-coordinates of those points, squaring their respective differences, and setting their sum equal to the radius/hypotenuse squared relates these quantities using the Pythagorean equation.

The end of this lesson is a good place for students to engage in some practice exercises to develop procedural fluency in writing the equation of a circle.

## ASSESS AND REFLECT ON THE LESSON

### FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

(a) Mason says that a point with the coordinates $(4, 4)$ lies on a circle centered at the origin with a radius of 4 units. Is Mason correct? Explain why or why not.

Mason is not correct. A circle whose radius is 4 and whose center is at the origin will have the equation $x^2 + y^2 = 16$. The point $(4, 4)$ does not make the equation true: $4^2 + 4^2 \neq 16$. (Alternatively, the distance from the origin to the point $(4, 4)$ is not 4, and therefore the point $(4, 4)$ cannot be on the circle.)

(b) Susan knows that the coordinates of the center of a circle and a point on the circle are $(-3, 2)$ and $(10, -2)$, but she can't remember which coordinate pair is which. What could be an equation of the circle?

The distance between $(-3, 2)$ and $(10, -2)$ is $\sqrt{185}$. The circle whose center is $(-3, 2)$ and that contains $(10, -2)$ will have the equation $(x+3)^2 + (y-2)^2 = 185$. The circle whose center is $(10, -2)$ and that contains $(-3, 2)$ will have the equation $(x-10)^2 + (y+2)^2 = 185$.

### HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

#### Handout 3.6.A: Unit Circle with Coordinates

This handout should be projected during Part 1 of the lesson. See lesson plan for details.

#### Handout 3.6.B: Unit Circle Without Coordinates

This handout should be projected during Part 2 of the lesson. See lesson plan for details.

LESSON 3.7

UNIT 3

# Intersections of Circles and Lines

## OVERVIEW

### LESSON DESCRIPTION

**Part 1: Watering the Lawn**
Students investigate a contextual scenario involving the circular range of a lawn sprinkler. The scenario leads to a system of equations involving a circle and a line that intersects it twice.

**Part 2: Solving a System of Equations**
Students solve the system of equations they developed from the scenario in Part 1. The algebra in this part of the lesson connects solving systems of equations with solving quadratic equations.

**Part 3: Determining the Area of the Wet Sidewalk**
Students use systems of equations and area formulas to determine a solution to the contextual scenario from Part 1.

### CONTENT FOCUS

In this lesson students are introduced to solving systems of equations that involve a line and a circle. This ties together the algebraic methods of solving a system of linear equations with solving quadratic equations. As with the solutions to systems of linear equations, the solutions to the systems in this lesson are the intersection points of the graphs involved. These graphs are a circle and line, rather than two lines, which leads to different numbers of possible solutions. Similar to quadratic equations or a system of equations involving a linear equation and a quadratic equation, there can be zero, one, or two solutions to the systems in this lesson.

### AREAS OF FOCUS

- Connections Among Multiple Representations
- Greater Authenticity of Applications and Modeling

### SUGGESTED TIMING

~45 minutes

### LESSON SEQUENCE

This lesson is part of a lesson sequence (~180 minutes total) with Lessons 3.6 and 3.8.

### HANDOUTS

**Lesson**
- 3.7.A: The Sprinkler

**Practice**
- 3.7.B: Lines Intersecting Circles

### MATERIALS

- graph paper
- straightedge/ruler
- calculator
- graphing utility

UNIT 3

## COURSE FRAMEWORK CONNECTIONS

| Enduring Understandings | |
|---|---|
| ▪ The geometry of a circle is completely determined by its radius. | |
| **Learning Objectives** | **Essential Knowledge** |
| **3.3.8** Solve a system of equations consisting of a linear equation and the equation of a circle. | **3.3.8a** The intersection of a line and a circle corresponds to an algebraic solution of the system of their corresponding equations.<br>**3.3.8b** An algebraic solution to a system of equations is an ordered pair that makes all equations true simultaneously. The system may have zero, one, or two solutions. |

## FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

Determine the intersection points of the circle and line algebraically. Use a graphing utility to verify your calculations.

$$\begin{cases}(x-3)^2+(y-1)^2=169\\ y=-2.4x+8.2\end{cases}$$

## PART 1: WATERING THE LAWN

The lesson opens with a scenario involving the range of a garden sprinkler with a circular spray pattern. The context is intended to have students generate equations that model one edge of the sidewalk and the range of the sprinkler. In the second part of the lesson, students will learn how to solve a system involving a circle and a line.

- Begin by displaying the sprinkler scenario for students. The scenario is also on **Handout 3.7.A: The Sprinkler** if you would prefer students to have their own copies.
- Allow students some time to closely observe and analyze the scenario and diagram. Ask students to make some suggestions about how they might estimate the area of the sidewalk that is wet.

  Students might suggest that the wet section of the sidewalk is roughly in the shape of a trapezoid.

- You may have to focus students' attention on estimating the area of the wet section of sidewalk with a trapezoid. Students might suggest estimating the area using a different shape, like a rectangle, whose area would likely be easier to calculate. However, a trapezoid is a better choice because it more closely approximates the wet area. Some students may object to a trapezoid, saying that a trapezoid is not the exact shape of the wet section of sidewalk. This is true, and it is important to validate that objection. One characteristic of mathematical modeling is making choices about the scenario being modeled and analyzing the result. Mathematics is exact, but the real world is usually far less precise. As a result, mathematical models often include some error. The class will have to decide if the error of the trapezoid is tolerable.

> **Meeting Learners' Needs**
> Some students will need to see a trapezoid superimposed on the wet section of sidewalk before they are convinced that a trapezoid will result in a good approximation of the area. You can project or display an image of the figure from **Handout 3.7.A: The Sprinkler** and have a student draw a trapezoid.

- Once students agree that a trapezoid is the best figure for approximating the area, ask them some questions like the ones below to help them identify the necessary steps for estimating the area of the wet section of sidewalk.
  - Since we are using a trapezoid to estimate the area of the wet section of sidewalk, what information about the trapezoid do we need? (In other words, what information do we need to calculate the area of a trapezoid?)

    To determine the area of a trapezoid, we need the height and the lengths of the two bases.

- What parts of the diagram represent the height and the two bases of the trapezoid?

  The height of the trapezoid is the width of the sidewalk. The two bases of the trapezoid are the parallel edges of the wet section of sidewalk.

- How could we determine the lengths we need to find the area of the wet section of sidewalk?

  Students might suggest measuring the lengths of the edges of the wet sidewalk using a ruler.

**Guiding Student Thinking**

If students suggest measuring the distances of the sidewalk's wet edges with a ruler, encourage them to think about what kind of ruler they would need. There is a scale given in the scenario (1 box side = 5 feet), but the side length of a box does not necessarily conform to a standard unit of measure like centimeters. Students might suggest that they could make their own ruler using the side length of a box as the standard unit and then scaling it by 5 feet. It is important to validate this suggestion, because it is similar to the indirect measurement tasks that students explored in the beginning of Unit 2. Encourage students to think about how much estimating they might have to do when using their manufactured ruler. Diagonal distances are often difficult to determine when the unit of measure cannot be subdivided into smaller units. That is, students could probably estimate one-half of a side length of a box but would have much more difficulty estimating one-third or one-fifth of a side length of a box. Because the unit of measure in the diagram makes this estimation difficult, algebra is a better option for determining the lengths of the edges of the wet section of sidewalk.

- Record in a central location the suggestions that students make about determining the length of the longer edge of the wet section of sidewalk. You may need to steer the discussion away from "eyeballing" the lengths, as that would likely yield an imprecise estimate. Encourage students to think of ways to get better estimates of the length.
- If no students suggest writing an equation for the circle and the longer side of the wet section of sidewalk, you may have to introduce that idea to them. You can ask them some questions like the ones below to encourage them to connect the graph to algebra:
  - If we can't determine where the circle and the line intersect from the graph, is there any other method we can use to figure it out?

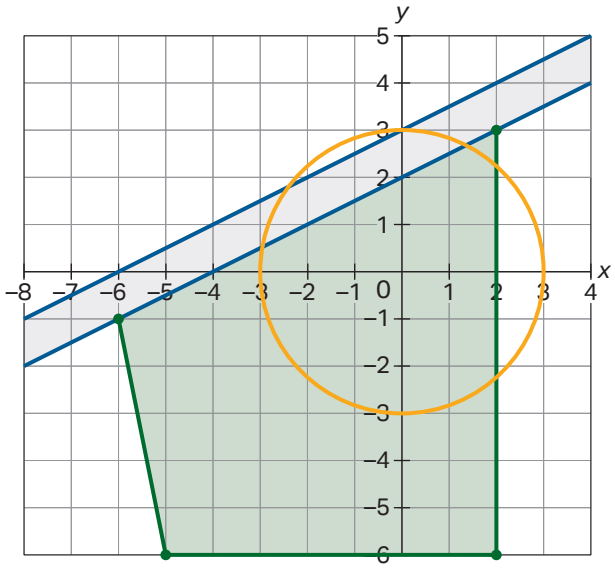
    We could write equations for the circle and for the line and then use algebra to find out the $x$- and $y$-values of their intersection points.
- After the class determines that algebra is a good option for this task, allow students some time to write an equation for the circle and an equation for the inside edge of the sidewalk. You can have students work in pairs.

**Instructional Rationale**

There are many different ways to write these equations, because the origin of the coordinate grid is not specified. Allowing students to set their own origin will encourage a more authentic modeling experience. Some choices for the location of the origin will lead to easier algebra than others. It is OK for students to make an inefficient choice, because it will help them understand the importance of exploring a few options before making a final decision.

- Circulate around the room while students write the equations. As they work, you might need to ask them some questions about their choices:
  - Where is your origin? Why did you choose that point?
  - What do you need to write the equation of a circle? What coordinate pair is the center of your circle? What is the radius of the circle?
  - What do you need to write the equation of a line? What points could you use to determine the slope of the line? What points could you use to write the equation of the line?
- Have students share their equations. If students used different origins, then their equations will differ.

  If students set the center of the circle as the origin and use the interval between each gridline as one unit, then a set of equations could be $x^2+y^2=9$ and $y=\frac{1}{2}x+2$.



## PART 2: SOLVING A SYSTEM OF EQUATIONS

In this part of the lesson, students learn how to solve the system of equations they wrote in Part 1. In Algebra 1, students learned how to solve systems of equations using the substitution method, as well as how to solve quadratic equations. Solving the system they wrote in Part 1 involves combining these two techniques.

- Begin this part of the lesson by asking students to closely observe and analyze the equations they wrote in Part 1. You can ask them some higher-order questions like the ones below to encourage them to make connections between their Algebra 1 course work and the system of equations.
  - When have you seen two equations with two variables before?

    In Algebra 1 we solved systems of two linear equations with two variables.
  - How is this system of equations similar to the ones you explored in Algebra 1? How is it different from the ones you explored in Algebra 1?

    This system is similar to the ones in Algebra 1 because there are two equations and two variables. This system is different than the ones in Algebra 1 because one of the equations has two squared terms, so they are not both linear equations.
  - What techniques do we know for solving a system of equations?

    We could graph them and find the intersection points by inspection. We could make a table of values and find where they have the same $x$- and $y$-values. We could use an algebraic technique like substitution or elimination.
  - Which techniques that we know might be good to try here? Which might be bad to try here?

    We already have a graph, and it does not show us exact values for the intersection points by inspection. A table of values could take a long time, because the intersection points are not integer values. We should use an algebraic technique like substitution. Elimination works best for systems of linear equations.
- Once students identify that substitution is the best technique to use, allow them some time to set up the first step of their substitution. Their first step should yield an equation with one variable.

$$x^2 + \left(\frac{1}{2}x + 2\right)^2 = 9$$

**Classroom Ideas**

You can have students complete their work on small whiteboards. When they finish, they can hold up their work for you and the class to see. This gives you an opportunity to quickly check the work of all students and for students to analyze their peers' work. It also allows you to target feedback to particular students who need more guidance.

- You can let students proceed with the rest of the algebra. Students should square the binomial and combine like terms. Once they complete those steps, you may need to focus their attention on what kind of equation they have and what techniques they know to solve it.

$$x^2+\frac{1}{4}x^2+2x+4=9$$
$$\frac{5}{4}x^2+2x+4=9$$

**Meeting Learners' Needs**
If students are struggling with squaring the binomial, this is an opportunity for a just-in-time review of the algebraic technique.

- Ask students to think about what kind of equation they notice in the final step of their work. If students need support, you might need to ask them to recall the kinds of equations they solved in Algebra 1.
  - What kind of equation did we get? Have we seen these kinds of equations in previous classes?

    We got a quadratic equation. We solved those equations in Algebra 1.
  - What techniques do we know for solving a quadratic equation?

    We could try to factor, complete the square, or use the quadratic formula.
  - Does the equation require any additional manipulation before we solve it?

    We should algebraically move all the terms to one side so that it has the form $ax^2+bx+c=0$.
- Allow students time to work through the algebra of solving the quadratic equation. As you circulate around the room while students work, you can provide necessary support and guidance to students who need it. When they finish solving the quadratic, they can share their work with the class.

$$\frac{5}{4}x^2+2x+4=9$$
$$\frac{5}{4}x^2+2x-5=0$$
$$x=\frac{-2\pm\sqrt{4-4\left(\frac{5}{4}\right)(-5)}}{2\left(\frac{5}{4}\right)}$$
$$x=\frac{-2\pm\sqrt{29}}{\frac{5}{2}}$$
$$x\approx-2.95,\ 1.35$$

- Ask students some questions like the ones below to have them analyze the solutions they obtained:
  - What do these numbers represent?

    The solutions we found are the $x$-values of the intersection points of the line and the circle.
  - If those are the $x$-values of the intersection points, what should we do to get the $y$-values of the intersection points?

    We should substitute each $x$-value into one of the equations to get the corresponding $y$-value.
- Provide students with some time to perform the substitutions. Like before, have students share their results when they have the intersection points.

  If $x \approx -2.95$, then $y \approx \frac{1}{2}(-2.95)+2$

  $y \approx 0.53$

  If $x \approx 1.35$, then $y \approx \frac{1}{2}(1.35)+2$

  $y \approx 2.68$

  The intersection points are $(-2.95, 0.53)$ and $(1.35, 2.68)$.
- Once students have the intersection points, ask them why knowing the intersection points will help them solve the problem.

  Now we can find the distance between the intersection points to determine the longer edge of the sidewalk that gets wet.
- Have students determine the distance between intersection points to find the longer edge of the sidewalk that gets wet.

  The distance between the intersection points is $\sqrt{(1.35-(-2.95))^2+(2.68-0.53)^2} \approx 4.80$ units. Because each unit represents 5 feet, the length of the longer edge of the wet section of sidewalk is approximately 24 feet.

## PART 3: DETERMINING THE AREA OF THE WET SIDEWALK

In this part of the lesson, students solve another system of equations to determine the length needed to calculate the area of the wet sidewalk.

- Once students determine the length of the longer edge of the wet section of the sidewalk, you can ask them to determine the length of the shorter edge of the sidewalk. This part of the task should proceed in a manner similar to Part 2. Some students will be able to work independently on this aspect of the task, having already done something similar. You can focus your attention on students who are struggling with the concept of a system of equations or who need assistance with the mechanics of solving the equations.
- Using the same origin as in Part 2, the shorter edge of the wet section of sidewalk can be modeled by the equation $y=\frac{1}{2}x+3$. The equation of the circle will be the same, so students should solve the system of equations $\begin{cases} x^2+y^2=9 \\ y=\frac{1}{2}x+3 \end{cases}$. The solutions to this system are $(-2.4,\ 1.8)$ and $(0,\ 3)$. The length of the shorter edge of the wet section of the sidewalk is approximately 2.68 units, or about 13.4 feet.
- Once students determine the length of the shorter edge of the wet section of sidewalk, you can turn their attention to the width of the sidewalk. It is challenging to determine the width of the sidewalk. Since the critical part of the lesson was learning how to solve a system of equations involving the equations of a circle and a line, you could simply tell students that the width of the sidewalk is about 0.89, units or 4.45 feet. A standard sidewalk is about 4 feet wide, so this sidewalk is slightly wider. The area of the wet sidewalk is $\frac{1}{2}(4.45)(24+13.4)\approx 83.22$ square feet.
- At this point, you can provide some time for students to practice solving systems of equations. **Handout 3.7.B: Lines Intersecting Circles** has some sample problems that students could solve after this lesson. The problems could be assigned to be completed within the class period if you think students need additional support or outside of class if they are ready to solve the systems without help.

**Meeting Learners' Needs**

To provide a challenge for students who can benefit from one, you could ask them to calculate the width of the sidewalk using algebra. This would entail writing the equation of a line that is perpendicular to the lines representing the longer and shorter edges of the wet section of sidewalk, determining the intersection points of the perpendicular line with the lines representing the edges of the sidewalk, and using the distance formula to calculate the distance between these intersection points.

UNIT 3

## ASSESS AND REFLECT ON THE LESSON

### FORMATIVE ASSESSMENT GOAL

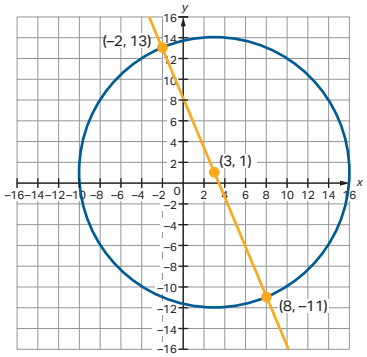
When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

Determine the intersection points of the circle and line algebraically. Use a graphing utility to verify your calculations.

$$\begin{cases}(x-3)^2+(y-1)^2=169\\ y=-2.4x+8.2\end{cases}$$

The intersection points are (–2, 13) and (8, –11).

$$(x-3)^2+(-2.4x+8.2-1)^2=169$$
$$(x-3)^2+(-2.4x+7.2)^2=169$$
$$x^2-6x+9+5.76x^2-34.56x+51.84=169$$
$$6.76x^2-40.56x+60.84=169$$
$$6.76x^2-40.56x-108.16=0$$
$$x=\frac{40.56\pm\sqrt{(-40.56)^2-4(6.76)(-108.16}}{2(6.76)}$$
$$x=-2 \text{ and } 8$$
$$y=13 \text{ and } -11$$




### HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

**Handout 3.7.A: The Sprinkler**

See lesson for details.

### Handout 3.7.B: Lines Intersecting Circles

These problems are intended to give students some practice in the mechanics of solving systems of equations involving circles and lines.

1. The intersection points are (5, –2) and (0, –7).
2. The intersection points are (–1, 0) and (0.2, 2.4).
3. The intersection points are (–2, –3) and (0.6, 4.8).
4. The intersection points are approximately (–0.944, 2.887) and (2.544, –4.087).
5. The equation of the line is $y = -\frac{2}{3}x + 4$. The equation of the circle is $(x+1)^2 + (y+2)^2 = 49$. The intersection points are approximately (–1.48, 4.99) and (5.63, 0.25).

UNIT 3

LESSON 3.8

# Lines Tangent to a Circle

## OVERVIEW

### LESSON DESCRIPTION

**Part 1: The Sprinkler Revisited**

Students again investigate a contextual scenario involving the circular range of a lawn sprinkler. In this version of the sprinkler problem, Omar wants to position the sprinkler so it does not get the sidewalk wet.

**Part 2: Investigating the Slope of a Line Tangent to a Circle**

Students use a graphing utility to determine the relationship between the radius of a circle and a line tangent to the circle.

**Part 3: Placing the Sprinkler**

Students use what they learned in Part 2 to solve the sprinkler problem they explored in Part 1.

### CONTENT FOCUS

In this lesson students are introduced to lines tangent to a circle. The most important understanding for students to take from this lesson is that a line tangent to a circle is perpendicular to a radius of the circle at their point of intersection. Students use what they know about the slopes of perpendicular lines to solve problems involving circles and tangent lines in the coordinate plane.

### AREAS OF FOCUS

- Connections Among Multiple Representations
- Greater Authenticity of Applications and Modeling

### SUGGESTED TIMING

~45 minutes

### LESSON SEQUENCE

This lesson is part of a lesson sequence (~180 minutes total) with Lessons 3.6 and 3.7.

### HANDOUTS

**Lesson**

- 3.8.A: The Sprinkler Revisited
- 3.8.B: Lines Intersecting Circles

### MATERIALS

- graph paper
- straightedge/ruler
- compass
- calculator
- access to **Desmos.com**

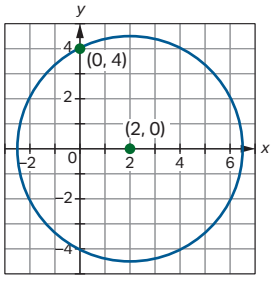
## COURSE FRAMEWORK CONNECTIONS

| Enduring Understandings | |
|---|---|
| ▪ The geometry of a circle is completely determined by its radius. | |
| **Learning Objectives** | **Essential Knowledge** |
| **3.3.7** Construct a line, ray, or line segment tangent to a circle. | **3.3.7a** A line, ray, or line segment tangent to a circle intersects the circle at exactly one point.<br>**3.3.7b** A line, ray, or line segment tangent to a circle is perpendicular to a radius of the circle at the point of intersection.<br>**3.3.7c** In the coordinate plane, the slope of the line, ray, or line segment tangent to the circle and the slope of the radius that intersects this tangent line, ray, or line segment will be opposite reciprocals, or one will be vertical and the other will be horizontal. |
| **3.3.8** Solve a system of equations consisting of a linear equation and the equation of a circle. | **3.3.8a** The intersection of a line and a circle corresponds to an algebraic solution of the system of their corresponding equations.<br>**3.3.8b** An algebraic solution to a system of equations is an ordered pair that makes all equations true simultaneously. The system may have zero, one, or two solutions. |

## FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

Draw a line tangent to the circle whose equation is $(x-2)^2 + y^2 = 20$ at the point (0, 4). Write an equation for this tangent line.



## PART 1: THE SPRINKLER REVISITED

The lesson opens by revisiting the sprinkler scenario from Lesson 3.7. In this version of the sprinkler situation, Omar wants to position the sprinkler so all the water from the sprinkler is confined to his front yard. This change in the scenario introduces students to the concept of a tangent line: a line that intersects the circle at only one point.

- Begin by displaying the revised sprinkler scenario for students. The scenario is also on **Handout 3.8.A: The Sprinkler Revisited** if you would prefer for students to have their own copies.
- Allow students some time to closely observe and analyze the scenario and diagram. You can ask students some questions to focus their analysis:
  - Is it possible for Omar to put the sprinkler somewhere so the water stays completely in his yard? Why or why not?

    It is possible, because the diameter of water sprayed by the sprinkler is less than the shortest distances between either pair of opposite sides of the yard.
  - Is it possible for Omar to put the sprinkler somewhere so the water it sprays stays completely in the yard but just reaches the top and right edges of the property? Why or why not?

    It is possible to place the sprinkler so the water just barely reaches the sidewalk and the neighbor's yard. The radius of the water sprayed by the sprinkler is small enough that the sprinkler could be moved away from both edges so the water just reaches the top and right edges of the property.
  - What does Omar need to figure out to solve this problem algebraically?

    He needs to figure out where the new center of the circle should be.
  - If the spray of the water just barely reaches to the edge of the sidewalk, then algebraically, how many intersection points would there be between the line that represents the bottom edge of the sidewalk and the circle that represents the range of the sprinkler?

    There would only be one intersection point.

**Guiding Student Thinking**

Some students might suggest that in practice, Omar would just move the sprinkler until the sidewalk and neighbor's yard are not getting wet. It is important to validate this pragmatic thinking. Encourage students to continue to complete the problem to determine the exact position for the sprinkler without having to move it multiple times and then checking to see if the water reaches to the edge of the yard.

- An important concept to bring out during the discussion is that the sidewalk edge and the water circle should only intersect once. This makes the problem different from the sprinkler problem in Lesson 3.7.
- You can provide a definition to students for this new concept:

  *A line that intersects a circle exactly once is called a tangent to the circle.*

**Guiding Student Thinking**

Students might recall the tangent ratio that they investigated in Unit 2 and wonder if there is a connection between these concepts. The Latin root word of tangent is *tangere*, which means "to touch." See Lesson 2.8 for an explanation of why the tangent ratio is named the way it is.

- Give students a little time to make some suggestions about where the center of the circle (the position of the sprinkler) should be and how they could figure it out. Record their conjectures in a central location.
- You can let students know that their next task will be to investigate some of the properties of tangent lines and circles.

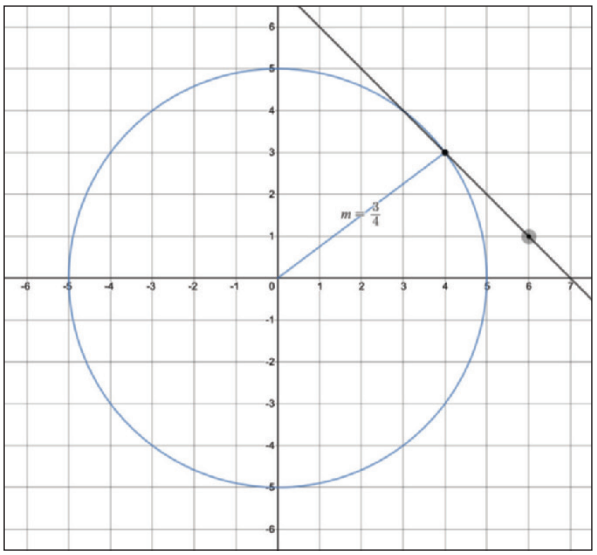
## PART 2: INVESTIGATING THE SLOPE OF A LINE TANGENT TO A CIRCLE

In this part of the lesson, students investigate the relationship between a line tangent to a circle and the radius of the circle at their point of intersection. By investigating the slopes of pairs of these geometric figures in the coordinate plane, students will observe that they are perpendicular. This part of the lesson involves students using a Desmos interactive to draw conclusions about the relationship between the radius and the tangent line.

- To begin this part of the lesson, pair students and direct them to **preap.org/tangent-lines**. Let students know that their task is to determine the relationship between the radius of the circle and the line tangent to the circle.
- Allow time for students to explore the interactive, which displays a circle whose center is at the origin and whose radius is 5 units. There is a point that can be moved to integer coordinates around the circle. The students' task is to move the point on the circle to any location and adjust the slope of the line that contains this point until the line is tangent to the circle.

**Guiding Student Thinking**

Some students might find the interactive challenging at first. You can recommend that they focus on the $x$- and $y$-intercepts first before trying other coordinates on the circle.




- Encourage students to record any observations they make. As you circulate around the room, ask students some questions so they continue to justify their observations:
  - What do you know about the relationship between horizontal and vertical lines?

    We know that horizontal and vertical lines are perpendicular to each other.
  - How do you know if lines that are not horizontal or vertical are perpendicular? What about the lines indicates that they are perpendicular?

    We know lines are perpendicular if their slopes are opposite reciprocals of each other.
- After students collect some observations, debrief with the entire class about the relationship between the slopes of the radii and the lines tangent to the circle at their points of intersection.
  - What do you observe about the radius and the tangent line?

    The radius and the tangent line are perpendicular to each other at their intersection point on the circle.
  - How could you tell that the radius and the tangent line were perpendicular?

    The slopes of the radius and the tangent line were opposite reciprocals of each other. When the slopes of lines or line segments are opposite reciprocals of each other, those lines or segments are perpendicular.

- Do we know for sure that this relationship will be true for all circles? How could we be sure?

  We do not know for sure that this relationship will be true for all circles. We would have to prove it to be sure.

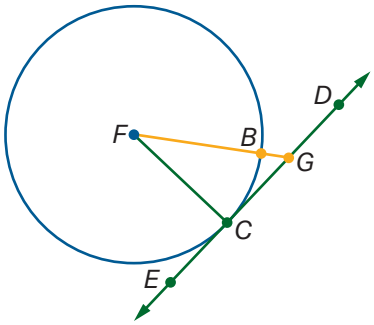
- You can have a student summarize the conjecture for the class:

  *A line tangent to a circle is perpendicular to a radius of the circle at the point where the line intersects the circle (called the point of tangency).*

- At this point, it could be useful to work through a proof with students to demonstrate that the conjecture is true. One version of the proof is provided here:

  Suppose that circle $F$ has a radius $\overline{FC}$, and $\overleftrightarrow{DE}$ is tangent to circle $F$ at point C, but we assume that $\overline{FC}$ is not perpendicular to $\overleftrightarrow{DE}$. Then a segment, $\overline{FG}$, can be drawn to point $G$ on $\overleftrightarrow{DE}$ such that $\overline{FG}$ is perpendicular to $\overleftrightarrow{DE}$. Let $\overline{FG}$ intersect circle $F$ at point $B$.

  If $\angle FGC$ is a right angle, then $\Delta FGC$ is a right triangle whose hypotenuse is $\overline{FC}$. If the hypotenuse is $\overline{FC}$, then $m\overline{FC} > m\overline{FG}$. But this can't be true, because $\overline{FC}$ and $\overline{FB}$ are radii of circle $F$, so $m\overline{FC} = m\overline{FB}$. Since $\overline{FG}$ extends outside the circle, then $m\overline{FG} > m\overline{FB}$. Therefore, it cannot be true that $\overline{FG}$ is perpendicular to $\overleftrightarrow{DE}$.




This proof by contradiction shows that a line tangent to a circle cannot be perpendicular to a segment drawn from the center of the circle that extends outside the circle. This means our assumption that the tangent line is not perpendicular to a radius at their point of intersection is incorrect. Therefore, the tangent line must be perpendicular to the segment drawn from the center of the circle to the point where the tangent line intersects the circle. That segment is a radius of the circle.

## PART 3: PLACING THE SPRINKLER

In this part of the lesson, students determine the placement of the sprinkler so the water just reaches the top and the right edges of the yard. Some students should approximate the placement using a graphing utility, while other students should determine the placement of the sprinkler algebraically.

- Begin by directing students' attention back to the sprinkler scenario. You can let them know that they will be using what they have learned about the relationship between a line tangent to a circle and a radius of the circle at the point of tangency.
- Arrange the students into groups of four. Within each group, two students will use a graphing utility to approximate the placement of the sprinkler. The other two students will determine the sprinkler placement using algebra. After each pair of students completes their part of the task, they can swap work and solutions and provide feedback and critique to their groupmates.
- Students using a graphing utility should input their own equations for the bottom edge of the sidewalk, the right edge of the yard, and the range of the sprinkler. They should adjust the location of the sprinkler until the water just reaches the right side of the yard and the sidewalk. Students should first identify that the $x$-value of the sprinkler is $-1$. This leads to the equation $(x+1)^2+(y-k)^2=9$. If students are familiar with the slider feature of Desmos, they can use it to translate the circle vertically until it appears to be tangent to the line representing the edge of the sidewalk, $y=\frac{1}{2}x+2$. The sprinkler should be at $(-1, -1.8)$.
- Students determining the location of the center of the circle using algebra will have to solve a system of equations. A worked solution is provided here: We know that the line $y=\frac{1}{2}x+2$ has to be perpendicular to a radius (or diameter) of the circle at the point of tangency. The equation of the circle is $(x+1)^2+(y-k)^2=9$. We know that the line containing the radius will have the form $y=-2(x+1)+k$, because that line must pass through the center of the circle and be perpendicular to the sidewalk edge. Solving the system of equations consisting of the two lines leads to:

$$\frac{1}{2}x+2=-2(x+1)+k$$

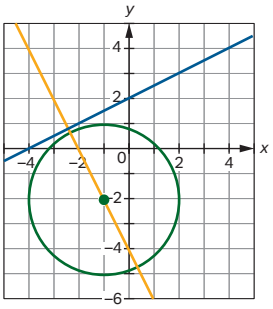
$$\frac{1}{2}x+2=-2x-2+k$$

$$k=4+\frac{5}{2}x$$

**Classroom Ideas**

If you want to create a third option or need to replace the graphing utility option, you could have students use a compass with their own diagram or the one on **Handout 3.8.A: The Sprinkler Revisited** to determine a placement of the sprinkler. A paper-and-pencil approach will likely yield the least precise placement of the sprinkler, as the scale of the graph is in units of 1. However, a paper-and-pencil approach might be helpful for some students struggling to understand the task.

This tells us that the $y$-coordinate of the center of the circle is related to the $x$-value of the intersection point between the tangent line and a line containing the radius of the circle.




The points of intersection of the circle and the line containing the radius perpendicular to the tangent can be determined by solving the system

$$\begin{cases}(x+1)^2+(y-k)^2=9\\ y=-2(x+1)+k\end{cases}.$$

Using substitution leads to

$$\begin{aligned}(x+1)^2+(-2(x+1)+k-k)^2&=9\\(x+1)^2+(-2x-2)^2&=9\\5x^2+10x+5&=9\end{aligned}$$

The equation can be manipulated to the form $5x^2+10x-4=0$ and solved using the quadratic formula: $x=\frac{-10\pm6\sqrt{5}}{10}=\frac{-5\pm3\sqrt{5}}{5}$. Because only one of the $x$-values is to the left of the center of the circle, we can use $x=\frac{-5-3\sqrt{5}}{5}$ in the equation $k=4+\frac{5}{2}x$ to determine the value of $k$: $k=4+\frac{5}{2}\left(\frac{-5-3\sqrt{5}}{5}\right)\approx-1.85$. Therefore, the sprinkler should be placed at approximately $(-1, -1.8)$ or $(-1, -1.9)$.

- Students might wonder why the graphing utility and the algebraic method produce slightly different solutions. This is due to the imprecision of the graphing utility, which may not be able to adjust the circle to the exact location of the center. As is often the case, algebra yields an exact answer while the technology provides a very close approximation.
- This is a good place to provide students with opportunities to practice finding intersection points for lines tangent to a circle. **Handout 3.8.B: Lines Intersecting Circles** has some sample problems that students can complete.

UNIT 3

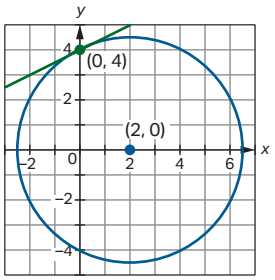
## ASSESS AND REFLECT ON THE LESSON

### FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

Draw a line tangent to the circle whose equation is $(x-2)^2+y^2=20$ at the point (0, 4). Write an equation for this tangent line. Explain your work.

The line tangent to the circle at (0, 4) is perpendicular to the radius at their point of intersection. The line containing the radius and the point of tangency, (0, 4), has slope –2. Therefore, the line tangent must have slope $\frac{1}{2}$. The $y$-intercept is given as (0, 4). Therefore, the equation of the line is $y=\frac{1}{2}x+4$, and its graph is shown below.




### HANDOUT ANSWERS AND GUIDANCE

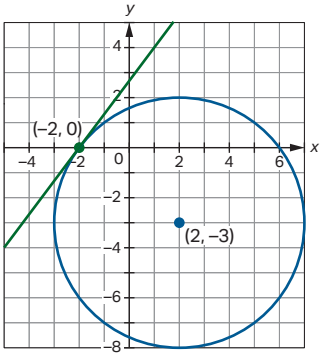
To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

#### Handout 3.8.A: The Sprinkler Revisited

See lesson for details.

#### Handout 3.8.B: Lines Intersecting Circles

1. The line tangent to the circle at (–2, 0) is perpendicular to the radius at their point of intersection. The line containing the radius and the point of tangency, (–2, 0), has slope $-\frac{3}{4}$. Therefore the tangent line must have slope $\frac{4}{3}$. The $x$-intercept is given as (–2, 0). Therefore, the equation of the line tangent to the circle whose equation is $(x-2)^2+(y+3)^2=25$ at the point (–2, 0) is $y=\frac{4}{3}(x+2)$.

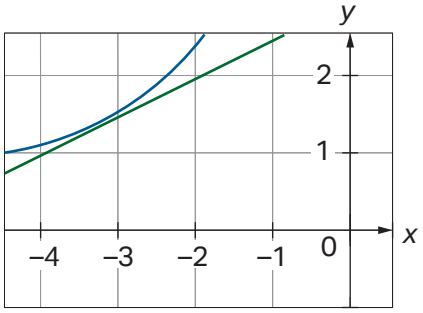



2. The line $y=-\frac{5}{6}x+9$ is not tangent to the circle $(x-4)^2+(y-2)^2=8$ at the point (6, 4). The slope of the radius at (6, 4) is 1. Because the slope of the line $y=-\frac{5}{6}x+9$ is not −1, the lines are not perpendicular, which means $y=-\frac{5}{6}x+9$ is not tangent to the circle.
3. The radius to the point of tangency and the tangent line must be perpendicular. The slope of the line containing the radius and the point of tangency is $-\frac{12}{5}$. Therefore, an equation for the line tangent to the circle at (2, −7) must be $y=\frac{5}{12}(x-2)-7$.
4. Solving the system $\begin{cases}(x+3)^2+(y+4)^2=17\\ y=4x+25\end{cases}$ using substitution yields:

$$\begin{aligned}(x+3)^2+(4x+25+4)^2&=17\\ x^2+6x+9+16x^2+232x+841&=17\\ 17x^2+238x+833&=0\\ x^2+14x+49&=0\\ (x+7)^2&=0\\ x&=-7\end{aligned}$$

Because $x=-7$, $y=4(-7)+25=-3$. The point of tangency is $(-7, -3)$.

5. The line is not tangent to the circle. Graphically, they are close to each other, but they do not intersect:




Algebraically, solving the system $\begin{cases}(x+5)^2+(y-5)^2=16\\ y=\frac{1}{2}x+3\end{cases}$ using substitution yields:

$$(x+5)^2+\left(\frac{1}{2}x+3-5\right)^2=16$$

$$x^2+10x+25+\frac{1}{4}x^2-2x+4=16$$

$$\frac{5}{4}x^2+8x+13=0$$

The value of the discriminant is –1, so the quadratic equation has no real solutions. This means that the line and the circle do not intersect, so the line cannot be tangent to the circle.

Unit 3

# Performance Task

PERFORMANCE TASK

# Olga's Walkie-Talkie

## OVERVIEW

### DESCRIPTION

In this performance task, students use their understanding of circles to answer questions about the range of a walkie-talkie.

### CONTENT FOCUS

This task is designed to assess students' understanding of the intersection of a circle and a line in context.

### AREAS OF FOCUS

- Greater Authenticity of Applications and Modeling
- Connections Among Multiple Representations

### SUGGESTED TIMING

~45 minutes

### HANDOUTS

Unit 3 Performance Task: Olga's Walkie-Talkie

### MATERIALS

- scientific calculator or graphing utility

## COURSE FRAMEWORK CONNECTIONS

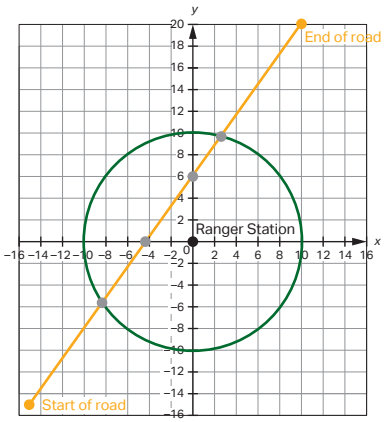
| Enduring Understandings | |
|---|---|
| ▪ The geometry of a circle is completely determined by its radius. | |
| **Learning Objectives** | **Essential Knowledge** |
| **3.3.2** Translate between the geometric and algebraic representations of a circle. | **3.3.2a** A circle is the set of all points equidistant from a given point.<br>**3.3.2b** In the coordinate plane, the graph of the equation $(x-h)^2+(y-k)^2=r^2$ is the set of all points located $r$ units from the point $(h,k)$. This is a circle with radius $r$ and center $(h,k)$. |
| **3.3.8** Solve a system of equations consisting of a linear equation and the equation of a circle. | **3.3.8a** The intersection of a line and a circle corresponds to an algebraic solution of the system of their corresponding equations.<br>**3.3.8b** An algebraic solution to a system of equations is an ordered pair that makes all equations true simultaneously. The system may have zero, one, or two solutions. |

## SCORING GUIDELINES

There are 9 possible points for this performance task.

**Student Stimulus and Part (a)**

(a) Construct a model on a coordinate plane of the scenario described.

| Sample Solutions | Points Possible |
|---|---|
| On a coordinate plane in which each unit represents one mile, Olga and the ranger station are at the origin. The range of Olga's walkie-talkie is represented by the area of a circle whose center is the origin and whose radius is 10 miles. The equation of the boundary of the region is $x^2 + y^2 = 100$. The road that Neil is driving on starts at the point $(-15, -15)$ and ends at the point $(10, 20)$. The straight road can be represented by a line segment whose endpoints are $(-15, -15)$ and $(10, 20)$.<br>The equation of the line that contains the segment is $y = \frac{7}{5}(x - 10) + 20$.<br>  <br> | **3 points maximum**<br>1 point for any model that includes a circle and a line segment<br>1 point for a circle with correct radius<br>1 point for a line segment with the correct distances relative to the ranger station<br>*Scoring note*: The ranger station does not need to be located at the origin. The ranger station could be any point. If each unit represents one mile, the start of the road must be 15 units to the left and 15 units below the station. The end of the road must be 10 units to the right and 20 units above the station. |

**Targeted Feedback for Student Responses**

If students struggle with part (a), it could be because they are not sure how to use the available information to construct a drawing and translate that drawing into algebraic equations.

TEACHER NOTES AND REFLECTIONS

**Student Stimulus and Part (b)**

(b) For approximately how many miles of road will Neil be able to receive Olga's weather updates?

| Sample Solutions | Points Possible |
|---|---|
| To find the intersection points of the circle and the line, set up and solve the system of equations: $x^2+y^2=100$, $y=1.4x+6$<br>Use substitution:<br>$x^2+(1.4x+6)^2=100$<br>$x^2+1.96x^2+16.8x+36=100$<br>$2.96x^2+16.8x-64=0$<br>Use the quadratic formula to solve:<br>$x=\frac{-16.8\pm\sqrt{(16.8)^2-4(2.96)(-64)}}{2(2.96)}$<br>$x\approx 2.61, -8.29$<br>Use the $x$-values to get the corresponding $y$-values:<br>$y_1\approx 1.4(2.61)+6$<br>$y_1\approx 9.65$<br>$y_2\approx 1.4(-8.29)+6$<br>$y_2\approx -5.61$<br>Find the distance between the intersection points (2.61, 9.65) and (−8.29, −5.61) with the distance formula:<br>$d\approx\sqrt{(2.61+8.29)^2+(9.65+5.61)^2}$<br>$d\approx 18.75$ miles | **3 points maximum**<br>1 point for setting up and solving the system<br>1 point for both intersection points<br>1 point for the distance between the intersection points |

| | |
|---|---|
| Olga can expect Neil to be able to receive weather updates for approximately 18.75 miles of road. | |
| **Targeted Feedback for Student Responses** | |
| If students struggle with part (b), it could be because they need extra practice solving a system of equations involving a quadratic equation, specifically using the quadratic formula to determine the $x$-values of points of intersection. | |

TEACHER NOTES AND REFLECTIONS

**Student Stimulus and Part (c)**

(c) Suppose that Neil drives at an average rate of 15 miles per hour. If Neil starts his trip at 8 a.m., at what time should Olga expect him to first be able to receive her weather updates?

| Sample Solutions | Points Possible |
|---|---|
| The distance between the start of the road and when Neil's receiver is within range of Olga's walkie-talkie is calculated by<br>$d \approx \sqrt{(-15+8.29)^2+(-15+5.61)^2}$<br>$d \approx 11.54$ miles<br>Since Neil is traveling at 15 miles per hour, it will take him $\frac{11.54}{15} \approx 0.77$ hours. Converting to minutes, 0.77 hours is about 46 minutes.<br>Therefore, Olga can expect Neil to first be able to receive her weather updates around 8:46 a.m. | **3 points maximum**<br>1 point for the distance between start of road and the range of walkie-talkie<br>1 point for the time it takes to travel the distance between the start of road and the range of the walkie-talkie<br>1 point for the time when Olga can expect Neil to first be able to receive her weather updates |

**Targeted Feedback for Student Responses**

If students struggle with part (c), it could be because they are unsure about how to use the distance formula to determine the length of the road, or because they are not sure how to use the speed that Neil is traveling to determine the length of time he is within range of Olga's walkie-talkie.

TEACHER NOTES AND REFLECTIONS

| Points Received | Appropriate Letter Grade (If Graded) | How Students Should Interpret Their Scores |
|---|---|---|
| 8 or 9 points | A | "I know all of this geometry really well." |
| 6 or 7 points | B | "I know all of this geometry well, but I made a few mistakes." |
| 4 or 5 points | C | "I know some of this geometry well, but not all of it." |
| 2 or 3 points | D | "I only know a little bit of this geometry." |
| 0 or 1 point | F | "I don't know much of this geometry at all." |

PERFORMANCE TASK

# Olga's Walkie-Talkie

Olga is a park ranger in a large national park. The park is in a rural area that has no cellular phone service. Olga has a walkie-talkie that allows her to her to transmit weather updates to other rangers in the park. Her walkie-talkie has a range of 10 miles in all directions. The other park rangers have receivers in their cars that can pick up her weather updates when they are in range of her walkie-talkie. On a particular day, Olga is at the ranger station in the center of the park. Her coworker Neil is doing a survey of the land by driving along a straight road through the park. The road that Neil is on starts 15 miles west and 15 miles south of the ranger station and ends 20 miles north and 10 miles east of the ranger station.

(a) Construct a model on a coordinate plane of the scenario described.

(b) For approximately how many miles of road will Neil be able to receive Olga's weather updates?

(c) Suppose that Neil drives at an average rate of 15 miles per hour. If Neil starts his trip at 8 a.m., at what time should Olga expect him to first be able to receive her weather updates?

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# Unit 4

# Unit 4
# Measurement in Two and Three Dimensions

## Overview

**SUGGESTED TIMING: APPROXIMATELY 7 WEEKS**

This unit deepens students' understanding of measurement by expanding the concept of measurement to two dimensions through the areas of planar figures and to three dimensions through the volumes of solid figures. One reason for studying area is that it often represents quantities that are otherwise difficult to compute. For example, the area under the graph of an object's speed over time corresponds to its total distance traveled. Therefore, techniques for calculating area can be adapted to find other quantities. Students likely have prior experience with calculating the areas of conventional figures and composites of those figures. Students may also have experience with calculating the volumes of conventional solids. The unit introduces students to Cavalieri's principle, which relates the area of a planar figure to its cross-sectional lengths and the volume of a solid to its cross-sectional areas. The focus of the unit is on justifying area and volume formulas with which students are already familiar and using area and volume concepts to model real-world physical scenarios.

Throughout Units 2-4, specific learning objectives require students to prove geometric concepts. Students' proofs can be organized in a variety of formats, such as two-column tables, flowcharts, or paragraphs. The format of a student's proof is not as important as their ability to justify a mathematical claim or provide a counterexample disproving one. They should develop an understanding that a mathematical proof establishes the truth of a statement by combining previously developed truths into a logically consistent argument.

UNIT 4

## ENDURING UNDERSTANDINGS

This unit focuses on the following enduring understandings:

- The area of a figure depends on its height and its cross-sectional widths.
- The volume of a solid depends on its height and its cross-sectional areas.
- The geometry of a sphere is completely determined by its radius.

## KEY CONCEPTS

This unit focuses on the following key concepts:

- 4.1: Area as a Two-Dimensional Measurement
- 4.2: Volume as a Three-Dimensional Measurement
- 4.3: Measurements of Spheres

## UNIT RESOURCES

The tables below outline the resources provided by Pre-AP for this unit.

| Lessons for Key Concept 4.1: Area as a Two-Dimensional Measurement | | | | |
|---|---|---|---|---|
| **Lesson Title** | **Learning Objectives Addressed** | **Essential Knowledge Addressed** | **Suggested Timing** | **Areas of Focus** |
| Lesson 4.1: Shear Transformations and Cavalieri's Principle | 4.1.1 | 4.1.1a | ~90 minutes | Engagement in Mathematical Argumentation |
| | The following Key Concept 4.1 learning objectives and essential knowledge statements are not addressed in Pre-AP lessons. Address these in teacher-developed materials.<br>▪ Learning Objectives: 4.1.2, 4.1.3<br>▪ Essential Knowledge Statements: 4.1.2a, 4.1.2b, 4.1.3a | | | |

| Lessons for Key Concept 4.2: Volume as a Three-Dimensional Measurement | | | | |
|---|---|---|---|---|
| **Lesson Title** | **Learning Objectives Addressed** | **Essential Knowledge Addressed** | **Suggested Timing** | **Areas of Focus** |
| Lesson 4.2: Volumes of Prisms and Cylinders | 4.2.1, 4.2.3, 4.2.5 | 4.2.1a, 4.2.1b, 4.2.3a, 4.2.3b, 4.2.5a | ~90 minutes | Engagement in Mathematical Argumentation, Greater Authenticity of Applications and Modeling |
| | The following Key Concept 4.2 learning objectives and essential knowledge statements are not addressed in Pre-AP lessons. Address these in teacher-developed materials.<br>▪ Learning Objectives: 4.2.2, 4.2.4, 4.2.6<br>▪ Essential Knowledge Statements: 4.2.2a, 4.2.2b, 4.2.4a, 4.2.4b, 4.2.6a | | | |

| Learning Checkpoint 1: Key Concepts 4.1–4.2 (~45 minutes) |
|---|
| This learning checkpoint assesses learning objectives and essential knowledge statements from Key Concepts 4.1 and 4.2. For sample items and learning checkpoint details, visit Pre-AP Classroom. |

| Practice Performance Task for Unit 4 (~45 minutes) |
|---|
| This practice performance task assesses learning objectives and essential knowledge statements addressed up to this point in the unit. |

| Lessons for Key Concept 4.3: Measurements of Spheres |
|---|
| There are no provided Pre-AP lessons for this key concept. As with all key concepts, this key concept is addressed in a learning checkpoint. |

UNIT 4

**Learning Checkpoint 2: Key Concepts 4.2–4.3 (~45 minutes)**

This learning checkpoint assesses learning objectives and essential knowledge statements from Key Concepts 4.2 and 4.3. For sample items and learning checkpoint details, visit Pre-AP Classroom.

**Performance Task for Unit 4 (~45 minutes)**

This performance task assesses learning objectives and essential knowledge statements from the entire unit.

LESSON 4.1

UNIT 4

# Shear Transformations and Cavalieri's Principle

## OVERVIEW

### LESSON DESCRIPTION

**Part 1: Connecting the Areas of Rectangles and Parallelograms**
In this part of the lesson, students use dot paper to explore and define shear transformations through the relationship between the areas of a rectangle and a parallelogram that share certain characteristics.

**Part 2: Exploring the Areas of Triangles Under Shear Transformations**
Students use geoboards or dot paper to explore the areas of triangles under shear transformations.

**Part 3: Investigating Cavalieri's Principle**
Students formally define Cavalieri's principle and use it to determine and compare the areas of figures.

### AREA OF FOCUS

- Engagement in Mathematical Argumentation

### SUGGESTED TIMING

~90 minutes

### HANDOUTS

**Lesson**

- 4.1.A: Quadrilaterals With Equal Area
- 4.1.B: Shearing Triangles
- 4.1.C: Which Figures Have Equal Area?
- 4.1.D: Creating Figures With Equal Area

### MATERIALS

- rulers
- access to the Pre-AP Platform

### CONTENT FOCUS

In this lesson, students explore shear transformations, a new category of transformations that preserves the area of the preimage, but may not preserve all angle measures or distances between points. Observations of shear transformations lead to Cavalieri's principle, which establishes that if two planar figures have congruent bases and equal heights, and line segments in the interiors of those figures that are parallel to, and equal distances from the base are congruent, then the two figures have equal area. Through this lesson, students delve into the connection between the two-dimensional measurement of a planar figure, the area, and a one-dimensional measurement of a planar figure, the distance between the sides parallel to its base.

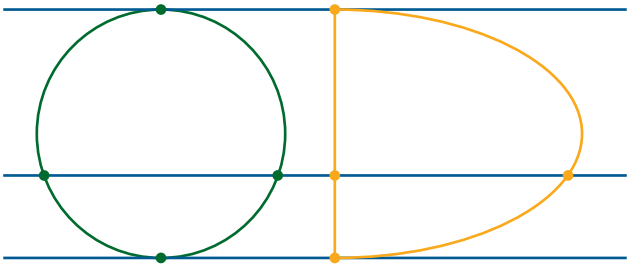
## COURSE FRAMEWORK CONNECTIONS

| Enduring Understandings | |
|---|---|
| ▪ The area of a figure depends on its height and its cross-sectional widths. | |
| **Learning Objectives** | **Essential Knowledge** |
| **4.1.1** Use Cavalieri's principle to solve problems involving the areas of figures. | **4.1.1a** If two figures have congruent bases and equal heights, and line segments in the interiors of those figures that are parallel to, and equal distances from the base are congruent, then the figures will have equal area. |

## FORMATIVE ASSESSMENT GOAL

This lesson should prepare students to complete the following formative assessment activity.

1. Based on Cavalieri's principle, what conditions must be met for the areas of two figures in the image below to be equal?
2. Determine whether the areas of the two figures in the image below are equal. Explain your answer using Cavalieri's principle.



## PART 1: CONNECTING THE AREAS OF RECTANGLES AND PARALLELOGRAMS

In this part of the lesson, students use dot paper to explore shear transformations through the relationship between the areas of a rectangle and a parallelogram that share certain characteristics. Students develop a deeper understanding of the area formula of a parallelogram.

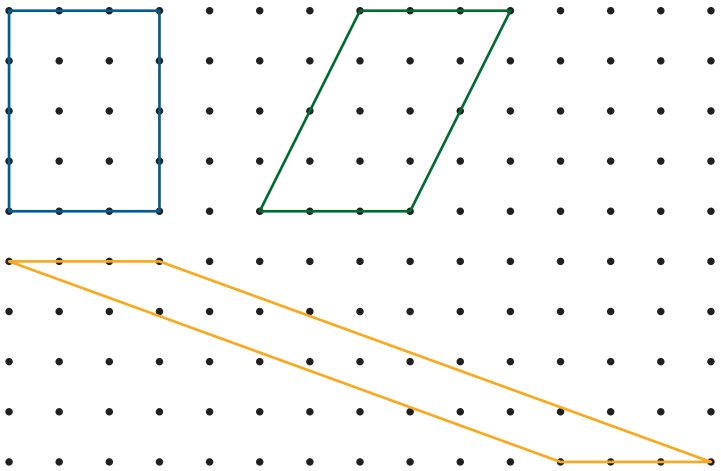
- Begin by posing this question to motivate the lesson:
  - Do you think it's possible to use the transformations you know to map a preimage to an image so that the area of the image is equal to the area of preimage, but the image is NOT congruent to the preimage?
- Give students a moment to think about the question. Encourage them to consider whether translations, rotations, reflections, and/or dilations can preserve the area of a figure but affect the figure so the image is not congruent to the preimage.
- After students write down their answers, have them share their ideas with a partner.
- Select one or two students to share their answers with the class. Expect students to state that because rigid motion transformations map preimages to congruent images, and that dilations map preimages to similar images, none of the four transformations they know preserves the area of the preimage but not the angles and distances between points of the preimage.
- Let students know that they are going to explore a new kind of transformation that does not change the area of a figure, but does change the angle measures and distances between points.
- Pair the students and provide each pair with a copy of **Handout 4.1.A: Quadrilaterals with Equal Area**. In the activity on the handout, students will create a rectangle and then create two new figures with straight sides that have congruent bases and equal heights but are not rectangles. Through this activity they will explore shear transformations.

**Classroom Ideas**
Students could do this activity with physical or virtual geoboards instead of the dot paper on Handout 4.1.A. One virtual geoboard can be found at **apps.mathlearningcenter.org/geoboard/**.

### Guiding Student Thinking

Many students will construct rectangles whose sides are vertical and horizontal segments, but some may draw rectangles whose sides are not parallel to the sides of the dot paper. For this activity, it will be easier to observe the relationships between the quadrilaterals if students start with rectangles that have horizontal and vertical sides.

- As you circulate around the room, make sure that students observe that the areas of all three quadrilaterals are equal. If the figures do not have equal areas, then students have miscalculated the areas or incorrectly created the figures. Here is an example of three noncongruent quadrilaterals with equal area:

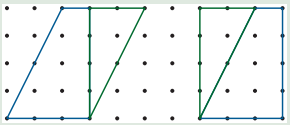


- If students are struggling to determine why the quadrilaterals have equal areas, you can help focus them on the base length and height of each figure.
  - Does it make sense that all the shapes you created have the same area? Why or why not?

    The parallelograms have equal heights and bases of equal length, so they enclose equal areas.

**Meeting Learners' Needs**

Some students might benefit from decomposing the area of one of their parallelograms by "slicing" a triangle off one side and relocating it on the other side of the figure, like this:

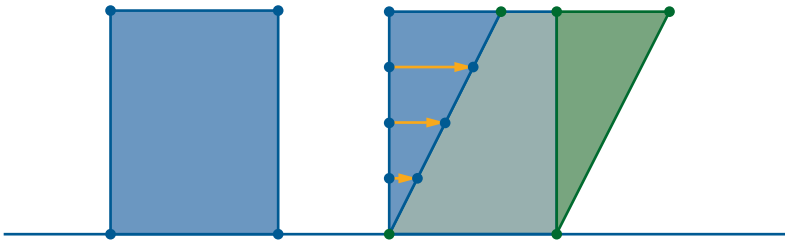


- Once pairs of students have completed problem 5 by making a conjecture about the area of a fourth quadrilateral that has an equal height and a base of equal length as the other three quadrilaterals they drew, have pairs join together to create groups of four. If possible, create groups where the pairs have sets of quadrilaterals with different dimensions. Ask each pair to share their quadrilaterals and their areas with the other members of their group. This will give students an opportunity to engage in an academic conversation about the similarities and differences among their figures.
- To debrief this part of the lesson, ask students to reflect back on the question from the beginning of the class.
  - Do you think it's possible to use the transformations you know to map a preimage to an image so that the area of the image is equal to the area of preimage, but the image is NOT congruent to the preimage? Why or why not?

    It is not possible using any of the transformations we learned about before this lesson.
  - Is it possible to think of any parallelogram as a rectangle that has been transformed? If so, how would it be transformed?

    A nonrectangular parallelogram is like a rectangle that is leaning over so adjacent sides are no longer perpendicular.
- Let students know that this is an example of a new transformation, called a *shear transformation*. You can provide a definition for this term:

  *A shear transformation is a transformation that holds the points on one line constant and shifts every other point in a direction parallel to the line by a distance proportional to its perpendicular distance from the line.*



- To illustrate how shear transformations affect a polygon, you can show students this Desmos interactive: **preap.org/shear-transformation**.
- Let students know they will explore how shear transformations affect triangles in the next part of the lesson.

## PART 2: EXPLORING THE AREAS OF TRIANGLES UNDER SHEAR TRANSFORMATIONS

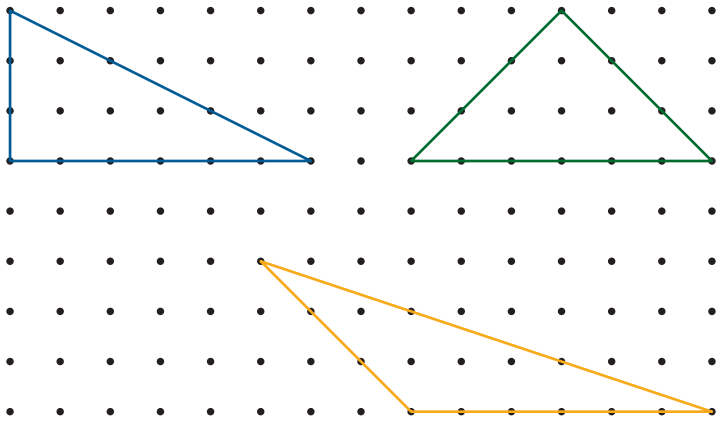
In this part of the lesson, students use dot paper to explore the effect of a shear transformation on the area of a triangle. Through the two investigations in this lesson, students gain a deeper understanding of the formulas for the areas of a parallelogram and a triangle.

- This part of the lesson should proceed in a manner similar to Part 1. This time, however, the figures students examine are triangles.
- Have students return to their original pairs and give each pair a copy of **Handout 4.1.B: Shearing Triangles**.

**Guiding Student Thinking**

Many students will create triangles whose bases are horizontal, but that is not absolutely necessary for this part of the activity. A base that is vertical would work equally well.

- As you circulate around the room, make sure that students observe that the areas of all three triangles they create are equal. If their triangles do not have equal areas, then students have miscalculated the areas or incorrectly created the figures. Here is an example of three noncongruent triangles with equal area:



- You can focus students' attention on the areas of their triangles by asking them some questions as you observe their work. Some sample questions are provided here:
  - What do you notice about the areas of the triangles you created?

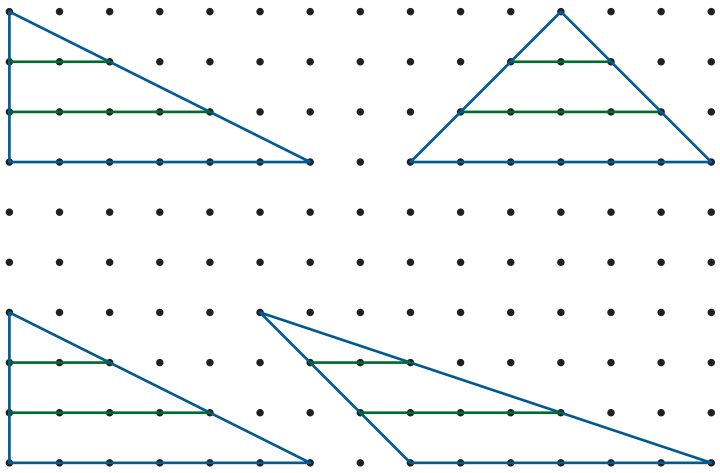
    The areas are equal.
  - Does it make sense that all the triangles you created have the same area? Why or why not?

    The triangles have equal heights and bases of equal length, so they enclose an equal amount of area.

**Instructional Rationale**

The goal of the last three questions on the handout is for students to realize that line segments drawn within the figures parallel to the base and at equal heights above the base have equal lengths. This is a key insight for understanding Cavalieri's principle.

- Problems 5, 6, and 7 on Handout 4.1.B ask students to draw line segments in the interior of each triangle parallel to the base and at equal heights above the base. Students may want rulers or compasses to measure the lengths of the segments.
- Because the segments are drawn on dot paper, it is possible for students to determine the lengths of the segments without a ruler. An example of such line segments are shown here:



- As you circulate around the room, make sure that students observe that line segments drawn at equal heights above the base are congruent. If the line segments are not congruent, then students have incorrectly drawn the segments or miscalculated their lengths.
- Once pairs of students reach problem 8 and have made conjectures about the areas of sheared triangles and the lengths of interior segments at equal heights above their bases, have pairs join together to create groups of four. If possible, create groups where the pairs have sets of triangles with different dimensions. Ask each pair to share their triangles and their areas with the other members of their group. Students should conclude that triangles with congruent bases and equal heights have the same area, and line segments drawn in the interiors of such triangles at equal heights above their bases are congruent.
- To debrief this part of the lesson, ask a student to share their conclusions about the areas of triangles that have congruent bases and equal heights. Then ask another student to share their conclusions about the lengths of line segments in the interior of such triangles at equal heights above their bases.

**Instructional Rationale**

It is important to connect this part of the lesson back to the rectangles and parallelograms from Part 1. The next set of questions will help students see that they can compare areas of figures with congruent bases, equal heights, and line segments drawn between the sides of those figures parallel to their bases and at equal heights above the bases.

- Have students look back to their work on Handout 4.1.A to determine if line segments drawn parallel to and at equal heights above the bases in their quadrilaterals are congruent. Students should recognize that the relationship between interior line segments that they observed in the triangles on Handout 4.1.B is also true in the quadrilaterals from Handout 4.1.A. Students may also observe that for a parallelogram, all line segments in its interior drawn parallel to the base are congruent.

**Meeting Learners' Needs**

Because these questions require students to think deeply, some students will benefit from engaging in a quickwrite activity, in which everyone works quietly to jot down an answer for the question you posed.

- You can conclude this part of the lesson by having students make a conjecture about the relationship between the areas of figures and the lengths of line segments drawn parallel to their bases in the interiors of these figures. Student responses should converge on the following observation:

  If two figures have congruent bases and equal heights, and line segments drawn parallel to and at equal heights above their bases are congruent, then the figures will have equal area.

## PART 3: INVESTIGATING CAVALIERI'S PRINCIPLE

In this part of the lesson, students extend what they learned about shear transformations in Parts 1 and 2 to nonpolygonal figures. A key insight of Cavalieri's principle is that if two figures have congruent bases and equal heights, and line segments drawn parallel to and at equal distances above their bases have equal lengths, then the figures have equal areas. The lesson concludes with a formal statement of Cavalieri's principle in two dimensions.

- Before formally stating Cavalieri's principle, take a few minutes to let students explore some problems involving area. Students should work individually on the problems on **Handout 4.1.C: Which Figures Have Equal Area?**
- Once they have answered all problems on the handout, they can pair up with a classmate to engage in an academic conversation about their answers.
- As you circulate around the room, listen for students to refer to the congruent bases of the figures, the equal heights of the figures, and the lengths of line segments drawn parallel to and at equal distances above the bases. See the Assess and Reflect section of the lesson for solutions to the handout.

**Instructional Rationale**

It is important for students to share their solutions to the problems on the handout with the class. This is an opportunity for students to critique the reasoning of others, which supports the development of students' abilities in mathematical argumentation.

- After discussing the solutions for the handout, you can use the following Desmos interactive to demonstrate how Cavalieri's principle can be used to compare the areas of two figures: **preap.org/cavalieri**.

**Instructional Rationale**

One feature of the Desmos interactive is that the number of segments in the interior of the figure can be increased or decreased. This kind of animation can help prepare students for the concepts involved in limits in calculus.

- Ask students some questions like the ones below during the demonstration:
  - What do you notice about the figure as the value of $T$ increases or decreases?

    A change in the value of $T$ changes the shape of the object from a rectangle to a parallelogram to a nonpolygonal figure.
  - What do you notice about the figure as the value of $n$ increases and decreases?

    A change in the value of $n$ changes the number of cross-sectional segments that are shown. When $n$ is large, the cross-sectional segments look like they fill the interior of the figure, which we measure as its area.
  - What can we conclude about two figures that do not look the same, but have congruent bases and equal heights, and would have cross-sectional segments of equal length at equal heights parallel to and above their bases?

    Those two figures have equal area.
- Share with students that they have been exploring an important property known as Cavalieri's principle. Provide this formal definition of the principle:

  *Cavalieri's Principle (for two dimensions) states that if two figures in the plane have congruent bases, equal heights, and would have cross-sectional interior segments of equal length parallel to and at equal heights above their bases, then the figures have equal areas.*
- To make sure students understand applications of Cavalieri's principle, have small groups of students work together on **Handout 4.1.D: Creating Figures with Equal Area**. On this handout, students must use the properties of Cavalieri's principle to complete a figure so that it has an area equal to a given figure. Students may need to use rulers to compare the lengths of segments in the interiors of the figures that are parallel to their bases at corresponding heights. See the Assess and Reflect section of this lesson for solutions to the handout.
- You can conclude the lesson by letting students know that Cavalieri's principle can also be applied in three dimensions to compare and find the volumes of solids. They will explore that relationship in the next key concept.

## ASSESS AND REFLECT ON THE LESSON

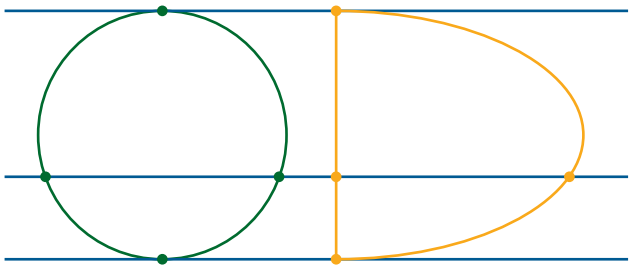
### FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

1. Based on Cavalieri's principle, what conditions must be met for the areas of two figures in the image below to be equal?

   If two figures have congruent bases, equal heights, and cross-sectional interior segments of equal length parallel to and at equal heights above their bases, then the figures have equal areas.

2. Determine whether the two figures in the image below have the same area. Explain your answer using Cavalieri's principle.



   The areas of the figures are equal because the figures have congruent bases (which, in this case, are single points) and equal heights, and because the lengths of interior line segments drawn parallel to and at equal heights above the straight line that includes the points that serve as bases are equal.

**Guiding Student Thinking**

If students struggle to answer question 2, you could ask them how Cavalieri's principle can help them determine whether the two figures have the same area. They might need to draw more lines parallel to the ones given in the image to understand how it applies. Students might also be initially confused by the concept of a base for a circular figure.

## HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

### Handout 4.1.A: Quadrilaterals with Equal Area

See lesson for answers.

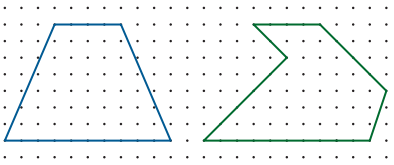
### Handout 4.1.B: Shearing Triangles

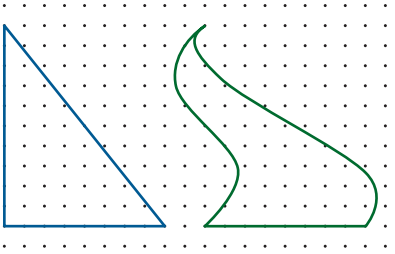
See lesson for answers.

### Handout 4.2.C: Which Figures Have Equal Areas?

1. Students should conclude that the areas are the same, as the two figures have the same base length and height and have congruent cross-sectional segments parallel to and at corresponding heights above the base.
2. Students should conclude that the areas are not the same, as the bases of the figures are not congruent.
3. Students should conclude that the areas are the same, as the two figures have the same base length (a single point) and height, and have congruent cross-sectional segments parallel to and at corresponding heights above the bases.
4. Students should conclude that the areas are the same, as the two figures have congruent bases, the same height, and congruent cross-sectional segments parallel to and at corresponding heights above the bases.
5. Students should conclude that the areas are the same, as the two figures have congruent bases, the same height, and congruent cross-sectional segments parallel to and at corresponding heights above the bases.
6. Students should conclude that the areas are not the same, because cross-sectional segments at corresponding heights above the bases are not congruent.

### Handout 4.2.D: Creating Figures with Equal Area

1. 

2. 

LESSON 4.2

UNIT 4

# Volumes of Prisms and Cylinders

## OVERVIEW

### LESSON DESCRIPTION

**Part 1: Investigating the Evolution Tower**
The lesson begins with a real-world example of a modern building with a twisted structure that differs from the traditional shape of a right prism. Students recall their earlier work with finding the volume of a solid, and consider alternative methods for determining volume.

**Part 2: Exploring Nonright Prisms**
In this part of the lesson, students examine nonright prisms to investigate how Cavalieri's principle can be applied in three dimensions. They then find the volume of the Evolution Tower from Part 1.

**Part 3: Applying Cavalieri's Principle to Cylinders**
In this part of the lesson, students extend what they know about Cavalieri's principle to cylinders and use it to determine the volumes of right and nonright cylinders.

### AREAS OF FOCUS

- Engagement in Mathematical Argumentation
- Greater Authenticity of Applications and Modeling

### SUGGESTED TIMING

~90 minutes

### HANDOUTS

**Lesson**

- 4.2: Exploring Stacks of Sticky Notes

### MATERIALS

- blank paper
- ruler/straightedge
- a pad of sticky notes (optional)

### CONTENT FOCUS

In this lesson, students explore how Cavalieri's principle provides a way to determine the volume of a prism when the area of a cross section of the solid is known. That is, the volume of a prism is equal to the product of the area of the base (or cross section) and the height of the prism. Through this lesson and Lesson 4.1, students come to understand that Cavalieri's principle for a planar figure relates a two-dimensional measurement, area, to a one-dimensional measurement, the length of a line segment parallel to the base, and for a solid, it relates a three-dimensional measurement, volume, to a two-dimensional measurement, the area of a cross section.

UNIT 4

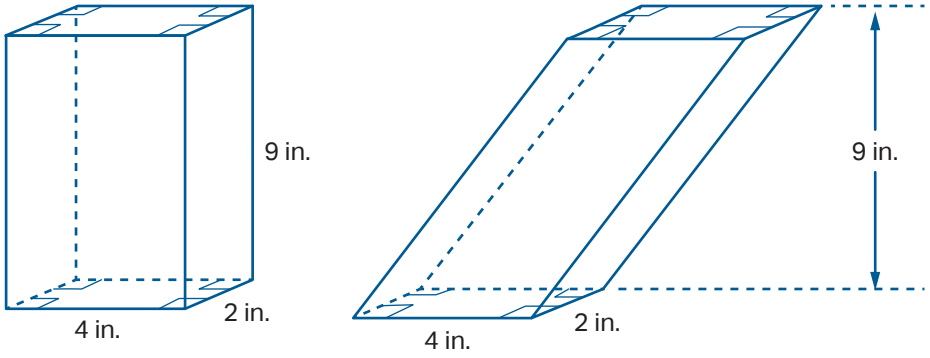
## COURSE FRAMEWORK CONNECTIONS

| Enduring Understandings | |
|---|---|
| ▪ The volume of a solid depends on its height and its cross-sectional areas. | |
| **Learning Objectives** | **Essential Knowledge** |
| **4.2.1** Justify the volume formula for a right prism. | **4.2.1a** The cross section of a right prism is the polygon formed by the intersection of the solid with a plane parallel to its base.<br>**4.2.1b** The volume of a right prism is equal to the product of the height of the solid and the area of its base. |
| **4.2.3** Justify the volume formula for a right cylinder. | **4.2.3a** The cross section of a right cylinder is the circle formed by the intersection of the solid with a plane parallel to its base.<br>**4.2.3b** The volume of a right cylinder is equal to the product of the height of the solid and the area of its base. |
| **4.2.5** Use Cavalieri's principle to solve problems involving volumes of solids. | **4.2.5a** If two solid figures have congruent bases and equal heights, and cross sections that are parallel to and equal distances from each base are congruent, then the solids have equal volume. |

**FORMATIVE ASSESSMENT GOAL**

This lesson should prepare students to complete the following formative assessment activity.

1. If the bases of these rectangular prisms are parallel, which prism has a greater volume? Explain your answer.




2. Luis is comparing the volumes of a narrow cylinder and a wide cylinder. The narrow cylinder has a radius of 2 inches and the wide cylinder has a radius of 6 inches. He observes that the wide cylinder's radius is three times the length of the narrow cylinder's radius. He says that if the narrow cylinder is three times as tall as the wide cylinder, Cavalieri's principle states that the two cylinders will have the same volume. What is wrong with his reasoning?

## PART 1: INVESTIGATING THE EVOLUTION TOWER

The lesson begins with a real-world example of a modern building with a twisted structure that differs from the traditional shape of a right prism. Students recall their earlier work with finding the volume of a solid, and consider alternative methods for determining volume.

- Display the photo shown on this page for students. Some students might express skepticism that this building actually exists. You can let them know that this is an actual office building in Moscow, Russia, called the Evolution Tower. At this point, refrain from providing more details about the building.
- Ask students what they notice and what they wonder. You could have them share their observations and questions with a partner before asking several students to share their thoughts with the whole class. Make sure to record their statements in a central location.

(Alamy / Leonid Serebrennikov)

- Once students have shared their observations and questions, you can provide some more information about the Evolution Tower. It is 722 feet tall and has 55 rectangular floors. Each floor is rotated about the center by an angle of 3 degrees relative to the floor below it. This rotation gives the building its characteristic swirled appearance. (The tower is listed as 807 feet tall to the tip, but the height to the top of its highest floor is 722 feet.)

- Let students know that this building was constructed in a location that requires both heating and air conditioning. There are many considerations for determining the optimal heating and cooling systems for the building, including its overall volume. Let students know that their goal will be to determine the volume of the Evolution Tower.
- Ask students to share everything they remember about volume. Expect students to state definitions and formulas. Through this brainstorm, make sure that students define volume as the amount of space a solid figure occupies.

**Meeting Learners' Needs**

Some students might need a brief review of the formula for the volume of a rectangular prism before being able to engage with the next part of the activity. This is a good opportunity to provide students who need a refresher with some practice problems involving the volumes of prisms and cylinders.

- Ask students some questions like the ones below to help them analyze the available information to determine if they know enough to find the volume of the building.
  - What shape is the building? Do we know how to find the volume of a shape like this?

    The building is not a familiar shape, like a prism. We do not have a formula for how to find the volume of this kind of shape.
  - What measurements might we need to determine the volume of this building?

    We might need to know the height of the building (722 feet). We also might need to know the area of each floor.

**Instructional Rationale**

The purpose of the second question above is to get students thinking about how to determine the volume of a solid figure that has an unfamiliar shape. Students may not be able to articulate an answer to the second question above, or they might suggest measurements that will not be helpful in determining the volume. That is acceptable at this point. It is more important that students make suggestions about what measurements they might need. They will revisit their suggestions at the end of Part 2 when they determine the volume of the Evolution Tower.

- Let students know that they will start by investigating some similar, but easier, problems to develop techniques they can use to find the volume of the Evolution Tower.

**Guiding Student Thinking**

Working on an easier problem to develop an approach for solving a harder problem is a very powerful problem-solving strategy. Students can sometimes struggle with how to make a problem easier while retaining some or all of the challenging aspects of the problem. In this case, the swirled shape of the building needs to be kept but the size of the building can be scaled down. In Part 2, students explore a stack of sticky notes that has been twisted in manner similar to the building. This will help them figure out that twisting a prism into a swirl does not affect its volume, as long as each slice of the stack—that is, each sticky note in the pad or each floor of the building—is congruent.

## PART 2: EXPLORING NONRIGHT PRISMS

In this part of the lesson, students examine nonright prisms to investigate how Cavalieri's principle can be applied in three dimensions. Then they find the volume of the Evolution Tower from Part 1.

- Begin this part of the lesson by displaying the two images of the stacks of sticky notes for students. Ask students to compare and contrast these stacks of sticky notes to each other and to the Evolution Tower. Provide students with a minute of quiet work time so they can brainstorm some thoughts, and then give them another minute to share their ideas with a classmate.

Stack 1

Stack 2

(Can Stock Photo / Anton Starikov)

**Instructional Rationale**

The compare-and-contrast activity that begins this part of the lesson is an important first step in helping students understand why the sticky note problem can help them solve the Evolution Tower problem. The images of the stacks of sticky notes are also on Handout 4.2. You may want to provide students with their own copies of these images at this time. The handout also includes questions for students to answer, and students may feel compelled to answer the questions on the handout rather than engage in the compare-and-contract activity. Use your discretion as to whether it would be better to distribute the handout first or wait until after the activity is completed.

- Have one or two students share their observations about what is the same and what is different between the two stacks of sticky notes. Students may suggest that that the sticky notes look like they might be the same size, or that the stacks seem to be the same height. They may observe that the stack on the right appears to be twisted, or that each note is rotated by some angle about a center point, similar to the floors of the Evolution Tower. Students may wonder if the stacks have the same number of sticky notes.
- At this point, you can pair up students and distribute **Handout 4.2: Exploring Stacks of Sticky Notes** to each pair. As you circulate around the room, monitor how students are answering the questions and identify the questions with which students are struggling. It might be worthwhile to address these questions as a whole class. See the Assess and Reflect section of the lesson for answers to the handout questions.
- Once most students have made conjectures about the volumes of prisms with equal heights and slices of equal volume (problem 9 on the handout), you can bring the class back together to summarize the main points of the lesson.
- Have students share their conjectures about the volumes of prisms that have equal heights and slices of equal volume. Students should state that two prisms with equal heights and slices of equal length and width and thickness have equal volume.

**Meeting Learners' Needs**

Some students will need more support to answer the handout questions. Some of the handout questions ask students to make conjectures and support their conjectures with evidence. You can best support struggling students by having them offer any initial conjecture, even if it is off-base, and then helping them refine their conjecture by asking probing questions that get them to think more deeply about the volume of the sticky notes.

- Ask students some questions like the ones that follow to get them thinking about the thickness of a slice of a solid.
  - Does the volume of the stack of sticky notes change if the stack is shaped like a right prism or twisted? Why or why not?

    A twisted stack can be untwisted so it has the shape of a right prism. So the volumes of the sticky notes in a right prism shape and the sticky notes in a twisted stack must be equal.
  - What is the total height of each stack of sticky notes?

    Students should be able to calculate that, because there are 500 single notes and each note is 0.004 inches thick, the height of each stack is 2 inches.
  - Assuming that the stack is 2 inches tall, how many single sticky notes would be in the stack if each note was 0.002 inches thick, which is half as thick as the original note? How do you know?

    There would be 1,000 notes in the stack because $\frac{2\text{ in}}{0.002\text{ in}} = 1{,}000$.
  - Would the volume of the stack be different if each note was half as thick as the notes in the original stack, but there were twice as many notes? Why or why not?

    The volume would be the same as the volume of the original stack. Even though the volume of each note would be half the volume of each original note, there are twice as many notes.
  - Again, assuming that the stack is 2 inches tall, how many single sticky notes would be in the stack if each note was 0.001 inches thick, which is one-fourth as thick as the notes in the original stack? Would the volume of the stack be different than the volume of the original stack?

    There would be 2,000 notes in the stack because $\frac{2\text{ in}}{0.001\text{ in}} = 2{,}000$. The volume will be the same as the volume of the original stack. Even though the volume of each individual note would be one-fourth the volume of the original note, there are four times as many notes.
  - Suppose that we continue this process of making each individual sticky note thinner without changing the total height of the stack. What could we conclude about the number of single sticky notes in the stack? What could we conclude about how the volume of the stack with ever-thinner sticky notes compared to the volume of the original stack?

We could conclude that the number of individual sticky notes would increase. We could also conclude that the volume of the stack would be equal to the volume of the original stack.

- If the individual sticky notes became so thin that that they appeared to have almost no thickness, would we still be able to determine the area of the note? Why or why not? Could we use the area of each thin sticky note to determine the volume of the stack?

  If each individual note was so thin that it had almost no thickness, we would still be able to determine its area because the length and width of the note would not have changed. Even though each note has almost no visible thickness, stacking these notes on top of each other would produce a stack with a measurable height. We can use the area of each note and multiply it by the height of the stack to determine the volume of the stack.

- Use the area of a single sticky note and the height of the stack to determine the volume of the stack. Does it agree with what you calculated on the handout?

  The volume would be (3 in)(3 in)(2 in) = 18 $in^3$ which is the same as our calculation on the handout.

**Instructional Rationale**

Some students might struggle with the idea of a stack made of thinner and thinner individual sticky notes that includes more and more notes. The idea that a solid figure can be thought of as infinitely many infinitesimally thin slices is central to determining the volume of a solid using calculus. While some students may not completely understand the idea, it is important to expose them to it at this point in their mathematical experience.

- You can take a moment to summarize the preceding discussion with students. You can let them know that they have demonstrated Cavalieri's principle for three dimensions: If two prisms have the same height and congruent bases, and each slice parallel to and at the same height from their bases have the same area, then the prisms have equal volume.

- Now, show the image of the Evolution Tower again:

(Alamy / Leonid Serebrennikov)

- Ask students some questions to help them connect the sticky note activity to the Evolution Tower.
  - How can we apply what we learned from our investigation of the stacks of sticky notes to help us figure out the volume of the Evolution Tower?

    Students should recognize that if they know the area of each congruent floor, they can multiply that by the height of the building to find its volume, just as if it were a right rectangular prism.

  - We know that the Evolution Tower has a height of 722 feet. If each floor of the tower has an area of about 16,000 square feet, what is the volume of the building?

    The volume of the building is about $(722 \text{ ft})(16{,}000 \text{ ft}^2) \approx 11{,}552{,}000 \text{ ft}^3$.

- Before moving on the next part of the lesson, students should understand two important concepts about volume: 1) The volume of a solid prism can be determined by knowing the area of the base and multiplying it by the height of the prism, and 2) Cavalieri's principle states that if two prisms have congruent bases, equal heights, and the areas of slices parallel to the base, called cross sections, are equal, then the volume of the prisms will be equal.

## PART 3: APPLYING CAVALIERI'S PRINCIPLE TO CYLINDERS

In this part of the lesson, students extend what they know about Cavalieri's principle to cylinders and use it to determine the volume of right and nonright cylinders. Students work through a cylinder problem using an approach similar to the one they followed for the prism problems in Part 2 to see that Cavalieri's principle applies to both prisms and cylinders.

- Present the following problem to students and work through it together as a class:

  *Thirty quarters are stacked with each quarter placed exactly on top of the one below it to form a cylinder. Each quarter has a diameter of 24.26 mm and a thickness of 1.75 mm. What is the volume of the cylinder formed by 30 quarters?*

**Classroom Ideas**
You can use a stack of any thin, congruent cylindrical objects to model these questions. Some students may benefit from seeing a physical demonstration with a stack of quarters or other similar objects.

We can find the area of the base, which is a circle. Since the quarter has a diameter of 24.26 mm, we can calculate the area of the face of the quarter to be $\left(\frac{24.26 \text{ mm}}{2}\right)^2 \pi$, which is approximately 462.24 $\text{mm}^2$. Next, we need to find the height of the stack. Each quarter has a thickness of 1.75 mm and there are 30 quarters, so the height of the stack is (1.75 mm)(30) = 52.5 mm. The volume of the stack is the area of the face of a quarter multiplied by the height of the stack: (462.24 $\text{mm}^2$)(52.5 mm) = 24,267.6 $\text{mm}^3$.

- Next, present the following image and problem:

  *Suppose that the stack of quarters has been moved and the quarters are no longer stacked perfectly aligned in a column. Will the stack of quarters have the same volume as it originally did? Why or why not? Use Cavalieri's principle to justify your answer.*

Students should recognize that although the stack of quarters changed shape, the area of the face of a quarter and the height of the stack did not change, so the stacks have equal volume. Cavalieri's principle tells us that if the two prisms or cylinders are the same height and each cross section parallel to and at equal heights above the base have the same area, then the solids have equal volume.

**Guiding Student Thinking**

Some students may not identify that the slices, or cross sections, need to be made at equal distances above the base. In the cases of prisms and cylinders, all cross sections parallel to the base are congruent. Therefore, it is easy to overlook the distance that the cross section is away from the base. This will be more important as students apply Cavalieri's principle to pyramids, cones, and other solids.

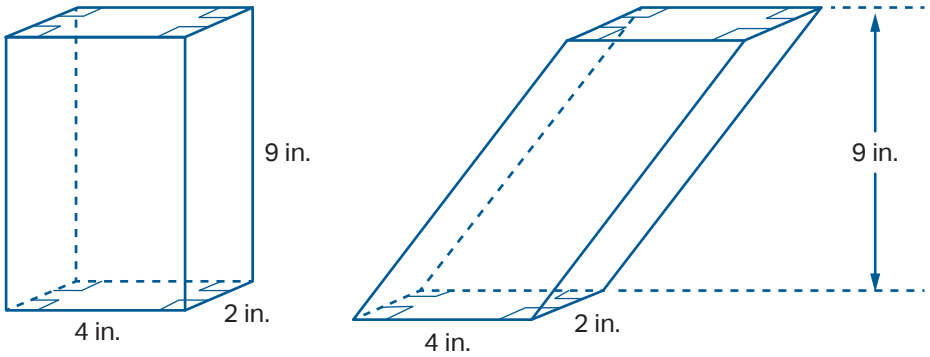
While this lesson does not explore how Cavalieri's principle applies to other solid figures such as pyramids, cones, and spheres, students could explore those applications as well. The key observation for students to make is that solids with congruent bases and equal heights that have the same cross-sectional areas parallel to and at corresponding heights above the base have equal volumes.

## ASSESS AND REFLECT ON THE LESSON

### FORMATIVE ASSESSMENT GOAL

When your students have completed the lesson, you can use this task to gain valuable feedback on and evidence of student learning.

1. If the bases of these rectangular prisms are parallel, which prism has a greater volume? Explain your answer.




Neither prism has a greater volume, because their volumes are equal. Because the bases are congruent, all cross sections made parallel to a base are congruent, and because the heights are also equal, the prisms will have equal volume.

2. Luis is comparing the volumes of a narrow cylinder and a wide cylinder. The narrow cylinder has a radius of 2 inches and the wide cylinder has a radius of 6 inches. He observes that the wide cylinder's radius is three times the length of the narrow cylinder's radius. He says that if the narrow cylinder is three times as tall as the wide cylinder, Cavalieri's principle states that the two cylinders will have the same volume. What is wrong with his reasoning?

Cavalieri's principle only applies when the two solids have congruent cross sections parallel to the base at equal heights. Because the cylinders have different radii and different heights, Cavalieri's principle does not apply.

**Guiding Student Thinking**

If students struggle with problem 1, they may need help with understanding that the cross sections of the prism parallel to the base have equal area for every distance from the base, therefore the volumes are equal. If students cannot articulate the mistakes in Luis's reasoning in problem 2, they may not understand that Cavalieri's principle only applies under specific conditions.

## HANDOUT ANSWERS AND GUIDANCE

To supplement the information within the body of the lesson, additional answers and guidance on the handouts are provided below.

### Handout 4.2: Exploring Stacks of Sticky Notes

1. The areas of the faces of the sticky notes from each stack will be equal because the faces of the sticky notes have the same dimensions, 3 inches by 3 inches.
2. The stacks have the same volume because each individual sticky note has the same volume, so 500 sticky notes together in either stack have the same volume.
3. A single sticky note has volume because it has a third dimension of height, even though its measure is much less than that of either its length or width.
4. Multiplying the volume of one sticky note by 500 will yield the volume of an entire stack.
5. The volume of a single sticky note in stack 1 is (3 in)(3 in)(0.004 in) = 0.036 $in^3$.
6. Since the volume of a single sticky note is 0.036 $in^3$, the volume of the entire stack of sticky notes is (0.036 $in^3$)(500) = 18 $in^3$.

**Guiding Student Thinking**

It is important for students to report their answers for problems 5 and 6 using the correct units. In problem 5, each of the three dimensions is given in inches. The product of these three measurements yields the volume, in cubic inches, of a single sticky note. The units to use in the answer to problem 6 may be less clear to students, who may have difficulty making sense of the units of "500." It may be helpful to use dimensional analysis methods write the calculation as:

$$\left(0.036\,\frac{\text{in}^3}{\text{sticky note}}\right)\left(500\,\frac{\text{sticky notes}}{\text{stack}}\right)=18\,\frac{\text{in}^3}{\text{stack}}.$$

7. The volume of a single sticky note in stack 2 is (3 in)(3 in)(0.004 in) = 0.036 $in^3$. The volume of the entire stack of sticky notes is (0.036 $in^3$)(500) = 18 $in^3$.
8. The volumes of stacks 1 and 2 are equal.
9. Prisms of equal height that have congruent cross-sectional slices have equal volumes.

UNIT 4

PRACTICE PERFORMANCE TASK

# Digging a Ditch

## OVERVIEW

### DESCRIPTION

In this practice performance task, students model a landscaping project using a trapezoidal prism. The task requires them to utilize what they know about the area of planar figures and the relationship between the cross section of a solid and its volume.

### CONTENT FOCUS

This task is designed to assess students' understanding of the relationships between area, surface area, and volume. In this task, students engage in a real-world scenario involving digging a ditch to relieve flooding in a person's yard. The diagram of the ditch is not provided for students, so they must make choices about the shape of the ditch based on the information given. The ditch will not be the exact geometric shape that students use to model the situation, which increases the authenticity and challenge of the modeling aspects of the task.

### AREAS OF FOCUS

- Engagement in Mathematical Argumentation
- Greater Authenticity of Applications and Modeling

### SUGGESTED TIMING

~45 minutes

### HANDOUTS

Unit 4 Practice Performance Task: Digging a Ditch

### MATERIALS

- scientific calculator or graphing utility

## COURSE FRAMEWORK CONNECTIONS

| Enduring Understandings | |
|---|---|
| ▪ The area of a figure depends on its height and its cross-sectional widths.<br>▪ The volume of a solid depends on its height and its cross-sectional areas. | |
| **Learning Objectives** | **Essential Knowledge** |
| **4.2.1** Justify the volume formula for a right prism. | **4.2.1a** The cross section of a right prism is the polygon formed by the intersection of the solid with a plane parallel to its base.<br>**4.2.1b** The volume of a right prism is equal to the product of the height of the solid and the area of its base. |
| **4.2.6** Solve contextual problems involving volume of solid figures. | **4.2.6a** Physical objects in many real-world scenarios can be modeled by solid geometric figures such as prisms, pyramids, cylinders, and cones. |

## SUPPORTING STUDENTS

### BEFORE THE TASK

In this practice performance task, students are expected to use their understanding of area, surface area, and volume in a contextual scenario involving digging a ditch. You should rely on your classroom customs regarding formulas when implementing this task. That is, if students usually have a formula sheet or you usually display important formulas for students, then you should continue that practice for this task as well. However, if you do not usually provide formulas to students, then you should not feel obligated to make formulas available for this task.

### DURING THE TASK

Because this is a practice performance task, you could choose to have students engage in the task differently than a conventional assessment. You could choose an implementation strategy such as one of the following:

- Students could work in pairs to complete the task. It is not recommended for students to work in small groups. While there is ample work and enough potential discussion areas for two students, there may not be enough for each student to meaningfully engage in with larger groups.
- You could chunk the task into its three parts and have student pairs complete one part at a time. Students could check their solutions with you or the scoring guidelines before moving on to the next part. During the check, spend a few moments discussing the solution with the student. Focus on what changes, if any, students could make to their solution to craft a more complete response the next time they engage in a performance task.
- Alternatively, you could have students complete the task individually and then distribute the scoring guide to students to have them score their own responses or those of a classmate.
- Finally, have students reflect on their solution and the scoring guidelines to make recommendations to themselves about what they could do to craft a more complete response the next time they engage in a performance task.

During the task, students might ask about the shape of the ditch. The problem is designed to give students an opportunity to engage in an authentic modeling situation. This means that they should make decisions about the shape of the ditch based on the information they have. If they suggest that the shape of the cross section could be a trapezoid, ask them about whether a trapezoid will satisfy the boss's requirements. If they decide that it does, then it is a good model for the ditch.

## AFTER THE TASK

Whether you decide to have students score their own solutions, have students score their classmates' solutions, or score the solutions yourself, the results of the practice performance task should be used to inform instruction.

Students should understand that converting their score into a percentage does not provide a useful measure of how they performed on the task. You can use the suggested scoring conversion guide that follows the scoring guidelines to discuss their performance.

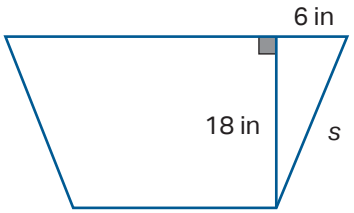
UNIT 4

## SCORING GUIDELINES

There are 12 possible points for this performance task.

**Student Stimulus and Part (a)**

(a) Your boss tells you that you need to line the ditch with plastic sheeting. Estimate the minimum amount of plastic you will need to line the ditch. Explain how you determined your answer.

| Sample Solutions | Points Possible |
|---|---|
| To find the amount of plastic needed to line the bottom of the ditch, we can find the area of the bottom of the ditch. The bottom can be modeled by a rectangle whose width is 12" and length is 96". The amount of plastic is 1,152 square inches.<br><br>A reasonable assumption is that the ends of the ditch are isosceles trapezoids. The area of each trapezoid will be $\frac{1}{2}(18)(12+24)=324$ square inches, so the total area of both ends is 648 square inches.<br><br>To find the amount of plastic for the sides of the ditch, we need to find the slant height of the trapezoid. If the trapezoid is isosceles, then the slant height can be found with the Pythagorean theorem. Note that this figure is not drawn to scale.<br><br>  <br><br><br>$s=\sqrt{18^2+6^2}$<br>$s\approx 19$ in | **4 points maximum**<br><br>1 point for the area of the bottom of the ditch<br><br>1 point for the area of both ends of the ditch<br><br>1 point for the area of both sides of the ditch<br><br>1 point for the total amount of plastic needed to line the ditch |

| | |
|---|---|
| The amount of plastic needed for the two sides is approximately $2(19)(96) = 3{,}648$ square inches.<br>The total amount of plastic needed to line the ditch is approximately $1{,}152 + 648 + 3{,}648 = 5{,}448$ square inches. | |

**Targeted Feedback for Student Responses**

Students may need support in part (a) to determine the slant height of the ditch. You can help them see the right triangle in the cross section of the trapezoid by suggesting that a carefully placed auxiliary segment could reveal a familiar and useful figure.

TEACHER NOTES AND REFLECTIONS

UNIT 4

## Student Stimulus and Part (b)

(b) Estimate the maximum amount of water that the ditch can hold without overflowing. Explain how you determined your answer.

| Sample Solutions | Points Possible |
|---|---|
| The ditch can be modeled by a trapezoidal prism. The volume of the trapezoidal prism model of the ditch expresses how much water the ditch can hold. The volume of the ditch is calculated by multiplying the area of the trapezoid base by the length of the ditch.<br>The area of the trapezoid is given by $\frac{1}{2}(18)(12+24)=324$ square inches.<br>The length of the ditch is 8 feet, or 96 inches. The volume is $(324)(96) = 31{,}104$ cubic inches. | **2 points maximum**<br>1 point for the cross-sectional area of the ditch<br>1 point for the volume of the ditch |
| **Targeted Feedback for Student Responses** | |
| If students make mistakes in part (b), it may be because they do not understand that the volume of the model of the ditch will help determine how much water could fill the ditch, or because they are unsure of how to calculate the volume of an unfamiliar prism using the area of the base of the prism. You can help them understand that the volume of a solid is a measure of the amount of space that a solid figure takes up. In this case, the solid figure is the filled ditch. | |

TEACHER NOTES AND REFLECTIONS

### Student Stimulus and Part (c)

(c) If you increase the length of the ditch by 1 foot, by how much does the volume of the ditch change?

| Sample Solutions | Points Possible |
|---|---|
| One approach would be to use the calculation procedure from part (b) to determine the volume of a ditch that is 9 feet long and subtract the volume of a ditch that is 8 feet long.<br>$V = A_{trapezoid} \bullet \text{length}$<br>$V = 324 \bullet 108$<br>$V = 34{,}992$<br>Therefore, the additional volume is $34{,}992 - 31{,}104 = 3{,}888$ cubic inches.<br>A second approach would be to divide the volume calculated in part (b) by 8: $\frac{31{,}104}{8} = 3{,}888$ cubic inches.<br>A third approach would be to calculate the volume of a trench that is only 1 foot (12 inches) long:<br>$V = A_{trapezoid} \bullet \text{length}$<br>$V = 324 \bullet 12$<br>$V = 3{,}888$ cubic inches | **3 points maximum**<br>1 point for the correct approach<br>1 point for the correct volume<br>1 point for the correct units |

**Targeted Feedback for Student Responses**

If students make a mistake in part (c), it could be that they calculated the volume of a ditch that is 9 feet long but then did not subtract the volume of the 8-foot-long ditch. You could help students understand that the difference between the volumes can be determined by calculating the volume of a ditch that has a congruent trapezoid base and is 1 foot long.

TEACHER NOTES AND REFLECTIONS

UNIT 4

## Student Stimulus and Part (d)

(d) The client decides that they want to line the ditch with a $\frac{1}{2}$ inch layer of rocks to make it look nicer. Estimate how much rock you would need, in cubic inches.

| Sample Solutions | Points Possible |
|---|---|
| Because a $\frac{1}{2}$ inch layer of rocks will be placed along the sides, bottom, and ends of the ditch, the calculation for the amount of plastic needed to cover the ditch can be used to estimate the amount of rock needed.<br>Because the rocks need to cover 5,448 square inches and have a depth of $\frac{1}{2}$ inch, you would need about 2,724 cubic inches of rocks. | **3 points maximum**<br>1 point for a reasonable method of estimating the volume of rocks needed<br>1 point for a reasonable estimate of the volume of rocks needed<br>1 point for the correct units<br>*Scoring note*: Students do not have to use their value from part (a) to answer this question. They may use a different approach, but if the approach is appropriate and valid then they should receive the first point. |

**Targeted Feedback for Student Responses**

If students make mistakes in part (d), it may be because they are not sure how to use surface area to estimate volume. You can help them see that because the rocks will have a uniform thickness, the surface area can be helpful in an estimate of volume. Some students may choose to directly calculate the volume of the rocks. It is a valuable discussion for students to have to reflect on the advantages and disadvantages of each approach.

TEACHER NOTES AND REFLECTIONS

| Points Received | Appropriate Letter Grade (If Graded) | How Students Should Interpret Their Score |
|---|---|---|
| 11 or 12 points | A | "I know all of this geometry really well." |
| 8 to 10 points | B | "I know all of this geometry well, but I made a few mistakes." |
| 5 to 7 points | C | "I know some of this geometry well, but not all of it." |
| 2 to 4 points | D | "I only know a little bit of this geometry." |
| 0 or 1 point | F | "I don't know much of this geometry at all." |

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Unit 4

# Performance Task

PERFORMANCE TASK

# Star Energy

## OVERVIEW

### DESCRIPTION

In this performance task, students use the surface area and volume of a sphere to answer questions about the amount of power generated and emitted by a star.

### CONTENT FOCUS

This task is designed to assess students' understanding of the surface area and volume of a sphere. Spheres are helpful models for three-dimensional round objects, such as stars and planets. The dimensions in this task are intentionally large to give students an opportunity to utilize scientific notation in context.

### AREA OF FOCUS

- Greater Authenticity of Applications and Modeling

### SUGGESTED TIMING

~45 minutes

### HANDOUTS

Unit 4 Performance Task: Star Energy

### MATERIALS

- scientific calculator or graphing utility

## COURSE FRAMEWORK CONNECTIONS

| Enduring Understandings | |
|---|---|
| ▪ The geometry of a sphere is completely determined by its radius. | |
| **Learning Objectives** | **Essential Knowledge** |
| **4.3.2** Justify the surface area formula for a sphere. | **4.3.2a** The surface area of a sphere is given by the formula $SA = 4\pi r^2$, where $r$ represents the length of the radius of the sphere. |
| **4.3.3** Justify the volume formula for a solid sphere. | **4.3.3a** The volume of a solid sphere is given by the formula $V = \frac{4}{3}\pi r^3$, where $r$ represents the length of the radius of the sphere. |
| **4.3.4** Solve contextual problems using spheres. | **4.3.4a** Round physical objects in real-world scenarios can be modeled by spheres. |

## SUPPORTING STUDENTS

### BEFORE THE TASK

In this performance task, students are expected to solve an equation of the form $Ax^3 = Bx^2$, where $A$ and $B$ are constants and $x > 0$. If students have not solved equations like this before, you could have them engage in a short warmup activity. Some sample problems are provided here:

- Solve these equations: $5x^4 = 10x^3$; $4y^5 = 36y^3$; $8z^{12} = 6z^{10}$

  $x = 0$, or $x = 2$; $y = 0$, or $y = \pm 3$; $z = 0$, or $z = \pm\frac{\sqrt{3}}{2}$

- What techniques did you use to solve these equations? Why do you think your technique worked?

  I realized that if the variable is equal to zero, then the products on both sides of the equal sign would be zero, so zero is a solution to the equation. Also, I divided both sides of the equation by the variable term with the smaller exponent. Then I divided both sides of the equation by one of the coefficients to isolate the variable. I took the square root of both sides for the second and third equations.

In the performance task, the solution to the equation in part (c) will not be equal to zero, because the solution indicates the radius of a star in equilibrium. For this reason, the equation in the performance task can be solved by dividing both sides of the equation by the term with the smaller exponent. This technique would not be appropriate if the solution were allowed to be zero. In Algebra 2, students will learn how to manipulate polynomial equations so that all terms are on one side of the equation. Students can then write the polynomial as a product of its factors.

UNIT 4

## SCORING GUIDELINES

There are 9 possible points for this performance task.

**Student Stimulus and Part (a)**

(a) Suppose the star has a radius of about $7 \times 10^8$ cm. How much power would the interior of the star generate? How much power would the star's surface shine into space? Which quantity is greater?

| Sample Solutions | Points Possible |
|---|---|
| If the star has a radius of $7 \times 10^8$ cm, then the volume is determined by $V = \frac{4}{3}\pi\left(7 \times 10^8\right)^3$. The volume of the star is approximately $1.4 \times 10^{27}$ cubic centimeters. To determine the power generated by the interior of the star, multiply the volume by the number of watts/cm$^3$: $1.4 \times 10^{27}\,\text{cm}^3 \bullet 3 \times 10^{-7}\,\frac{\text{watts}}{\text{cm}^3} \approx 4.2 \times 10^{20}$ watts. If the star has a radius of $7 \times 10^8$ cm, then the surface area is determined by $A = 4\pi\left(7 \times 10^8\right)^2$. The surface area of the star is approximately $6.2 \times 10^{18}$ square centimeters. To determine the power that the surface of the star shines into space, multiply the surface area by the number of watts/cm$^2$: $6.2 \times 10^{18}\,\text{cm}^2 \bullet 6{,}000\,\frac{\text{watts}}{\text{cm}^2} \approx 3.7 \times 10^{22}$ watts. The power that the surface of the star shines into space is greater than the power generated by the interior of the star. | **3 points maximum**<br>1 point for the power generated by the interior of the star<br>1 point for the power the star shines into space<br>1 point for the conclusion that the power the star shines into space is greater than the power generated by the interior of the star<br>*Scoring note*: Students should receive a point for a conclusion that is consistent with their calculations, even if their calculations are incorrect. |

**Targeted Feedback for Student Responses**

If students make mistakes in part (a), it may be because of the scientific notation. Students sometimes struggle with entering numbers written in scientific notation into their calculators, especially because different models of calculators may use different notations. It is possible that students understand the concepts of surface area and volume but cannot use their tools effectively.

TEACHER NOTES AND REFLECTIONS

## Student Stimulus and Part (b)

(b) Suppose the star has a radius of about $5\times10^{12}$ cm. How much power would the interior of the star generate? How much power would the star's surface shine into space? Which quantity is greater?

| Sample Solutions | Points Possible |
|---|---|
| If the star has a radius of $5\times10^{12}$ cm, then the volume is determined by $V=\frac{4}{3}\pi\left(5\times10^{12}\right)^3$. The volume of the star is approximately $5.2\times10^{38}$ cubic centimeters. To determine the power generated by the interior of the star, multiply the volume by the number of watts/cm$^3$:<br>$5.2\times10^{38}\,\text{cm}^3\bullet3\times10^{-7}\,\frac{\text{watts}}{\text{cm}^3}\approx1.6\times10^{32}$ watts.<br>If the star has a radius of $5\times10^{12}$ cm, then the surface area is determined by $A=4\pi\left(5\times10^{12}\right)^2$. The surface area of the star is approximately $3.1\times10^{26}$ square centimeters. To determine the power that the surface of the star shines into space, multiply the surface area by the number of watts/cm$^2$:<br>$3.1\times10^{26}\,\text{cm}^2\bullet6{,}000\,\frac{\text{watts}}{\text{cm}^2}\approx1.9\times10^{30}$ watts.<br>The power generated by the interior of the star is greater than the power that the surface of the star shines into space. | **3 points maximum**<br>1 point for the power generated by the interior of the star<br>1 point for the power the star shines into space<br>1 point for the conclusion that the power generated by the interior of the star is greater than the power the star shines into space<br>*Scoring note*: Students should receive a point for a conclusion that is consistent with their calculations, even if their calculations are incorrect. |

UNIT 4

| Targeted Feedback for Student Responses |
|---|
| If students make mistakes in part (b), it may be because it is similar to part (a) and they have repeated a conceptual or procedural error from the previous problem. Additionally, students sometimes ignore the units in their calculations. Attending to the units of the values in the calculation can help students make sense of their work and their solutions. |

TEACHER NOTES AND REFLECTIONS

## Student Stimulus and Part (c)

(c) Life on Earth depends on our Sun maintaining equilibrium, neither growing nor shrinking. Likewise, in this conceptual model of the star, if the star is too small, then the number of watts leaving the surface will exceed the number of watts generated in the interior, and the star will cool down. If the star is too large, then the number of watts generated in the interior will exceed the number of watts leaving the surface, and the star will heat up. What is the radius of the star for which the power leaving the surface equals the power generated in the interior?

| Sample Solutions | Points Possible |
|---|---|
| To determine the radius at which the power leaving the surface equals the power generated in the interior, this equation should be solved for $r$:<br>$(6{,}000)(4\pi r^2) = (3\times10^{-7})\left(\frac{4}{3}\pi r^3\right)$<br>The equation simplifies to<br>$2.4\times10^4 r^2 = 4\times10^{-7} r^3$.<br>Because the radius will be positive, we can divide both sides by $r^2$ and by $4\times10^{-7}$ to obtain the radius: $r = 6\times10^{10}$ cm. | **3 points maximum**<br>1 point for setting up the equation<br>1 point for the correct value of the radius<br>1 point for the correct units |

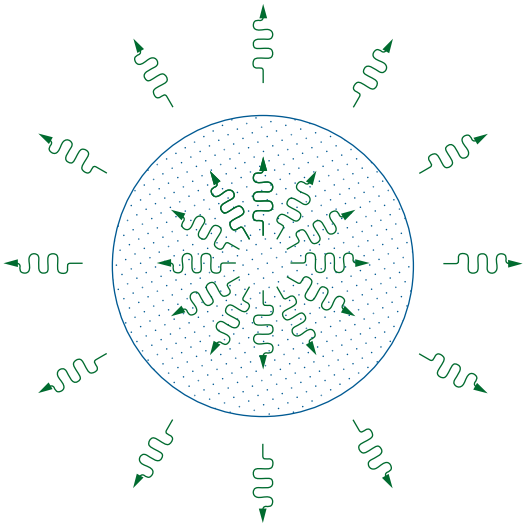
| Targeted Feedback for Student Responses |
|---|
| Students may be unsure how to set up or solve the equation in part (c). They may need additional support for solving equations of monomials. |

TEACHER NOTES AND REFLECTIONS

| Points Received | Appropriate Letter Grade (If Graded) | How Students Should Interpret Their Score |
|---|---|---|
| 8 or 9 points | A | "I know all of this geometry really well." |
| 6 or 7 points | B | "I know all of this geometry well, but I made a few mistakes." |
| 4 or 5 points | C | "I know some of this geometry well, but not all of it." |
| 2 or 3 points | D | "I only know a little bit of this geometry." |
| 0 or 1 point | F | "I don't know much of this geometry at all." |

UNIT 4

# Star Energy



The diagram above shows a conceptual model of a spherical star that generates and emits light energy. Here are the assumptions we will make about the star:

- Each cubic centimeter of the star's interior generates $3 \times 10^{-7}$ watts of power.
- Each square centimeter of the star's surface shines 6,000 watts of power into space.

(a) Suppose the star has a radius of about $7 \times 10^{8}$ cm. How much power would the interior of the star generate? How much power would the star's surface shine into space? Which quantity is greater?

(b) Suppose the star has a radius of about $5 \times 10^{12}$ cm. How much power would the interior of the star generate? How much power would the star's surface shine into space? Which quantity is greater?

(c) Life on Earth depends on our Sun maintaining equilibrium, neither growing nor shrinking. Likewise, in this conceptual model of the star, if the star is too small, then the number of watts leaving the surface will exceed the number of watts generated in the interior, and the star will cool down. If the star is too large, then the number of watts generated in the interior will exceed the number of watts leaving the surface, and the star will heat up. What is the radius of the star for which the power leaving the surface equals the power generated in the interior?